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Polyhedral Results for the Pickup and Delivery Travelling Salesman Problem†

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Abstract. In this paper we consider a variant to the classical symmetric travelling salesman problem (TSP), the TSP with extra precedence constraints on vertex pairs. We call this problem the pickup and delivery travelling salesman problem (PDTSP). We conduct a polyhedral analysis of the PDTSP. We determine the size of the associated polytope, we propose new valid inequalities, and we identify several classes of inequalities that are facets for the PDTSP polytope.

Keywords. Travelling salesman problem, precedence constraints, pickup and delivery, facets.

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1 Introduction

The travelling salesman problem, both in its symmetric (TSP) as well as its asymmetric version (ATSP), consists of finding the shortest Hamiltonian cycle that visits all vertices in a graph that has lengths associated with its edges/arcs. Although the TSP has direct applications, in many real-life problems additional constraints are needed. In many vehicle routing, scheduling, or distribution management applications, additional constraints are needed to impose precedence: some vertices need to be visited before others. In our paper we will study the TSP with precedence constraints, which we will refer to as the pickup and delivery travelling salesman problem (PDTSP). The PDTSP can be used to model pickup and delivery problems with only one vehicle, when no time windows or vehicle capacity are involved. The PDTSP has direct applicability to urban courier services.

The PDTSP is related to the Dial-a-Ride Problem (DARP). In the case of DARP the nodes in the graph represent customer requests and a depot. The customers can be serviced by one or more vehicles of limited capacity. Time windows may be associated with each customer request. Quality service constraints may also be imposed. For a recent survey on the DARP we refer the reader to [3].

Another problem that is related to the PDTSP is the TSP with backhauls where an additional constraint is imposed: all pickup customers must be visited before any of the delivery customers (see for example [4]).

In our paper we will study the PDTSP defined on a graph that has an origin vertex, a destination vertex, and the set of all other vertices split into two disjoint subsets of equal size. The two subsets are defined by the precedence constraints. One of the subsets contains what we call *pickup vertices* and the other contains *delivery vertices*, terminology used by analogy with routing problems in which an item has to be first picked up somewhere, then delivered at a different location. The precedence constraints that we consider in this paper are very simple: each pickup node has to preceed exactly one delivery vertex, specified a priori. Therefore each pickup vertex has one corresponding delivery vertex. The PDTSP is NP-hard since it can be reduced to the travelling salesman problem [9].

The PDTSP has received received relatively little attention so far, less than its asymmetric counterpart. While several papers propose exact algorithms that solve various variants of the PDTSP [6, 7, 12], most of them focus on heuristic solution methods [5, 9, 10, 13, 14]. As far as we are aware only Ruland and Rodin have looked at the polyhedral structure of the PDTSP [11, 12]. The PDTSP they studied is exactly the one that we consider in our paper. However, apart from the validity of several classes of constraints Ruland and Rodin give no other proof. In our paper, we will fill in the gaps in the literature and prove some new results. We will determine the dimension of the PDTSP polytope, we will introduce new valid inequalities, and we will show under what conditions several classes of valid inequalities are facets for the PDTSP polytope.

2 Definitions and notation

The results presented in this paper are built on top of the theory presented by Ruland [11] and Ruland and Rodin [12]. For this reason we will keep their modelling conventions without repeating here the integer programming model proposed by them. For modelling convenience, the depot is described by two depot vertices: a start depot and a destination depot. Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. The set of vertices is partitioned into three subsets: two sets of equal size, V^p and V^d (corresponding to the set of pickup respectively delivery vertices), and a set that contains the two depot vertices. We will denote the start depot by 0 and the destination depot by $2n + 1$. It is clear that since in reality only one depot exists, 0 and $2n + 1$ are always connected by an edge. The pickup vertices will be those in $V^p = \{1, \ldots, n\}$ and the destinations vertices those in $V^d = \{n + 1, \ldots, 2n\}$. The destination corresponding to any pickup vertex $i \in V^p$ is $n + i \in V^d$.

We denote an edge between i and j by (i, j). Clearly, since the graph is undirected, $(i, j) = (j, i)$. We use the notation $(i_0, i_1, \ldots, i_{2n+1})$ for the tour $((i_0, i_1), (i_1, i_2), \ldots, (i_{2n}, i_{2n+1}), (i_{2n+1}, i_0))$.

Given a set of vertices $S \subseteq V$ we denote by $\Pi(S)$ the set of all sequences that are permutations of vertices in S (paths that visit all vertices in S exactly once). If $S \subseteq V^p$, we emphasize the fact that all vertices are *pickup* vertices by writing any permutation $\sigma \in \Pi(S)$ by σ^p. Similarly, if $S \subseteq V^d$, we denote any permutation $\sigma \in \Pi(S)$ by σ^d. Given a sequence of vertices σ we will denote its first vertex by σ_s and its last vertex by σ_e. (By extension, we will have $\sigma^p_s, \sigma^p_e, \sigma^d_s, \sigma^d_e$.) In what follows, if in the description of a tour we write both σ^p and σ^d, we mean by that that σ^d is the delivery vertex sequence corresponding to the pickup vertex sequence σ^p. In other words, the k-th element of σ^d is the delivery corresponding to the k-th element of σ^p (i.e. $\sigma^d_k = n + \sigma^o_k$), for any $k = 1, \ldots, l(\sigma^p)$, where $l(\sigma^p) = l(\sigma^d)$ is the number of vertices contained in σ^p and σ^d. We will also use the notation $\overleftarrow{\sigma}$ to denote the sequence of vertices σ written in reverse order, i.e if $\sigma = (i_1, \ldots, i_l)$, $\overleftarrow{\sigma} = (i_l, i_{l-1}, \ldots, i_1)$. Clearly, $\sigma_s = \overleftarrow{\sigma}_e$ and $\sigma_e = \overleftarrow{\sigma}_s$. We note that if a permutation is defined on the empty set, then it will not contain any vertices. If such a permutation appears in the description of a tour, we will read the tour *without* that permutation.

Assuming that the set E is ordered, let B^E be the set of binary vectors, with components indexed by E. We associate an incidence vector $x \in B^E$ with every tour. The vector x is defined as follows: $x_{ij} = 1$ if $(i, j) \in E$ and $x_{ij} = 0$ otherwise. For notational convenience we do not distinguish between a tour and its incidence vector. We also perform arithmetic on tours. For example $(1, 2, 3)-(2, 3, 5) = ((1, 2), (2, 3), (3, 1))-((2, 3), (3, 5), (5, 2)) = (1, 2)+(1, 3)-(3, 5)-(2, 5)$. The incidence vector corresponding to $(1, 2, 3) - (2, 3, 5)$ will have 1 on the positions corresponding to $(1, 2), (1, 3), (3, 5)$, and $(2, 5)$ and 0 on the positions corresponding to every other edge.

Definition 2.1 *Let $\mathcal{T}$ be the set of all feasible tours of the PDTSP. We call the polytope*

$$\mathcal{P}_{PDTSP} = conv\{x \in B^E : x \in \mathcal{T}\}$$

the pickup and delivery travelling salesman problem polytope*, or the* PDTSP polytope.

If what follows we use the standard notation $\delta(i) = \{(i,j) : (i,j) \in E\}$, where $i \in V$, to denote is the set of edges in E that contain i.

Assumption 2.2 *We make the following assumptions:*

1. $\delta(0) = \{(0,1),(0,2),\ldots,(0,n)\}$ *and* $\delta(2n+1) = \{(n+1,2n+1),\ldots,(2n-1,2n+1)\}$.
2. *Let G' be the subgraph of G induced by $V^p \cup V^d$. G' is a complete graph.*

We note that the first assumption simply means that the graph G we consider is the graph obtained at the end of a preprocessing step. The graph G cannot be further reduced.

3 Dimension of the PDTSP polytope

In order to establish the dimension of the PDTSP polytope, we need to introduce an order on the set of edges. The order that we will use is derived from the lexicographic order. We mention that for the following definition, and for all subsequent order definitions, we consider the edges written in increasing order of their vertices. For example, if $i < j$, we will write the edge between i and j as (i,j), not as (j,i).

Definition 3.1 *Let $E^0 = \{(0,2n+1)\}$ and $E^1 = E \setminus (E^0 \cup E^2)$, where $E^2 = (\delta(0) \cup \delta(2n+1) \cup \{(n,2n)\}) \setminus E^0$. Let $\prec_{E^1}$ be the lexicographic order on the set E^1 and $\prec_{E^2}$ the lexicographic order on the set E^2. We define a relation of total order $\prec$ on the set of edges E as follows:*

i. *for any $(i,j) \in E \setminus E^0$, $(0,2n+1) \prec (i,j)$;*

ii. *the restriction of $\prec$ to E^1 is $\prec_{E^1}$;*

iii. *the restriction of $\prec$ to E_2 is $\prec_{E^2}$;*

iv. *for any $(i,j) \in E^1$ and $(k,l) \in E^2$, $(i,j) \prec (k,l)$.*

Definition 3.2 *Let v be an incidence vector of a feasible tour in the PDTSP polytope or a vector obtained after performing arithmetic on feasible tours in the PDTSP polytope. If the components of v are denoted by $v_{(i,j)}$, $(i,j) \in E$, we say that the edge (k,l) is* leading *for v, if it $v_{(k,l)} \neq 0$ and $v_{(i,j)} = 0, \forall (i,j) \prec (k,l)$.*

Remark 3.3 *The number of edges in E is $2n^2 + n + 1$.*

An upper bound on the dimension of the PDTSP polytope is given in the following proposition, proved by Ruland [11].

Proposition 3.4 *The dimension of the PDTSP polytope is at most $2n^2 - n - 2$.*

Proof. The rank of the equality constraints is $2n + 3$ (see [11]), so by Proposition 2.4 from Chapter I.4 of Nemhauser and Wolsey [8] the polytope has dimension at most $|E| - (2n + 3) = 2n^2 - n - 2$. ■

Theorem 3.5 *The PDTSP polytope has dimension* $2n^2 - n - 2, \forall n \geq 2$.

Proof.

From Proposition 3.4 we know that the dimension of the PDTSP polytope is at most $2n^2 - n - 2$, i.e.

$$dim(\mathcal{P}_{PDTSP}) \leq 2n^2 - n - 2. \tag{1}$$

We will prove that there are $(2n^2-n-2)+1$ lineary independent feasible tours in the PDTSP polytope. These tours will form a set of $2n^2 - n - 1$ affinely independent elements of the PDTSP polytope and therefore the dimension of the polytope is at least $2n^2 - n - 2$, i.e.

$$dim(\mathcal{P}_{PDTSP}) \geq 2n^2 - n - 2. \tag{2}$$

From 1 and 2 it follows that the dimension of the PDTSP polytope is $2n^2 - n - 2$.

We now need to find $2n^2 - n - 1$ linear combinations of feasible tours in the PDTSP polytope. We take each feasible tour and consider it a row in a matrix, in which every column corresponds to an edge (ordered increasingly with respect to the order introduced in Definition 3.1). By row operations we will find $2n^2 - n - 1$ linearly independent vectors that are linear combinations of rows in the matrix (i.e. feasible tours) and form an upper triangular matrix. The rank of the upper triangular matrix will be $2n^2 - n - 1$ and so the rank of the initial matrix (the one that has *all* the feasible tours as its rows) will be at least $2n^2 - n - 1$. Therefore there are $2n^2 - n - 1$ linearly independent rows of that matrix. Hence there are $2n^2 - n - 1$ linearly independent feasible tours of the PDP polytope, which is what we need to show.

We will group the linear combinations of feasible tours in several sets: T_i, $i = 0, \ldots, 7$. The union of the sets will provide the linearly independent linear combinations of feasible tours needed. Each vector in a set T_i will have a leading element corresponding to a different edge from the first $2n^2 - n - 1$ edges (ordered according to the order defined on the set of edges).

We now describe the sets T_i, $i = 0, \ldots, 7$.

0. Leading edge $(\mathbf{0}, \mathbf{2n+1})$:

We define the set T_0 that contains only one element: $T_0 = \{(0, 1, 2, \ldots, 2n, 2n + 1)\}$.

1. Leading edges $(\mathbf{1}, \mathbf{i}), \mathbf{i} = \mathbf{2}, \ldots, \mathbf{n}$:

We construct the vectors m_i as linear combinations of feasible tours in the PDTSP polytope such that their leading edges are $(1, i)$.

- $\mathbf{i = 2, \ldots, n-1}$.
  Let $\sigma^p \in \Pi(V^p \setminus \{1, i, n\})$.

$$m_i = (0,1,i,n,n+1,n+i,2n,\sigma^p,\sigma^d,2n,2n+1) - (0,1,n,i,n+1,n+i,2n,\sigma^p,\sigma^d,2n,2n+1)$$
$$= (1,i) + (n,n+1) - (1,n) - (i,n+1)$$

  Clearly, the leading edge of m_i is $(1,i)$.

- $\mathbf{i = n}$.
  Let $\sigma^p \in \Pi(V^p \setminus \{1, n\})$.

$$\begin{aligned} m_n &= (0,1,n,n+1,2n,\sigma^p,\sigma^d,2n+1) \\ &\quad - (0,1,n+1,n,2n,\sigma^p,\sigma^d,2n+1) \\ &\quad = (1,n) - (1,n+1) + (n+1,2n) - (n,2n). \end{aligned} \tag{3}$$

  The leading edge of m_n is $(1,n)$.

We define the set T_1 as the set of all vectors m_i, i.e. $T_1 = \{m_i : i = 2, \ldots, n\}$.

2. Leading edges $\mathbf{(1, n+i), i = 1, \ldots, n}$:
We construct the vectors a_i as linear combinations of feasible tours such that their leading edges are $(1, n+i)$.

- $\mathbf{i = 1}$.
  Let $\sigma^p \in \Pi(V^p \setminus \{1, n\})$.

$$a_1 = (0,\sigma^p,\sigma^d,n,1,n+1,2n,2n+1) - (0,\sigma^p,\sigma^d,n,1,2n,n+1,2n+1)$$
$$= (1,n+1) - (1,2n) + (2n,2n+1) - (n+1,2n+1).$$

  The leading edge of a_1 is $(1, n+1)$.

- $\mathbf{i \neq 2}$.
  Let $\sigma^p \in \Pi(V^p \setminus \{1, i, n\})$.

$$a_i = (0,i,\sigma^p,\sigma^d,n+i,1,n+1,2n+1) - (0,1,n+1,i,\sigma^p,\sigma^d,n+i,2n+1)$$
$$= (1,n+i) + (i,n+1) + (n+1,2n+1) - (n+i,2n+1) + (0,i) - (0,1).$$

  Clearly, the tours involved in the calculation of a_i, $i = 2, \ldots, n-1$ are feasible. The leading edge of any such a_i is $(1, n+i)$.

We define the set T_2 to be the set of all vectors a_i, i.e. $T_2 = \{a_i : i = n+1, \dots, 2n\}$.

3. Leading edges $(\mathbf{i}, \mathbf{j}), \mathbf{i} = \mathbf{2}, \dots, \mathbf{n-2}, \mathbf{j} = \mathbf{i+1}, \dots, \mathbf{n-1}$:
We construct the vectors b_{ij} as linear combinations of feasible tours. Let $\sigma^p \in \Pi(V^p \setminus \{i, j\}$.

$$\begin{aligned} b_{ij} &= (0, \sigma^p, \sigma^d, i, j, n+i, n+j, 2n+1) - (0, \sigma^p, \sigma^d, i, n+i, j, n+j, 2n+1) \\ &= (i, j) - (i, n+i) - (j, n+j) + (n+i, n+j) \end{aligned}$$

The leading edge of any such b_{ij} is (i, j).
We define the set T_3 to be the set of all vectors b_{ij}, i.e. $T_3 = \{b_{ij} : i = 2, \dots, n-1, j = i+1, \dots, n\}$.

4. Leading edges $(\mathbf{i}, \mathbf{n+j}), \mathbf{i} = \mathbf{2}, \dots, \mathbf{n-1}, \mathbf{j} = \mathbf{1}, \dots, \mathbf{n}$:
We construct the vectors c_{ij} as linear combinations of feasible tours, such that their leading edges are $(i, n+j)$.

- $\mathbf{i \neq j, j \neq n}$
  Let $\sigma^p \in \Pi(V^p \setminus \{i, j, n\})$.

$$\begin{aligned} c_{ij} &= (0, j, n+j, i, n, 2n, n+i, \sigma^p, \sigma^d, 2n+1) - (0, j, n+j, n, i, 2n, n+i, \sigma^p, \sigma^d, 2n+1) \\ &= (i, n+j) - (i, 2n) - (n, n+j) + (n, 2n) \end{aligned}$$

  Clearly, the leading edge of any such c_{ij} is $(i, n+j)$.

- $\mathbf{i = j}$.
  Since $i = j$ and $i = 2, \dots, n-1$, we have $j \neq n$. Let $\sigma^p \in \Pi(V^p \setminus \{i, n\})$.

$$\begin{aligned} c_{ii} =& (0, n, i, n+i, 2n, \sigma^p, \sigma^d, 2n+1) - (0, i, n, n+i, 2n, \sigma^p, \sigma^d, 2n+1) \\ =& (i, n+i) - (n, n+i) + (0, n) - (0, i). \end{aligned}$$

  The leading edge of c_{ii} is $(i, n+i)$.

- $\mathbf{j = n}$.
  Let $\sigma^p \in \Pi(V^p \setminus \{i, n\})$.

$$\begin{aligned} c_{in} &= (0, n, i, 2n, n+i, \sigma^p, \sigma^d, 2n+1) - (0, i, n, 2n, n+i, \sigma^p, \sigma^d, 2n+1) \\ &= (i, 2n) - (n, 2n) + (0, n) - (0, i) \end{aligned}$$

  The leading edge of any c_{in} is $(i, 2n)$.

We define the set T_4 to be the set of all vectors c_{ij}, i.e. $T_4 = \{c_{ij} : i = 2, \dots, n-1, j = 1, \dots, n\}$.

5. Leading edges $(\mathbf{n}, \mathbf{n}+\mathbf{i}), \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}-\mathbf{1}$:
We construct the vectors d_i as linear combinations of feasible tours in the PDTSP polytope, such that their leading edges are $(n, n+i)$.

Let $\sigma^p \in \Pi(V^p \setminus \{i, n\})$.

$$\begin{aligned} d_i &= (0, \sigma^p, \sigma^d, i, n, n+i, 2n, 2n+1) - (0, \sigma^p, \sigma^d, i, n, 2n, n+i, 2n+1) \\ &= (n, n+i) + (2n, 2n+1) - (n, 2n) - (n+i, 2n+1). \end{aligned}$$

Since $n + i < 2n, \forall i = 1, \ldots, n-1$, the leading edge of any d_i is $(n, n+i)$.

We define the set T_5 as the set of all vectors d_i, i.e. $T_5 = \{d_i : i = 1, \ldots, n-1\}$.

6. Leading edges $(\mathbf{n}+\mathbf{i}, \mathbf{n}+\mathbf{j}), \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}-\mathbf{2}, \mathbf{j} = \mathbf{i}+\mathbf{1}, \ldots, \mathbf{n}-\mathbf{1}$:
We construct the vectors e_{ij} as linear combinations of feasible tours in the PDTSP polytope such that their leading edges are $(n+i, n+j)$.

Let $\sigma^p \in \Pi(V^p \setminus \{i, j, n\})$.

$$\begin{aligned} e_{ij} &= (0, \sigma^p, \sigma^d, i, j, n, n+i, n+j, 2n, 2n+1) - (0, \sigma^p, \sigma^d, i, j, n, n+i, 2n, n+j, 2n+1) \\ &= (n+i, n+j) - (n+i, 2n) + (2n, 2n+1) - (n+j, 2n+1) \end{aligned}$$

Since $i < j$, it follows that $n + i < n + j$. Also, since $i < n$, we have $n + i < 2n$. Therefore, the leading edge of any vector e_{ij} is $(n+i, n+j)$.

We define the set T_6 to be the set of all vectors e_{ij}, i.e. $T_6 = \{e_{ij} : i = 1, \ldots, n-2, j = i+1, \ldots, n-1\}$.

7. Leading edges $(\mathbf{n}+\mathbf{i}, \mathbf{2n}), \mathbf{i} = \mathbf{1}, \mathbf{n}-\mathbf{2}$:
We construct the vectors f_i as linear combinations of feasible tours in the PDTSP polytope such that their leading edges are $(n+i, 2n)$.

Let $\sigma^p \in \Pi(V^p \setminus \{i, n-1, n\})$.

$$\begin{aligned} f_i &= (0, \sigma^p, \sigma^d, n-1, n, i, 2n, n+i, 2n-1, 2n+1) - (0, \sigma^p, \sigma^d, n-1, n, i, 2n, 2n-1, n+i, 2n+1) \\ &= (n+i, 2n) - (2n-1, 2n) + (2n-1, 2n+1) - (n+i, 2n+1) \end{aligned}$$

The leading edge of any vector f_i is $(n+i, 2n)$.

We define the set T_7 as being the set of all vectors f_i, i.e. $T_7 = \{f_i : i = 1, \ldots, n-2\}$.

Finally, we define the set T to be the union of all sets T_i, i.e. $T = \cup_{i=0}^{7} T_i$. The size of the union set is given by $|T| = |T_0| + |T_1| + \cdots + |T_7|$. From the definition of the sets T_i we have that $|T_0| = 1$, $|T_1| = n-1$, $|T_2| = n$, $|T_3| = \frac{(n-3)(n-2)}{2}$, $|T_4| = n(n-2)$, $|T_5| = n-1$, $|T_6| = \frac{(n-2)(n-1)}{2}$, and $|T_7| = n-2$. Therefore, $|T| = 2n^2 - n - 1$.

The vectors in T have distinct leading edges, which are the first $2n^2 - n - 1$ edges with respect to the order introduced (Definition 3.1). Therefore, modulo row interchanging, they form an upper triangular matrix with $2n^2 - n - 1$ rows and thus the rank of this matrix is $2n^2 - n - 1$. Hence, the rank of the original matrix (which has *all* the feasible tours of the PDTSP polytope as its rows) is at least $2n^2 - n - 1$. In other words, there are at least $2n^2 - n - 1$ linearly independent rows of the matrix (i.e. feasible tours of the PDTSP polytope), which concludes the proof. ■

4 Valid inequalities for the PDTSP polytope

We start this section by showing that in any valid constraint for the PDTSP the role of the pickup and delivery vertices can be reversed. We will describe some classes of valid inequalities that appear in the literature, point out which ones are not facets, and give an extension for one of the classes. We will also introduce some new valid inequalities for the PDTSP.

First, we need to introduce some notation. For any set of vertices $S \subseteq V$, we define $\mathcal{P}(S)$, the set of *predecessors of* S, as follows:

$$\mathcal{P}(S) = \{i \in V^p : n + i \in S\} \cup \{0\}$$

Similarly, we define the set of *successors of* S as being

$$\mathcal{S}(S) = \{n + i \in V^d : i \in S \cup V^p\} \cup \{2n + 1\}. \tag{4}$$

For any $S \subseteq V$, we denote by $\bar{S}$ the set $V \setminus S$. For any $S, T \subseteq V$, we denote by $[S; T]$ the set $\{(i, j) \in E : i \in S, j \in T\}$. We define $x([S, T]) = \sum_{(i,j)\in[S,T]} x_{i,j}$. For simplicity, we will use the notation $x(S)$ instead of $x([S, S])$.

We note that if we reverse the precedence constraints so that any vertex $n + i \in V^d$ preceeds $i \in V^p$ and so that 0 and $2n + 1$ are the destination, repectively the origin vertex, we obtain a new PDTSP that has its feasible tours in a one-to-one correspondence with the feasible tours of the original PDTSP. An immediate consequence of this observation is the following proposition.

Proposition 4.1 *If $ax \leq b$ is a valid inequality for the PDTSP, then there exists another valid inequality $a'x \leq b$, where $a'_{i,j} = a_{n+i,n+j}$, $a'_{i,n+j} = a_{n+i,j}$, $a'_{n+i,n+j} = a_{i,j}$, $a'_{n+i,j} = a_{i,n+j}$, $a'_{0,i} = a_{n+i,2n+1}$, $a'_{n+i,2n+1} = a_{0,i}$, and $a'_{0,2n+1} = a_{0,2n+1}$, for all $i, j \in V^p$. If $ax \leq b$ is facet for the PDTSP polytope, then $a'x \leq b$ is also a facet.*

4.1 Valid inequalities in the literature

Ruland proved the following result for the PDTSP polytope ([11]).

Proposition 4.2 Order Constraints: *For any $i_1, i_2 \in V^p$, $i_1 \neq i_2$, the inequality*

$$x_{i_1,n+i_2} + x_{n+i_1,i_2} \leq 1 \tag{5}$$

is valid and a proper face of the PDTSP polytope.

Proof. See [11]. ■

Balas et al. introduced a new class of valid inequalities for the asymmetric case [1], a generalisation of the order constraints of Ruland [11]. These inequalities can be extended to the symmetric case. We note that the Balas et al. inequalities can be also found in [11], where it is proved that the inequalities define proper faces of the PDTSP polytope. We merge the two results as follows:

Proposition 4.3 Generalised Order Constraints: *Let $S_1, \ldots, S_m \subseteq V^p \cup V^d$ be disjoint sets such that $S_i \cap \mathcal{P}(S_{i+1}) \neq \emptyset, \forall i = 1, \ldots, m$, where $S_{m+1} = S_1$.*

Then the inequality

$$\sum_{i=1}^{m} x(S_i) \leq \sum_{i=1}^{m} |S_i| - m - 1 \tag{6}$$

is valid and defines a proper face of the PDTSP polytope.

Proof. See [11]. ■

Observation 4.4 *The generalised order constraints (6) do not define facets for the PDTSP polytope.*

This observation is easy to check using Porta for small values of n. For example, for $n = 2$, the dimension of the generalised order constraint (which, in this case, is exactly the order constraint) is 1, while the dimension of the PDTSP polytope is 4. An inequality is facet for the PDTSP polytope for $n = 2$ if its dimension is 3.

Another class of valid inequalities for the PDTSP are inequalities lifted from the order matching constraints for the TSP and are described in the following theorem.

Proposition 4.5 Order Matching Constraints: *For any $i_1, \ldots, i_m \in V^p$ and $H \subset V \setminus \{n+i_1, \ldots, n+i_m, 0, 2n+1\}$ such that $\{i_1, \ldots, i_m\} \subset H$, the constraint*

$$x(H) + \sum_{j=1}^{m} x_{i_j,n+i_j} \leq |H| \tag{7}$$

is valid and a proper face for the PDTSP polytope.

The order matching constraints were introduced by Ruland [11] who proved (and stated) the theorem only for the case m even. However, we note that if a different kind of induction is used, Ruland's proof can be extended to cover both cases: m odd and m even. After the checking step (for $m = 2$ the constraints are the simple order constraints), the induction step that needs to be applied is: "for an arbitrary k, $2 < k < n$, we assume that the theorem is true for any $m = 2, \dots, k$. We prove that (7) is valid for $m = k + 1$." With the induction step so modified and used in the proof, the rest of the demonstration remains unchanged.

Cordeau has proved that a lifting of the order matching constraints is valid for the DARP [2]. However, due to the symmmetry, the constraints and the proof hold for the case of the PDTSP too.

Proposition 4.6 Lifted Order Matching Constraints: *Let $i_1, \dots, i_m \in V^p$, $2 \leq m \leq n$, and let $H \subset V^p \cup V^d$ and the disjoint sets of vertices $T_h \subset V^p \cup V^d$, for all $h = 1, \dots, m$, such that $\{i_h, n+i_h\} \subseteq T_h$ and $H \cap T_h = \{i_h\}$, for all $h = 1, \dots, m$. Then the constraint*

$$x(H) + \sum_{h=1}^{m} x(T_h) \leq |H| - \sum_{h=1}^{m} |T_h| - 2m \tag{8}$$

is valid for the PDTSP.

Ruland proved that the following result is true for the PDTSP polytope [11].

Proposition 4.7 Precedence Constraints: *For any $U \subseteq V$, $3 \leq |U| \leq |V| - 2$, with $0 \in U$, $2n+1 \notin U$, and for which there is a unique $i \in V^p, i \in \bar{U}, n+i \in U$, the inequality*

$$x([U : \bar{U}]) \geq 4 \tag{9}$$

is valid for the PDTSP polytope.

Proof. See [11]. ■

We now describe a set of valid inequalitites for the PDTSP polytope, which are a strenthening of the subtour elimination constraints for the TSP. These inequalities were first introduced by Balas et al. in [1].

Proposition 4.8 *For any $S \subseteq V \setminus \{2n+1\}$, the inequality*

$$x([S \setminus \mathcal{P}(S), \bar{S} \setminus \mathcal{P}(S)]) \geq 1 \tag{10}$$

is valid for the PDTSP polytope.

Proof. Let T be any feasible tour for the PDTSP. If we consider 0 to be the first vertex on the tour $\mathcal{T}$, let $\hat{s}$ be the last vertex from S on the tour $\mathcal{T}$. We note that $\hat{s} \in S \setminus \mathcal{P}(S)$. The successor of $\hat{s}$ in $\mathcal{T}$ is not in S and cannot be in $\mathcal{P}(S)$, therefore it must belong to $\bar{S} \setminus \mathcal{P}(S)$. It follows that the edge between $\hat{s}$ and its successor in $\mathcal{T}$ links $S \setminus \mathcal{P}(S)$ and $\bar{S} \setminus \mathcal{P}(S)$. ■

Observation 4.9 *The inequalities introduced in Proposition 4.8 do not define facets for the PDTSP polytope.*

This observation is easy to check with Porta for small values of n.

4.2 New valid inequalities

We now introduce a new class of valid inequalities that lift of the subtour elimination constraints for the TSP.

Proposition 4.10 Lifted Subtour Elimination Constraints: *Let $S \subseteq V^p \cup V^d$ with the property that there exists $i \in V^p$ such that $i \in S$ and $n+i \in S$. Then the inequality*

$$x(S) + \sum_{\substack{j \in S, \\ n+j \notin S}} x_{i,n+j} \leq |S| - 1 \tag{11}$$

is valid for the PDTSP polytope.

Proof. The subtour elimination constraint for the TSP applied to S is:

$$x(S) \leq |S| - 1. \tag{12}$$

Therefore, (11) is a lifting of the subtour elimination constraint.

It is clear that if $x_{i,n+j} = 0$ for any $j \in S$ such that $n+j \notin S$, the inequality (11) is valid.

We now look at the situation when there is $l \in S$ for which $n+j \notin S$, such that $x_{i,n+l} = 1$.

From the degreee equation for i and $x_{i,n+l} = 1$, it follows that

$$x_{i,n+i} + \sum_{\substack{j \in S, j \neq l, \\ n+j \notin S}} x_{i,n+j} + \sum_{j \in S} x_{i,j} \leq 1 \tag{13}$$

Now let $S_1 = \{i, n+l\}$ and $S_2 = S \setminus \{i\}$. The generalised order constraint applied to S_1 and S_2 (see Proposition 4.3) is:

$$x_{i,n+l} + x(S_2) \leq |S| - 2. \tag{14}$$

We remark that we have the following equality:

$$x(S) + \sum_{\substack{j \in S, \\ n+j \notin S}} x_{i,n+j} = x(S_2) + x_{i,n+l} + x_{i,n+i} + \sum_{j \in S} x_{i,j} + \sum_{\substack{j \in S, j \neq l, \\ n+j \notin S}} x_{i,n+j} \tag{15}$$

From (15), (13), and (14) we obtain:

$$x(S) + \sum_{\substack{j \in S, \\ n+j \notin S}} x_{i,n+j} \leq |S| - 2 + 1 = |S| - 1$$

Therefore, inequality (11) is valid for the PDTSP polytope. ■

Next, we introduce a class of inequalities that work when three pickup vertices and their corresponding delivery vertices are involved.

Proposition 4.11 *If $n \geq 3$ and $i_1, i_2, i_3 \in V^p$, the inequality*

$$x_{i_1,i_2}+x_{i_1,i_3}+x_{i_1,n+i_1}+x_{i_2,i_3}+x_{i_2,n+i_1}+x_{i_3,n+i_1}+x_{i_1,n+i_2}+x_{i_1,n+i_3}+x_{i_2,n+i_2}+x_{i_2,n+i_3}+x_{i_3,n+i_2} \leq 4 \quad (16)$$

is valid for the PDTSP polytope.

Proof. The inequality obtained from Proposition 4.10 for $S = \{1, 2, 3, n + 1\}$ is:

$$x_{i_1,i_2} + x_{i_1,i_3} + x_{i_1,n+i_1} + x_{i_2,i_3} + x_{i_2,n+i_1} + x_{i_3,n+i_1} + x_{i_1,m+i_2} + x_{i_1,n+i_3} \leq 3 \quad (17)$$

The simple order constraint for i_2 and i_3 (Proposition 4.2) is

$$x_{i_2,n+i_3} + x_{i_3,n+i_2} \leq 1 \quad (18)$$

By adding up the inequalities (17) and (18) we obtain

$$x_{i_1,i_2} + x_{i_1,i_3} + x_{i_1,n+i_1} + x_{i_2,i_3} + x_{i_2,n+i_1} + x_{i_3,n+i_1} + x_{i_1,m+i_2} + x_{i_1,n+i_3} + x_{i_2,n+i_3} + x_{i_3,n+i_2} \leq 4 \quad (19)$$

We see that all we need to show is that by adding $x_{i_2,n+i_2}$ to the lefthandside term of (19), which gives us (16), the inequality remains valid.

From (19), it is clear that if $x_{i_2,n+i_2} = 0$, (16) is valid.

We now look at the case $x_{i_2,n+i_2} = 1$. In this case we distinguish three situations:

1. If $x_{i_2,n+i_3} + x_{i_3,n+i_2} = 0$ and $x_{i_2,n+i_2} = 1$, it is clear that (16) is valid.

2. If $x_{i_3,n+i_2} = 1$, it follows that the tour contains the ordered sequence of vertices $i_2, n + i_2, i_3$. From the degree constraint for $n + i_2$ we get $x_{i_1,n+i_2} = 0$. From the subtour elimination constraint for the set of vertices $\{i_2, i_3, n + i_2\}$, knowing that $x_{i_2,n+i_2} = x_{i_3,n+i_2} = 1$, we obtain $x_{i_2,i_3} = 0$.

   (a) If $x_{i_3,n+i_1} = 1$, then we have $x_{i_1,i_3} = 0$. Also, since i_3 is between $n + i_2$ and $n + i_1$, it follows that i_1 must be before i_2 on the tour. The maximal total value for the variables is obtained when $x_{i_1,i_2} = 1$, $x_{i_1,n+i_1} = 0$, $x_{i_1,n+i_3} = 0$, and ${}_{i_2,n+i_1} = 0$. Therefore (16) is valid.

   (b) If $x_{i_3,n+i_1} = 0$, we have several situations:

      - If $x_{i_1,i_2} = 1$, it follows that $x_{i_1,i_3} = 0$ and $x_{i_1,n+i_3} = 0$. Since $n+i_1$ is after i_3 on the tour, we also have $x_{i_1,n+i_1} = x_{i_2,n+i_1} = 0$. Therefore (16) is valid.

- If $x_{i_1,i_3} = 1$, like above, we obtain $x_{i_1,n+i_1} = 0$. We know that the ordererd vertex sequence on the tour will be $i_2, n+i_2, i_3, i_1$. The vertices $n+i_1$ and $n+i_3$ will be after i_1 on the tour. Any of them can appear first. The corresponding values associated with the variables are: $x_{i_1,n+i_1} = 1$, $x_{i_1,n+i_3} = x_{i_2,n+i_1} = x_{i_3,n+i_1} = 0$ or $x_{i_1,n+i_3} = 1$, $x_{i_1,n+i_1} = x_{i_2,n+i_1} = x_{i_3,n+i_1} = 0$. In both cases (16) is valid.

(c) If $x_{i_2,n+i_2} = 1$, if fololows that the tour contains the ordered sequence of vertices $n+i_3, i_2, n+i_2$ (hence i_3 is before i_2 on the tour). From the degree constraint for i_2 we get $x_{i_1,i_2} + x_{i_2,i_3} + x_{i_2,n+i_1} = 0$.

- If $x_{i_3,n+i_1} = 0$, it follows that (16) is valid (by using the degree constraint for i_1).
- If $x_{i_3,n+i_1} = 1$, the ordered vertex sequences that we can have on the tour are: $(i_1, \sigma, n+i_1, i_3)$ or $(i_1, \sigma, i_3, n+i_1)$, where σ is a vertex sequence that does not contain $i_2, n+i_2$, or $n+i_3$. If $\sigma \neq \emptyset$, we have $x_{i_1,i_3} = x_{i_1,n+i_1} = x_{i_1,n+i_2} = x_{i_1,n+i_3} = 0$. If $\sigma = \emptyset$, we can have $n+i_3$. If $\sigma \neq \emptyset$, we have $x_{i_1,i_3} = x_{i_1,n+i_1} = x_{i_1,n+i_2} = x_{i_1,n+i_3} = 0$. If $\sigma \neq \emptyset$, we have $x_{i_1,n+i_1} = 1, x_{i_1,i_3} = x_{i_1,n+i_2} = x_{i_1,n+i_3} = 0$ or $x_{i_1,i_3} = 1, x_{i_1,n+i_1} = x_{i_1,n+i_2} = x_{i_1,n+i_3} = 0$. In any case (16) is valid.

We showed that in any case inequality (16) is valid for thr PDTSP polytope. ∎

Proposition 4.12 Generalised Order Matching Constraints: *Let $i_1, i_2, i_3 \in V^p$ and let $H \subseteq V^p \cup V^d$ and the disjoint sets of vertices $T_h \subset V^p \cup V^d$, $h = 1, \ldots, 3$ such that $T_h \cap \{n+i_1, n+i_2, n+i_3\} \neq \emptyset$ and $T_h \cap H = \{i_h\}$, for all $h = 1, \ldots, 3$. Then the constraint*

$$x(H) + \sum_{h=1}^{3} x(T_h) \leq |H| - \sum_{h=1}^{3} |T_h| - 6 \tag{20}$$

is valid for the PDTSP polytope.

5 Facets of the PDTSP polytope

In what follows we will look at several classes of valid inequalities and study the conditions under which they are facets of the PDTSP polytope.

5.1 Order matching constraints

In what follows we prove that a subset of the set of inequalities defined by (7) are facets for the PDTSP polytope.

We first need to introduce a new order on the set of edges.

Definition 5.1 *For any given $i_1, \ldots, i_m \in V^p$, $i_1 < \cdots < i_m$, we define $E^0 = \{(0, 2n+1)\}$ and $E^1 = E \setminus (E^0 \cup E^2)$, where $E^2 = (\delta(0) \cup \delta(2n+1) \cup \{(i_m, n+i_1), (n, 2n)\}) \setminus E^0$. Let $\prec_{E^1}$ be the lexicographic order on the set E^1 and $\prec_{E^2}$ the lexicographic order on the set E^2. We define a relation of total order $\prec$ on the set of edges E as follows:*

i. *for any $(i,j) \in E \setminus E^0$, $(0, 2n+1) \prec (i,j)$;*

ii. *the restriction of $\prec$ to E^1 is $\prec_{E^1}$;*

iii. *the restriction of $\prec$ to E^2 is $\prec_{E^2}$;*

iv. *for any $(i,j) \in E^1$ and $(k,l) \in E^2$, $(i,j) \prec (k,l)$.*

Notation 5.2 *If $S = \{i_1, , \ldots, i_m\}$, we denote by τ^p any permutation on S or subsets of S and by σ^p any permutation on $V^p \setminus S$ or subsets of $V^p \setminus S$. When τ^p and σ^p are used in the description of a tour, the subsets on which the permutations are defined will be S, respectively $V^p \setminus S$,* without *those vertices already used on the tour and explicitely written.*

Assumption 5.3 *Let $S = \{i_1, \ldots, i_m\} \subseteq V^p$. Without any loss of generality we can make the following assumptions.*

- *$i_1 < \cdots < i_m$,*
- *$i_1 = 1$ and $i_m = n$.*

Theorem 5.4 *Under our standing assumptions, for any $S = \{i_1, \ldots, i_m\} \subseteq V^p$, the constraint*

$$x(S) + \sum_{j=1}^{m} x_{i_j, n+i_j} \leq |S| \tag{21}$$

is facet for the PDTSP polytope.

Proof. It is easy to see that the face defined by (21) is proper, therefore its dimension is at most the dimension of the polytope minus one. In order to prove that (21) is facet defining for the PDTSP polytope, we need to show that the dimension of the convex hull of the feasible tours that satisfy (21) at equality is one less than the dimension of the PDTSP polytope. Since the face is proper, it would follow that the face defines a facet. Therefore, by Theorem 3.5, we need to prove that there are $2n^2 - n - 2$ affinely independent elements of the PDTSP polytope that satisfy (21) at equality.

We will prove this this by taking each tour in the PDTSP polytope that satisfies (21) at equality and considering it as a row in a matrix. We will demonstrate that this matrix has rank $2n^2 - n - 2$, and therefore there are $2n^2 - n - 2$ linearly independent rows of the matrix, i.e. $2n^2 - n - 2$ linearly independent tours of the PDTSP polytope that satisfy (21) at equality. Since linearly independence implies affinely independence, the result we needed is proved.

We will show that the rank of the matrix is $2n^2-n-2$ by using row operations. We will find $2n^2-n-2$ linear combinations of rows that will form an upper triangular matrix. We will explicitely describe eight sets of vectors, $T_0, T_1, \ldots, T_7$, that contain linear combinations of feasible tours of the PDTSP polytope satisfying (21) at equality. The construction of these sets ensures that they are disjoint. Their union is a set of $2n^2-n-2$ linearly independent vectors. Each vector from the union of the sets T_i, $i = 0, \ldots, 7$, will have a leading element corresponding to a different edge from the first $2n^2-n-2$ edges, according to the order defined in Definition 5.1.

We briefly describe the sets of edges that will correspond to leading elements of the vectors in the sets T_i, $i = 0, \ldots, 7$. We will denote the sets of edges by B_i, where B_i is associated with T_i, for any $i = 0, \ldots, 7$.

$B_0 = \{(0, 2n+1)\}$.

$B_1 = \{(1, j) : j = 2, \ldots, n\}$.

$B_2 = \{(1, n+j) : j = 2, \ldots, n\}$.

$B_3 = \{(j, k) : j = 2, \ldots, n-2, k = j+1, \ldots, n-1\} \cup \{(j, n) : j = 2, \ldots, n-1\}$.

$B_4 = \{(j, n+k) : j = 2, \ldots, n-1, k = 1, \ldots, n-1\} \cup \{(j, 2n) : j = 2, \ldots, n-1\}$.

$B_5 = \{(n, n+j) : j = 1, \ldots, n-1\} \setminus \{(n, n+1)\}$.

$B_6 = \{(n+j, n+k) : j = 1, \ldots, n-2, k = j+1, \ldots, n-1\}$.

$B_7 = \{(n+j, 2n) : j = 1, \ldots, n-2\}$.

0. Leading edge $(\mathbf{0}, \mathbf{2n+1})$:

We first define the set T_0 that will contain only one vector. The leading edge of this vector, with respect to the order defined on E, will correspond to the edge $(0, 2n+1)$.

Let $\sigma^p \in \Pi(V^p \setminus S)$ and $\tau^p \in \Pi(S \setminus \{1\})$.

We define $T_0 = \{(0, \sigma^p, \sigma^d, \tau^p, 1, n+1, \tau^d, 2n+1)\}$.

1. Leading edges $(\mathbf{1}, \mathbf{j}), \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}$:

We construct the vectors m_j as linear combinations of feasible vectors in the PDTSP polytope that satisfy (21) at equality. Their leading edges will be $(1, j)$.

1a. $\mathbf{j} \notin \mathbf{S}$

Let $\tau^p \in \Pi(S \setminus \{1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} m_j &= (0, \sigma^p, \sigma^d, j, 1, \tau^p, n, 2n, n+1, n+j, \tau^d, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, j, n, \overleftarrow{\tau}^p, 1, n+1, 2n, n+j, \tau^d, 2n+1) \\ &= (1, j) - (j, n) + (n, 2n) - (1, n+1) + (n+1, n+j) - (n+j, 2n). \end{aligned}$$

We note that if $\tau^p = \tau^d = \emptyset$, $m_j = (1, j) - (j, n) + (n, 2n) - (1, n+1)$.

In any case, the leading edge of m_j is $(1,j)$.

1b. $\mathbf{j \in S}$

- **$\mathbf{j \neq n}$**
  Let $\tau^p \in \Pi(S \setminus \{1,j\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} m_j &= (0,\sigma^p,\sigma^d,\tau^p,1,j,n+j,n+1,\tau^d,2n+1) \\ &\quad - (0,\sigma^p,\sigma^d,\tau^p,1,n+1,j,n+j,\tau^d,2n+1) \\ &= (1,j) - (1,n+1) + (n+1,n+j) - (j,n+1) + (n+1,\tau^d_s) - (n+j,\tau^d_s). \end{aligned}$$

  We note that if $\tau^p = \tau^d = \emptyset$, then $m_j = (1,j) - (1,n+1) + (n+1,n+j) - (j,n+1) + (n+1,2n+1) - (n+j,2n+1)$.

  In any case, the leading edge of m_j is $(1,j)$.

- **$\mathbf{j = n}$**
  Let $\tau^p \in \Pi(S \setminus \{1,j\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} m_j &= (0,\sigma^p,\sigma^d,1,j,n+j,n+1,\tau^p,\overleftarrow{\tau}^d,2n+1) \\ &\quad - (0,\sigma^p,\sigma^d,1,n+1,j,n+j,\tau^p,\overleftarrow{\tau}^d,2n+1) \\ &= (1,j) - (1,n+1) + (n+1,n+j) - (j,n+1) + (n+1,\tau^p_s) - (\tau^p_s,n+j). \end{aligned}$$

  We note that if $\tau^p = \tau^d = \emptyset$, we have $m_j = (1,j) - (1,n+1) + (n+1,n+j) - (j,n+1) + (n+1,2n+1) - (n+j,2n+1)$.

  Since $j = n$, we have that the leading edge of m_j is edge $(1,j)$.

We define the set T_1 as the set of all vectors m_j, i.e., $T_1 = \{m_j : j = 2,\ldots,n\}$.

2. Leading edges $\mathbf{(1,n+j), j = 1,\ldots,n}$:
We construct the vectors a_j, $j = 1,\ldots,n$ that are linear combinations of feasible tours that satisfy (21) at equality. Their leading edges will be $(1,n+j)$.

2a. $\mathbf{j \notin S}$

Let $\tau^p \in \Pi(S \setminus \{i_1\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} a_j &= (0,j,n+j,1,\tau^p,\overleftarrow{\tau}^d,n+1,\sigma^p,\sigma^d,2n+1) \\ &\quad - (0,1,\tau^p,\overleftarrow{\tau}^d,n+1,\sigma^p,\sigma^d,j,n+j,2n+1) \\ &= (1,n+j) - (j,n+\sigma^p_e) - (n+j,2n+1) + (n+\sigma^p_e,2n+1) + (0,j) - (0,1). \end{aligned}$$

We note that if $\sigma^p = \sigma^d = \emptyset$, we have $a_j = (1,n+j) - (j,n+1) - (n+j,2n+1) + (n+1,2n+1) + (0,j) - (0,1)$.

Since $j \notin S$, we have $j \neq 1$, therefore $1 < j$. The leading edge of a_j is $(1,n+j)$.

2b. $\mathbf{j \in S}$

- $\mathbf{j = 1}$

  Let $\tau^p \in \Pi(S \setminus \{1\})$ and $\sigma^p \in \Pi(V^o \setminus S)$.

$$\begin{aligned} a_1 &= (0, \tau^p, 1, n+1, \overleftarrow{\tau}^d, \sigma^p, \sigma^d, 2n+1) - (0, 1, \overleftarrow{\tau}^p, \tau^d, n+1, \sigma^p, \sigma^d, 2n+1) \\ &= (1, n+1) - (\tau^p_s, \tau^d_s) + (\tau^d_s, \sigma^p_s) - (\sigma^p_s, n+1) + (0, \tau^p_s) - (0, 1). \end{aligned}$$

  We note that in this case $\tau^p \neq \emptyset$, since $m \geq 2$. If $\sigma^p = \sigma^d = \emptyset$, we have $a_1 = (1, n+1) - (\tau^p_s, \tau^d_s) + (\tau^d_s, 2n+1) - (n+1, 2n+1) + (0, \tau^p_s) - (0, 1)$.

  In any case, the leading edge of a_1 is $(1, n+1)$.

- $\mathbf{j \notin \{1, n\}}$

  Let $\tau^p \in \Pi(S \setminus \{1, j\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} a_j &= (0, \tau^p, j, n+j, 1, n+1, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &\quad - (0, 1, n+1, j, \overleftarrow{\tau}^p, \tau^d, \sigma^p, \sigma^d, n+1, n+j, 2n+1) \\ &= (1, n+j) + (j, n+j) + (n+1, \tau^d_s) + (\sigma^d_e, 2n+1) + (0, \tau^p_s) \\ &\quad - (j, n+1) - (\sigma^d_e, n+1) - (n+1, n+j) - (n+j, 2n+1) - (0, 1). \end{aligned}$$

  We note that $\tau^p \neq \emptyset$, $(n \in S \setminus \{1, j\})$, and that if $\sigma^p = \emptyset$, we have $a_j = (1, n+j) + (j, n+j) + (n+1, \tau^d_s) + (\tau^d_e, 2n+1) + (0, \tau^p_s) - (j, n+1) - (\tau^d_e, n+1) - (n+1, n+j) - (n+j, 2n+1) - (0, 1)$.

  The leading edge of a_j is $(1, n+j)$.

- $\mathbf{j = n}$

  Let $\tau^p \in \Pi(S \setminus \{j, 1\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} a_j &= (0, \tau^p, j, n+j, 1, n+1, \overleftarrow{\tau}^d, \sigma^p, \sigma^d, 2n+1) - (0, 1, n+1, j, n+j, \tau^p, \overleftarrow{\tau}^d, \sigma^p, \sigma^d, 2n+1) \\ &= (1, n+j) + (\sigma^d_e, n+1) + (j, \tau^p_e) - (j, n+1) - (\tau^p_s, n+j) - (\tau^p_e, \tau^d_e) + (0, \tau^p_s) - (0, 1). \end{aligned}$$

  We note that if $\tau^p = \tau^d = \emptyset$, we have $a_j = (1, n+j) + (\sigma^p_s, n+1) - (j, n+1) - (\sigma^p_s, n+j) + (0, j) - (0, 1)$.

  We also note that if both $\tau^p = \sigma^p = \emptyset$, then $a_j = (1, n+j) - (j, n+1) + (n+1, 2n+1) - (n+j, 2n+1) + (0, j) - (0, 1)$.

  The leading edge of a_j is $(1, n+j)$.

We now define the set T_2 as the set of all vectors a_j, i.e. $T_2 = \{a_j : j = 1, \ldots, n\}$.

3. Leading edges $\mathbf{(j, k), j = 2, \ldots, n-1, k = j+1, \ldots, n}$:

We construct the vectors b_{jk} as linear combinations of feasible tours that satisfy (7) at equality and have the leading edges (j, k).

3a. $\mathbf{j, k \notin S}$

Let $\tau^p \in \Pi(S \setminus \{1\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j,k\}))$.

$$\begin{aligned} b_{jk} &= (0, j, k, n+j, n+k, 1, n+1, \tau^p, \overleftarrow{\tau}^d, \sigma^p, \sigma^d, 2n+1) \\ &\quad - (0, j, n+j, k, n+k, 1, n+1, \tau^p, \overleftarrow{\tau}^d, \sigma^p, \sigma^d, 2n+1) \\ &= (j,k) + (n+j, n+k) - (j, n+j) - (k, n+k). \end{aligned}$$

The leading edge of b_{jk} is (j,k).

3b. $\mathbf{j \in S, k \notin S}$

Let $\tau^p \in \Pi(S \setminus \{1,j\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{k\}))$.

$$\begin{aligned} b_{jk} &= (0, k, j, \tau^p, 1, n+1, \tau^d, n+j, n+k, \sigma^p, \sigma^d, 2n+1) \\ &\quad - (0, j, \tau^p, 1, n+1, k, \tau^d, n+j, n+k, \sigma^p, \sigma^d, 2n+1) \\ &= (j,k) + (n+1, \tau^d_s) - (k, n+1) - (k, \tau^d_s) + (0,k) - (0,j). \end{aligned}$$

We note that $\tau^o \neq \emptyset$ (since $n \in S, j \neq n$). The leading edge of b_{jk} is (j,k).

3c. $\mathbf{j \notin S, k \in S}$

Let $\tau^p \in \Pi(S \setminus \{1,k\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} b_{jk} &= (0, j, k, \tau^p, 1, n+1, n+k, n+j, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &\quad - (0, k, \tau^p, 1, n+1, n+k, j, n+j, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &= (j,k) + (n+k, n+j) - (j, n+k) - (j, n+j) + (0,j) - (0,k). \end{aligned}$$

The leading edge of b_{jk} is (j,k).

3d. $\mathbf{j \in S, k \in S}$

Let $\tau^p \in \Pi(S \setminus \{1,j,k\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} b_{ij} &= (0, j, k, \tau^p, 1, n+1, n+k, n+j, \sigma^p, \sigma^d, \tau^d, 2n+1) \\ &\quad - (0, k, \tau^p, 1, n+1, j, n+k, n+j, \sigma^p, \sigma^d, \tau^d, 2n+1) \\ &= (j,k) + (n+1, n+k) - (j, n+1) - (j, n+k) + (0,j) - (0,k). \end{aligned}$$

Since $j < k$, and $k < n+1$, $k < n+k$, we have that the leading edge of b_{jk} is (j,k).

We now define the set T_3 as the set of all vectors b_{jk}, i.e. $T_e = \{b_{jk} : j = 2, \ldots, n-1, k = j+1, \ldots, n\}$.

4. Leading edges $\mathbf{(j, n+k), j = 2, \ldots, n-1, k = 1, \ldots, n}$:

We construct the vectors c_{jk} as linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be $(j, n+k)$.

4-1. We first consider the case when $j \neq k$ and $k = 1, \ldots, n-1$.

4-1a. $\mathbf{j}, \mathbf{k} \notin \mathbf{S}$

Let $\tau^p \in \Pi(S \setminus \{1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j, k\}))$.

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \sigma^d, n, \tau^p, 1, n+1, \tau^d, k, n+k, j, n+j, 2n, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, n, \tau^p, 1, n+1, \tau^d, k, n+k, 2n, j, n+j, 2n+1) \\ &\quad = (j, n+k) + (n+j, 2n) + (2n, 2n+1) - (n+k, 2n) - (j, 2n) - (n+j, 2n+1). \end{aligned}$$

We have $n+k < 2n$, therefore $(j, n+k) \prec (j, 2n)$. It follows that the leading edge of c_{jk} is $(j, n+k)$.

4-1b. $\mathbf{j} \in \mathbf{S}, \mathbf{k} \notin \mathbf{S}$

Let $\tau^p \in \Pi(S \setminus \{1, j, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{k\}))$.

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \sigma^d, k, 1, \tau^p, n, 2n, n+k, j, n+j, n+1, \tau^d, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, k, 1, \tau^p, n, 2n, j, n+j, n+k, n+1, \tau^d, 2n+1) \\ &\quad = (j, n+k) + (n+1, n+j) + (n+k, 2n) - (j, 2n) - (n+j, n+k) - (n+1, n+k). \end{aligned}$$

Since $(j, n+k) \prec (j, 2n)$, the leading edge of c_{jk} is $(j, n+k)$.

4-1c. $\mathbf{j} \notin \mathbf{S}, \mathbf{k} \in \mathbf{S}$

- $\mathbf{k = 1}$

  Let $\tau^p \in \Pi(S \setminus \{k, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \sigma^d, \tau^p, n, k, n+k, j, n+j, 2n, \tau^d, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, \tau^p, n, k, n+k, j, 2n, n+j, \tau^d, 2n+1) \\ &\quad = (j, n+k) + (n+j, 2n) + (\tau^d_s, 2n) - (j, 2n) - (n+k, 2n) - (n+j, \tau^d_s). \end{aligned}$$

  We note that if $\tau^d = \emptyset$, we have $c_{jk} = (j, n+k) + (n+j, 2n) + (2n, 2n+1) - (j, 2n) - (n+k, 2n) - (n+j, 2n+1)$.

  Since $k < n$, we have $(j, n+k) \prec (j, 2n)$. Therefore, the leading edge of c_{jk} is $(j, n+k)$.

- $\mathbf{k \neq 1}$

  Let $\tau^p \in \Pi(S \setminus \{i_1, k, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \sigma^d, k, \tau^p, n, 1, n+1, j, n+k, n+j, 2n, \tau^d, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, k, \tau^p, n, 1, n+1, j, 2n, n+j, n+k, \tau^d, 2n+1) \\ &\quad = (j, n+k) + (2n, \tau^d_s) - (j, 2n) - (n+k, \tau^d_s). \end{aligned}$$

  We note that if $\tau^p = \tau^d = \emptyset$, we have $c_{jk} = (j, n+k) + (2n, 2n+1) - (j, 2n) - (n+k, 2n+1)$.

  Since $k < n$, we have $(j, n+k) \prec (j, 2n)$. Therefore, the leading edge of c_{jk} is $(j, n+k)$.

4-1d. $\mathbf{j}, \mathbf{k} \in \mathbf{S}$

- $\mathbf{k} \neq \mathbf{1}$

  let $\tau^p \in \Pi(S \setminus \{1, j, k, n\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} c_{jk} &= (0, 1, n+1, \tau^p, n, k, n+k, j, n+j, 2n, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &- (0, 1, n+1, \tau^p, n, k, n+k, 2n, j, n+j, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &= (j, n+k) + (n+j, 2n) + (2n, \tau^d_s) - (j, 2n) - (n+k, 2n) - (n+j, \tau^d_s). \end{aligned}$$

  We note that if $\tau^o = \tau^d = \emptyset$, we have $c_{jk} = (j, n+k) + (n+j, 2n) + (2n, 2n+1) - (j, 2n) - (n+k, 2n) - (n+j, 2n+1)$.

  The leading edge of c_{jk} is $(j, n+k)$.

- $\mathbf{k} = \mathbf{1}$

  Let $\tau^p \in \Pi(S \setminus \{1, j, k\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned} c_{jk} &= (0, n, \tau^p, k, n+k, j, n+j, 2n, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &- (0, n, \tau^p, k, n+k, 2n, j, n+j, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &= (j, n+k) + (n+j, 2n) + (2n, \tau^d_s) - (j, 2n) - (n+k, 2n) - (n+j, \tau^d_s). \end{aligned}$$

  We note that if $\tau^p = \tau^d = \emptyset$, we have $c_{jk} = (j, n+k) + (n+j, 2n) + (2n, 2n+1) - (j, 2n) - (n+k, 2n) - (n+j, 2n+1)$.

  The leading edge of c_{jk} is $(j, n+k)$.

4-2. We now consider the case when $j = k$. In other words we will construct linear combinations of feasible tours that satisfy (21) at equality. Their leading edges will be $(j, n+j)$.

4-2a. $\mathbf{j} \notin \mathbf{S}$

Let $\tau^p \in \Pi(S \setminus \{1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} c_{jj} &= (0, \sigma^p, \sigma^d, \tau^p, n, 1, n+1, \tau^d, j, n+j, 2n, 2n+1) \\ &- (0, \sigma^p, \sigma^d, \tau^p, n, 1, n+1, \tau^d, j, 2n, n+j, 2n+1) \\ &= (j, n+j) - (j, 2n) - (n+j, 2n+1) + (2n, 2n+1). \end{aligned}$$

Since $j < n$, we have $(j, n+j) \prec (j, 2n)$. Therefore, the leading edge of c_{jj} is $(j, n+j)$.

4-2b. $\mathbf{j} \in \mathbf{S}$

Let $\tau^p \in \Pi(S \setminus \{1, j\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{n\}))$.

$$\begin{aligned} c_{jj} &= (0, n, j, n+j, 2n, \tau^p, 1, n+1, \tau^d, \sigma^p, \sigma^d, 2n+1) \\ &- (0, j, n, 2n, \tau^p, 1, n+1, \tau^d, \sigma^p, \sigma^d, n+j, 2n+1) \\ &= (j, n+j) + (n+j, 2n) + (\sigma^d_e, 2n+1) + (0, n) - (n, 2n) - (n+j, \sigma^d_e) - (n+j, 2n+1) - (0, j). \end{aligned}$$

We note that if $\sigma^o = \sigma^d = \emptyset$, but $\tau^o \neq \emptyset$, we have $c_{jj} = (j, n+j) + (n+j, 2n) + (\tau_e^d, 2n+1) + (0, n) - (n, 2n) - (n+j, 2n+1) - (n+j, 2n+1) - (0, j)$.

If $\sigma^o = \sigma^d = \emptyset$ and $\tau^o = \tau^d = \emptyset$, we have $c_{jj} = (j, n+j) + (n+j, 2n) + (n+1, 2n+1) + (0, n) - (n, 2n) - (n+j, 2n+1) - (n+j, 2n+1) - (0, j)$.

In all cases, the leading edge of c_{jj} is $(j, n+j)$.

4-3. We now consider the case when $k = n$. In other words we construct the vectors c_{jn}, linear combinations of feasible tours that satisfy (21) at equality, and have leading edges $(j, 2n)$ (i.e. $(j, n+k), k = n$).

4-3a. $\mathbf{j \notin S}$

Let $\tau^p \in \Pi(S \setminus \{1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned}
c_{jn} &= (0, n, \tau^p, 1, n+1, \sigma^p, \sigma^d, 2n, j, n+j, \tau^d, 2n+1) \\
&\quad - (0, j, n+j, n, \tau^p, 1, n+1, \sigma^p, \sigma^d, 2n, \tau^d, 2n+1) \\
&\quad = (j, 2n) + (n+j, \tau_s^d) - (n, n+j) - (\tau_s^d, 2n) + (0, n) - (0, j).
\end{aligned}$$

We note that if $\tau^o = \tau^d = \emptyset$, we have $c_{jn} = (j, 2n) + (n+j, 2n+1) - (n, n+j) - (2n, 2n+1) + (0, n) - (0, j)$.

Since $j < n$, the leading edge of c_{jn} is $(j, 2n)$.

4-3b. $\mathbf{j \in S}$

Let $\tau^p \in \Pi(S \setminus \{1, j, n\})$ and $\sigma^p \in \Pi(V^p \setminus S)$.

$$\begin{aligned}
c_{jn} &= (0, n, 2n, j, n+j, \sigma^p, \sigma^d, \tau^p, 1, n+1, \tau^d, 2n+1) \\
&\quad - (0, j, n+j, \sigma^p, \sigma^d, \tau^p, 1, n+1, n, 2n, \tau^d, 2n+1) \\
&\quad = (j, 2n) + (n+1, \tau_s^d) - (n, n+1) - (2n, \tau_s^d) + (0, n) - (0, j).
\end{aligned}$$

We note that if $\tau^o = \tau^d = \emptyset$, we have $c_{jn} = (j, 2n) + (n+1, 2n+1) - (n, n+1) - (2n, 2n+1) + (0, n) - (0, j)$.

We recall that according to the order defined on the set of edges, $(j, 2n) \prec (n, n+1)$. Therefore, the leading edge of c_{jn} is $(j, 2n)$.

We define the set T_4 as the set of all vectors c_{ij}, i.e., $T_4 = \{c_{jk} : j = 2, \ldots, n-1, k = 1, \ldots, n\}$.

5. Leading edges $\mathbf{(n, n+j), j = 2, \ldots, n-1}$:

We now construct the vectors d_j as linear combinations of feasible tours that satisfy (21) at equality. Their leading edges will be $(n, n+j)$.

Let $\tau^p \in \Pi(S \setminus \{1, n\})$, if $j \notin S$ and $\tau^p \in \Pi(S \setminus \{1, j, n\})$, if $j \in S$. Let $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$, if $j \notin S$ and $\sigma^p \in \Pi(V^p \setminus S)$, if $j \in S$.

$$\begin{aligned} d_j &= (0, \sigma^p, \sigma^d, j, \tau^p, 1, n+1, n+j, n, 2n, \tau^d, 2n+1) \\ &- (0, \sigma^p, \sigma^d, j, \tau^p, 1, n+1, n, 2n, \tau^d, n+j, 2n+1) \\ &= (n, n+j) + (n+1, n+j) + (\tau_e^d, 2n+1) - (n, n+1) - (n+j, \tau_e^d) - (n+j, 2n+1). \end{aligned}$$

We note that if $\tau^p = \tau^d = \emptyset$, we have $d_j = (n, n+j) + (n+1, n+j) + (2n, 2n+1) - (n, n+1) - (n+j, 2n) - (n+j, 2n+1)$.

Since $(n, n+j) \prec (n, n+1)$, the leading edge of d_j is $(n, n+j)$.

We define the set T_5 as the set of all vectors d_j, i.e. $T_5 = \{d_j : j = 2, \ldots, n-1\}$.

6. Leading edges $(\mathbf{n+j}, \mathbf{n+k}), \mathbf{j = 1 \ldots, n-2}, \mathbf{k = j+1, \ldots, n-1}$:

We construct the vectors e_{jk} as linear combinations of feasible tours that satisfy (21) at equality. Their leading edges will be $(n+j, n+k)$.

6a. $\mathbf{j, k \notin S}$

Let $\tau^p \in \Pi(S \setminus \{i_1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j, k\})$.

$$\begin{aligned} e_{jk} &= (0, \sigma^p, \sigma^d, \tau^p, n, 1, n+1, \tau^d, j, k, n+j, n+k, 2n, 2n+1) \\ &- (0, \sigma^p, \sigma^d, \tau^p, n, 1, n+1, \tau^d, j, k, n+j, 2n, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

6b. $\mathbf{j \in S, k \notin S}$

- $\mathbf{j = 1}$

  Let $\tau^p \in \Pi(S \setminus \{j, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{k\}))$.

$$\begin{aligned} e_{jk} &= (0, \sigma^p, \sigma^d, k, \tau^p, n, j, n+j, n+k, \tau^p, 2n, 2n+1) \\ &- (0, \sigma^p, \sigma^d, k, \tau^p, n, j, n+j, 2n, \overleftarrow{\tau}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

  Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

- **$\mathbf{j \neq 1}$**
  Let $\tau^p \in \Pi(S \setminus \{j, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{k\})$.

$$\begin{aligned} e_{jk} &= (0, \sigma^p, \sigma^d, k, j, \tau^p, n, 1, n+1, n+j, n+k, \tau^d, 2n, 2n+1) \\ &- (0, \sigma^p, \sigma^d, k, j, \tau^p, n, 1, n+1, n+j, 2n, \overleftarrow{\tau}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

  Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

6c. $\mathbf{j \notin S, k \in S}$

Since $k \geq 2$ we have that $k \neq 1$. Let $\tau^p \in \Pi(S \setminus \{k\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j, n\})$.

$$\begin{aligned} e_{jk} &= (0, \sigma^p, \sigma^d, j, n, k, \tau^p, 1, n+1, n+k, n+j, \tau^d, 2n, 2n+1) \\ &- (0, \sigma^p, \sigma^d, j, n, k, \tau^p, 1, n+1, n+k, 2n, \overleftarrow{\tau}^d, n+j, 2n+1) \\ &= (n+j, n+k) - (n+k, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

6d. $\mathbf{j, k \in S}$

- **$\mathbf{j = 1}$**
  Let $\tau^p \in \Pi(S \setminus \{j, k, n\})$ and $\sigma^p \in \Pi(V \setminus S)$.

$$\begin{aligned} e_{jk} &= (0, \sigma^p, \sigma^d, k, n, \tau^p, j, n+j, n+k, \tau^d, 2n, 2n+1) \\ &- (0, \sigma^p, k, n, \tau^p, j, n+j, 2n, \overleftarrow{\tau}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

  Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

- **$\mathbf{j \neq 1}$**
  Let $\tau^p \in \Pi(S \setminus \{i_1, j, k\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{n\}))$.

$$\begin{aligned} e_{jk} &= (0, \sigma^p, \sigma^d, n, k, \tau^p, j, 1, n+1, n+j, n+k, \tau^d, 2n, 2n+1) \\ &- (0, \sigma^p, \sigma^d, n, n, k, \tau^p, j, 1, n+1, n+j, 2n, \overleftarrow{\tau}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

  Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

We define the set T_6 as the set of all vectors e_{jk}, i.e.$T_6 = \{e_{jk} : j = 1, \dots, n-1, k = 2, \dots, n-1\}$.

7. Leading edges $(\mathbf{n}+\mathbf{j}, \mathbf{2n}), \mathbf{j} = \mathbf{1}, \dots, \mathbf{n}-\mathbf{2}$:
We construct the vectors f_j as linear combinations of feasible tours that satisfy (21) at equality. Their leading edges will be $(n+j, 2n)$.

7a. $\mathbf{j} \notin \mathbf{S}$

- **$\mathbf{n}-\mathbf{1} \notin \mathbf{S}$**
  Let $\tau^p \in \Pi(S \setminus \{1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j, n-1\}))$.

$$\begin{aligned} f_j &= (0, \sigma^p, \sigma^d, j, n-1, \tau^p, n, 1, n+1, \tau^d, 2n, n+j, 2n-1, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, j, n-1, \tau^p, n, 1, n+1, \tau^d, 2n, 2n-1, n+j, 2n+1) \\ &\quad = (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1) \end{aligned}$$

  The leading element of f_j corresponds to the edge $(n+j, 2n)$.

- **$\mathbf{n}-\mathbf{1} \in \mathbf{S}$**
  We recall that $1, n \in S$. Since $j \notin S$, it follows that V^p contains at least three elements, i.e. $n \geq 3$. Hence we are in the case when $1 \neq n-1$.
  Let $\tau^p \in \Pi(S \setminus \{1, n-1, n\})$ and $\sigma^p \in \Pi(V^p \setminus (S \cup \{j\}))$.

$$\begin{aligned} f_j &= (0, \sigma^p, \sigma^d, j, \tau^p, n, n-1, 1, n+1, \tau^d, 2n, n+j, 2n-1, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, j, \tau^p, n, n-1, 1, n+1, \tau^d, 2n-1, n+j, 2n+1) \\ &\quad = (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1) \end{aligned}$$

7b. $\mathbf{j} \in \mathbf{S}$

- **$\mathbf{j} = \mathbf{1}$**
  Let $\tau^p \in \Pi(S \setminus \{j, n\})$, if $n-1 \notin S$, and $\tau^p \in \Pi(S \setminus \{j, n-1, n\})$, if $n-1 \in S$. Let $\sigma^p \in \Pi(V^p \setminus (S \cup \{n-1\}))$, if $n-1 \notin S$, and $\sigma^o \in \Pi(V^o \setminus S)$, if $n-1 \in S$.

$$\begin{aligned} f_j &= (0, \sigma^p, \sigma^d, n-1, \tau^p, j, n, 2n, n+j, \tau^d, 2n-1, 2n+1) \\ &\quad - (0, \sigma^p, \sigma^d, n-1, \tau^p, j, n, 2n, 2n-1, \overleftarrow{\tau}^d, n+j, 2n+1) \\ &\quad = (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1). \end{aligned}$$

- **$\mathbf{j} \neq \mathbf{1}$**
  Let $\tau^p \in \Pi(S \setminus \{1, j, n\})$, if $n-1 \notin S$, and $\tau^p \in \Pi(S \setminus \{1, j, n-1, n\})$, if $n-1 \in S$. Let

$\sigma^p \in \Pi(V^p \setminus S)$, if $n-1 \notin S$, and $\sigma^p \in \Pi(V^p \setminus S)$, if $n-1 \in S$. We define f_j as follows:

$$\begin{aligned} f_j &= (0, \sigma^p, \sigma^d, n-1, n, \tau^p, j, 1, n+1, \tau^d, 2n, n+j, 2n-1, 2n+1) \\ &- (0, \sigma^p, \sigma^d, n-1, n, \tau^p, j, 1, n+1, \tau^d, 2n, 2n-1, n+j, 2n+1) \\ &= (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1) \end{aligned}$$

In all cases described above, the leading edge of f_j is $(n+j, 2n)$.

We define the set T_7 as the set of all vectors f_j, $T_7 = \{f_j : j = 1, \dots, n-2\}$.
The vectors in $T = T_0 \cup T_1 \cup \dots \cup T_7$ are the vectors needed. ■

5.2 Precedence constraints

Next we prove that the inequalities (9) are facets of the PDTSP polytope under the conditions described in Proposition 4.7.

Notation 5.5 *In what follows we introduce some notation.*

- *Let* $U^p = \{i : i \in V^p \cap U\}$, $U_p = \{i : i \in U^p, n+i \notin U\}$, *and* $U_{pd} = \{i : i \in U^p, n+i \in U\}$. *Clearly,* $U^p = U_p \cup U_{pd}$.

  Similarly, let $\bar{U}^p = \{i : i \in V^p \cap \bar{U}\}$, $\bar{U}_{pd} = \{i : i \in \bar{U}^p, n+i \in \bar{U}\}$.

  We denote by σ^p *any permutation on* U^p *or subsets of* U^p, *by* τ^p *any permutation on* U^{pd} *or subsets of* U^{pd}, *and by* θ^p *any permutation on* $\bar{U}^{pd}$ *or subsets of* $\bar{U}^{pd}$.

- *We use the notation* $[i, j, k]$ *inside a tour to denote a sequence of vertices that contains* i, j, *and* k *such that the number of crossings between* U *and* $\bar{U}$ *is the number that we need in order to satisfy the inequality at equality* ($[i, j, k] \in \Pi(\{i, j, k\})$).

Assumption 5.6 *Without loss of generality we make the following assumptions:*

- $1 \in \bar{U}$, $n+1 \in U$.
- *The vertices in* U_p *and* U_{pd} *satisfy the following condition:* $\forall i \in U_p, j \in U_{pd}, i < j$.
- *Every pickup vertex in* $\bar{U}$ *is smaller than every pickup vertex in* U *(i.e.,* $\forall i \in U^p, j \in \bar{U}^p, i > j$*). We note that this implies that if* $\tau^d \neq \emptyset$, *then* $2n \in U$.

Figure 1 illustrates the most general structure of the sets U and $\bar{U}$, taking into account our assumptions.

In order to prove that the inequalities defined by (9) are facet defining we need to introduce a new order on the set of edges.

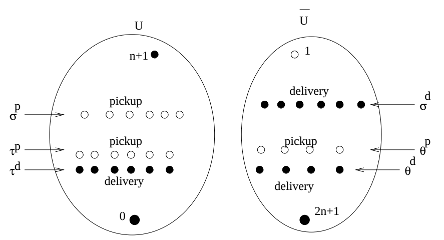

Figure 1: Illustration of our assumptions and notation for the set U.

Definition 5.7 *We define the sets* $E^0 = \{(0, 2n+1)\}$ *and* $E^1 = E \setminus (E^0 \cup E^2)$*, where* $E^2 = (\delta(0) \cup \delta(2n+1) \cup \{(n, 2n), (n+1, 2n)\}) \setminus E^0$*. Let* $\prec_{E^1}$ *be the lexicographic order on the set* E^1 *and* $\prec_{E^2}$ *the lexicographic order on the set* E^2*. We define a relation of total order* $\prec$ *on the set of edges* E *as follows:*

i. *for any* $(i, j) \in E \setminus E^0$*,* $(0, 2n+1) \prec (i, j)$*;*

ii. *the restriction of* $\prec$ *to* E^1 *is* $\prec_{E^1}$*;*

iii. *the restriction of* $\prec$ *to* E_2 *is* $\prec_{E^2}$*;*

iv. *for any* $(i, j) \in E^1$ *and* $(p, q) \in E^2$*,* $(i, j) \prec (p, q)$*.*

We now prove, under our standing assumptions, the following result.

Theorem 5.8 *The inequality (9) is facet defining of the PDTSP polytope for any* $U \subseteq V$*,* $3 \leq |U| \leq |V| - 2$*, with* $0 \in U, 2n+1 \notin U$*, for which there is a unique* $i \in V^p, i \in \bar{U}, n+i \in U$*.*

Proof. It is easy to see that the face defined by (9) is proper, therefore its dimension is strictly less that that of the entire polytope. In order to prove that (9) is facet defining for the PDTSP polytope, we need to show that the dimension of the convex hull of the feasible tours that satisfy (9) at equality is one less than the dimension of the PDTSP polytope. Therefore, by Theorem 3.5, we need to prove that there are $2n^2 - n - 2$ affinely independent elements of the PDTSP polytope that satisfy (9) at equality.

We will prove this this by taking each tour in the PDTSP polytope that satisfies (9) at equality and considering it as a row in a matrix. We will demonstrate that this matrix has rank $2n^2 - n - 2$, and therefore there are $2n^2-n-2$ linearly independent rows of the matrix, i.e. $2n^2-n-2$ linearly independent tours of the PDTSP polytope that satisfy (9) at equality. Since linearly independence implies affinely independence, the result we needed is proved.

We will show that the rank of the matrix is $2n^2-n-2$ by using row operations. We will find $2n^2-n-2$ linear combinations of rows that will form an upper triangular matrix. We will explicitely describe eight sets of vectors, $T_0, T_1, \ldots, T_7$, that contain linear combinations of feasible tours of the PDTSP polytope satisfying 9 at equality. The construction of these sets ensures that they are disjoint. Their union is a set of $2n^2 - n - 2$ linearly independent vectors. Each vector from the union of the sets T_i, $i = 0, \ldots, 7$, will have a leading element corresponding to a different edge from the first $2n^2 - n - 2$ edges, ordered according to the order defined in Definition 5.7.

We briefly describe the sets of edges that will correspond to leading elements of the vectors in the sets T_i, $i = 0, \ldots, 7$. We will denote the sets of edges by B_i, where B_i is associated with T_i, for any $i = 0, \ldots, 7$.

$B_0 = \{(0, 2n + 1)\}$.

$B_1 = \{(1, j) : j = 2, \ldots, n\}$.

$B_2 = \{(1, n + j) : j = 1, \ldots, n\}$.

$B_3 = \{(j, k) : j = 2, \ldots, n - 1, k = j + 1, \ldots, n\}$.

$B_4 = \{(j, n + k) : j = 2, \ldots, n - 1, k = 1, \ldots, n\}$.

$B_5 = \{(n, n + j) : j = 2, \ldots, n - 1\}$.

$B_6 = \{(n + j, n + k) : j = 1, \ldots, n - 2, k = j + 1, \ldots, n - 1\}$.

$B_7 = \{(n + j, 2n) : j = 2, \ldots, n - 2\}$.

We finally mention that we will not explicitely write the sets of vertices on which the permutations are defined. The description of the sets should be obvious to the reader.

0. Leading edge $(\mathbf{0}, \mathbf{2n} + \mathbf{1})$:

We first define the set T_0 that will contain only one vector. The leading edge of this vector, with respect to the order defined on the set of edges, will correspond to the edge $(0, 2n + 1)$.

$T_0 = \{(0, 1, n + 1, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 2n + 1)\}$.

1. Leading edges $(\mathbf{1}, \mathbf{j}), \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n}$:

We construct the vectors m_j as linear combinations of feasible vectors in the PDTSP polytope that satisfy (9) at equality. Their leading edges will be $(1, j)$.

$$
\begin{aligned}
m_j &= (0, \sigma^p, \tau^p, \tau^d, 1, j, n + 1, n + j, \sigma^d, \theta^p, \theta^d, 2n + 1) \\
&\quad - (0, \sigma^p, \tau^p, \tau^d, 1, n + 1, j, n + j, \sigma^d, \theta^p, \theta^d, 2n + 1) \\
&\quad = (1, j) - (1, n + 1) + (n + 1, n + j) - (j, n + j).
\end{aligned}
$$

Clearly, the leading edge of m_j is $(1, j)$.

We define $T_1 = \{m_j : j = 2, \ldots, n\}$.

2. Leading edges $(\mathbf{1}, \mathbf{n} + \mathbf{j}), \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}$:

We construct the vectors a_j that are linear combinations of feasible tours that satisfy (9) at equality. Their leading edges will be $(1, n + j)$.

2a. $\mathbf{j} \in \bar{\mathbf{U}}$

We consider two separate cases:

- $\mathbf{j} = \mathbf{1}$

$$\begin{aligned} a_j &= (0, n, 1, n+1, \sigma^p, \tau^p, \tau^d, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, 1, n, n+1, \sigma^p, \tau^p, \tau^d, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad = (1, n+1) - (n, n+1) + (0, n) - (0, 1). \end{aligned}$$

- $\mathbf{j} \neq \mathbf{1}$

$$\begin{aligned} a_j &= (0, j, 1, n+j, n+1, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, 1, j, n+j, n+1, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad = (1, n+j) - (j, n+j) + (0, j) - (0, 1). \end{aligned}$$

It is clear that in both cases, the leading edge of a_j is $(1, n+j)$.

2b. $\mathbf{j} \in \mathbf{U}$

We consider two separate cases:

- $\mathbf{n} + \mathbf{j} \in \bar{\mathbf{U}}$

$$\begin{aligned} a_j &= (0, j, \sigma^p, \tau^p, 1, n+j, n+1, \tau^d, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, 1, \overleftarrow{\tau}^p, \overleftarrow{\sigma}^p, j, n+1, \tau^d, \sigma^d, \theta^p, \theta^d, n+j, 2n+1) \\ &\quad = \begin{cases} (1, n+j) - (j, n+1) + (n+1, n+j) + (\theta^d_e, 2n+1) - (\theta^d_e, n+j) - (n+j, 2n+1), & \text{if } \theta^p \neq \emptyset \\ (1, n+j) - (j, n+1) + (n+1, n+j) + (\sigma^d_e, 2n+1) - (\sigma^d_e, n+j) - (n+j, 2n+1), & \text{if } \theta^p = \emptyset, \sigma^p \neq \emptyset \\ (1, n+j) - (j, n+1) + (n+1, n+j) + (\tau^d_e, 2n+1) - (\tau^d_e, n+j) - (n+j, 2n+1), & \text{if } \theta^p, \sigma^p = \emptyset \end{cases} \end{aligned}$$

  Since $1 < j$, the leading edge of any a_j is $(1, n+j)$.

- $\mathbf{n} + \mathbf{j} \in \mathbf{U}$

$$\begin{aligned} a_j &= (0, j, 1, n+j, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, 1, j, n+j, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad = (1, n+j) - (j, n+j) + (0, j) - (0, 1). \end{aligned}$$

In both cases, the leading edge of a_j is $(1, n+j)$.

We now define T_2 to be the set of all vectors a_j, i.e., $T_2 = \{a_j : j = 1, \ldots, n\}$.

3. Leading edges $(\mathbf{j}, \mathbf{k}), \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n-1}, \mathbf{k} = \mathbf{j+1}, \ldots, \mathbf{n}$:
We construct the vectors b_{jk} that are linear combinations of feasible tours that satisfy (9) at equality. Their leading edges will be (j, k).

3a. $\mathbf{j}, \mathbf{k} \in \bar{\mathbf{U}}$

$$\begin{aligned} b_{jk} &= (0, \sigma^p, \tau^p, \tau^d, 1, \sigma^d, \theta^p, \theta^d, j, k, n+j, n+k, n+1, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \sigma^d, \theta^p, \theta^d, j, n+j, k, n+k, n+1, 2n+1) \\ &= (j,k) - (j, n+j) + (n+j, n+k) - (k, n+k). \end{aligned}$$

3b. $\mathbf{j} \in \bar{\mathbf{U}}, \mathbf{k} \in \mathbf{U}$

$$\begin{aligned} b_{jk} &= (0, \sigma^p, \tau^p, \tau^d, 1, \sigma^d, \theta^p, \theta^d, j, k, n+1, n+k, n+j, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \sigma^d, \theta^p, \theta^d, j, n+1, k, n+k, n+j, 2n+1) \\ &= (j,k) - (j, n+1) + (n+1, n+k) - (k, n+k). \end{aligned}$$

3c. $\mathbf{j}, \mathbf{k} \in \mathbf{U}$

$$\begin{aligned} b_{jk} &= (0, 1, \sigma^p, \tau^p, \tau^d, j, k, n+1, [n+j, n+k], \theta^p, \theta^d, \sigma^d, 2n+1) \\ &- (0, 1, \sigma^p, \tau^p, \tau^d, j, n+1, k, [n+j, n+k], \theta^p, \theta^d, \sigma^d, 2n+1) \\ &= (j,k) + (n+1, n+j) - (j, n+1) - (k, n+j). \end{aligned}$$

3d. $\mathbf{j} \in \mathbf{U}, \mathbf{k} \in \bar{\mathbf{U}}$

- $\mathbf{n+j} \in \mathbf{U}$

$$\begin{aligned} b_{ij} &= (0, 1, \theta^p, \theta^d, k, j, n+1, n+j, \sigma^p, \tau^p, \tau^d, n+k, \sigma^d, 2n+1) \\ &- (0, 1, \theta^p, \theta^d, k, n+1, j, n+j, \sigma^p, \tau^p, \tau^d, n+k, \sigma^d, 2n+1) \\ &= (j,k) + (n+1, n+j) - (k, n+1) - (j, n+j). \end{aligned}$$

- $\mathbf{n+j} \in \bar{\mathbf{U}}$

$$\begin{aligned} b_{jk} &= (0, \sigma^p, \tau^p, \tau^d, j, k, n+j, n+k, \theta^p, \theta^d, \sigma^d, n+1, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, j, n+j, k, n+k, \theta^p, \theta^d, \sigma^d, n+1, 2n+1) \\ &= (j,k) + (n+j, n+k) - (k, n+k) - (j, n+j). \end{aligned}$$

We note that in all cases, since $j < k$, we have that the leading element of b_{jk} corresponds to the edge (j, k).

4. Leading edges $(\mathbf{j}, \mathbf{n}+\mathbf{k}), \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n}-\mathbf{1}, \mathbf{k} = \mathbf{1}, \ldots, \mathbf{n}$:
We construct the vectors c_{jk} that are linear combinations of feasible tours that satisfy (9) at equality. Their leading edges will be $(j, n+k)$.

4-1. We first consider the case when $\mathbf{j} \neq \mathbf{k}$ and $\mathbf{k} = \mathbf{2}, \ldots, \mathbf{n}-\mathbf{1}$.

4-1a. $\mathbf{j} \in \bar{\mathbf{U}}, \mathbf{n}+\mathbf{k} \in \bar{\mathbf{U}}$

- $\mathbf{2n} \in \mathbf{U}$

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \tau^p, \tau^d, n, k, \theta^p, \theta^d, \sigma^d, 1, j, n+k, n+j, 2n, n+1, 2n+1) \\ &\quad - (0, \sigma^p, \tau^p, \tau^d, n, k, \theta^p, \theta^d, \sigma^d, 1, j, 2n, n+1, n+k, n+j, 2n+1) \\ &= (j, n+k) + (n+j, 2n) + (n+1, 2n+1) - (n+1, 2n) - (j, 2n) - (n+j, 2n+1). \end{aligned}$$

- $\mathbf{2n} \in \bar{\mathbf{U}}$

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \tau^p, \tau^d, n, k, \theta^p, \theta^d, 1, j, n+k, n+j, \sigma^d, n+1, 2n, 2n+1) \\ &\quad - (0, \sigma^p, \tau^p, \tau^d, n, k, \theta^p, \theta^d, 1, j, 2n, n+k, n+j, \sigma^d, n+1, 2n+1) \\ &= (j, n+k) + (n+1, 2n) + (2n, 2n+1) - (n+k, 2n) - (j, 2n) - (n+1, 2n+1). \end{aligned}$$

  We have $n+k < 2n$, therefore $(j, n+k) \prec (j, 2n)$. It follows that the leading element of c_{jk} corresponds to the edge $(j, n+k)$.

In both cases, we have $n+k < 2n$, therefore $(j, n+k) \prec (j, 2n)$. It follows that the leading edge of c_{jk} is $(j, n+k)$.

4-1b. $\mathbf{j} \in \bar{\mathbf{U}}, \mathbf{n}+\mathbf{k} \in \mathbf{U}$

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \tau^p, \tau^d, n, k, \theta^p, \theta^d, 1, j, n+k, 2n, n+1, \sigma^d, n+j, 2n+1) \\ &\quad - (0, \sigma^p, \tau^p, \tau^d, n, k, \theta^p, \theta^d, 1, j, 2n, n+k, n+1, \sigma^d, n+j, 2n+1) \\ &= (j, n+k) + (n+1, 2n) - (j, 2n) - (n+1, 2n). \end{aligned}$$

Since $(j, n+k) \prec (j, 2n)$, the leading edge of c_{jk} is $(j, n+k)$.

4-1c. $\mathbf{j} \in \mathbf{U}, \mathbf{n}+\mathbf{k} \in \bar{\mathbf{U}}$

- $\mathbf{k} \in \mathbf{U}$

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \tau^p, \tau^d, k, n, j, n+k, \sigma^d, 1, \theta^p, \theta^d, n+j, 2n, n+1, 2n+1) \\ &\quad - (0, \sigma^p, \tau^p, \tau^d, k, n, j, 2n, n+k, \sigma^d, 1, \theta^p, \theta^d, n+j, n+1, 2n+1) \\ &= (j, n+k) - (j, 2n) + (n+j, 2n) + (n+1, 2n) - (n+k, 2n) - (n+1, n+j). \end{aligned}$$

- $\mathbf{k} \in \bar{\mathbf{U}}$

  - $\mathbf{n+j} \in \mathbf{U}$

$$\begin{aligned} c_{jk} &= (0, 1, k, \theta^p, \theta^d, n+k, j, n+1, n, 2n, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, 2n+1) \\ &- (0, 1, k, \theta^p, \theta^d, n+k, n, n+1, j, 2n, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, 2n+1) \\ &= (j, n+k) + (n, 2n) - (j, 2n) - (n, n+k). \end{aligned}$$

  - $\mathbf{n+j} \in \bar{\mathbf{U}}$

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \tau^p, \tau^d, 1, k, \theta^p, \theta^d, n+k, j, n, n+1, 2n, n+j, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, k, \theta^p, \theta^d, \sigma^d, n+k, n+1, n, j, 2n, n+j, 2n+1) \\ &= (j, n+k) + (n+1, 2n) - (j, 2n) - (n+1, n+k). \end{aligned}$$

In all cases, since $j < n$, we have $(j, n+k) \prec (j, 2n)$. Therefore, the leading element of c_{jk} corresponds to the edge $(j, n+k)$.

4-1d. $\mathbf{j} \in \mathbf{U}, \mathbf{n+k} \in \mathbf{U}$

$$\begin{aligned} c_{jk} &= (0, \sigma^p, \tau^p, \tau^d, k, n, j, n+k, 2n, n+j, 1, \sigma^d, \theta^p, \theta^d, n+1, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, k, n, j, 2n, n+k, n+j, 1, \sigma^d, \theta^p, \theta^d, n+1, 2n+1) \\ &= (j, n+1) + (n+j, 2n) - (j, 2n) - (n+k, 2n). \end{aligned}$$

The leading element of c_{jk} corresponds to the edge $(j, n+k)$.

4-2. We now consider the case when $\mathbf{j} \neq \mathbf{k}$ and $\mathbf{k} = \mathbf{1}$.

4-2a. $\mathbf{j} \in \bar{\mathbf{U}}, \mathbf{n+k} \in \mathbf{U}$

$$\begin{aligned} c_{j1} &= (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, j, n+1, n+j, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, j, n+j, n+1, 2n+1) \\ &= (j, n+1) - (j, n+j) + (n+j, 2n+1) - (n+1, 2n+1). \end{aligned}$$

Since $1 < j$, we have $(j, n+1) \prec (j, n+j)$. Therefore, the leading element of c_{j1} corresponds to the edge $(j, n+1)$.

4-2b. $\mathbf{j} \in \mathbf{U}, \mathbf{n+k} \in \mathbf{U}$

- $\mathbf{n+j} \in \mathbf{U}$

$$\begin{aligned} c_{j1} &= (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, j, n+1, n+j, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, j, n+j, n+1, 2n+1) \\ &= (j, n+1) - (j, n+j) + (n+j, 2n+1) - (n+1, 2n+1). \end{aligned}$$

- $\mathbf{n+j \in \bar{U}}$

$$\begin{aligned} c_{j1} &= (0, 1, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n+1, j, n, 2n, n+j, \sigma^d, 2n+1) \\ &- (0, 1, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n+1, n, j, 2n, n+j, \sigma^d, 2n+1). \\ &= (j, n+1) + (n, 2n) - (n, n+1) - (j, 2n). \end{aligned} \tag{22}$$

In both cases, the leading element of c_{j1} corresponds to the edge $(j, n+1)$.

4-3. We now consider the case when $\mathbf{j = k}$. We recall that $\mathbf{j \neq 1}$ and $\mathbf{j \neq n}$.

4-3a. $\mathbf{j \in U}$

- $\mathbf{n+j \in U}$, which implies that $2n \in U$.

$$\begin{aligned} c_{jj} &= (0, n, j, n+j, 2n, \sigma^p, \tau^p, \tau^d, \theta^p, \theta^d, \sigma^d, 1, n+1, 2n+1) \\ &- (0, j, n, n+j, 2n, \sigma^p, \tau^p, \tau^d, \theta^p, \theta^d, \sigma^d, 1, n+1, 2n+1) \\ &= (j, n+j) - (n, n+j) + (0, n) - (0, j). \end{aligned}$$

- $\mathbf{n+j \in \bar{U}}$.

$$\begin{aligned} c_{jj} &= (0, n, j, n+j, \theta^p, \theta^d, 1, n+1, \sigma^p, \tau^p, \tau^d, 2n, \sigma^d, 2n+1) \\ &- (0, j, n, n+j, \theta^p, \theta^d, 1, n+1, \sigma^p, \tau^p, \tau^d, 2n, \sigma^d, 2n+1) \\ &= (j, n+j) - (n, n+j) + (0, n) - (0, j). \end{aligned}$$

In both cases, since $j < n$, the leading element of c_{jj} corresponds to the edge $(j, n+j)$.

4-3b. $\mathbf{j \in \bar{U}}$, which implies that $n+j \in \bar{U}$.

- $\mathbf{2n \in U}$

$$\begin{aligned} c_{jj} &= (0, \sigma^p, \tau^p, \tau^d, n, 1, \theta^p, \theta^d, \sigma^d, j, n+j, n+1, 2n, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, n, 1, \theta^p, \theta^d, \sigma^d, j, 2n, n+1, n+j, 2n+1) \\ &= (j, n+j) - (j, 2n) + (2n, 2n+1) - (n+j, 2n+1). \end{aligned}$$

- $\mathbf{2n \in \bar{U}}$

$$\begin{aligned} c_{jj} &= (0, \sigma^p, \tau^p, \tau^d, n, 1, j, n+j, \sigma^d, 2n, n+1, \theta^p, \theta^d, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, n, 1, j, 2n, \overleftarrow{\sigma}^d, n+j, n+1, \theta^p, \theta^d, 2n+1) \\ &= (j, n+j) - (j, 2n) + (n+1, 2n) - (n+1, n+j). \end{aligned}$$

In both cases the leading element of c_{jj} corresponds to the edge $(j, n+j)$.

4-4. We now consider the case when $\mathbf{k = n}$.

4-4a. $\mathbf{j \in U}$

- $\mathbf{2n \in U}$

$$\begin{aligned} c_{jn} &= (0, n, j, 2n, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, \theta^p, \theta^d, 1, n+1, 2n+1) \\ &- (0, j, n, 2n, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, \theta^p, \theta^d, 1, n+1, 2n+1) \\ &= (j, 2n) - (n, 2n) + (0, n) - (0, j). \end{aligned}$$

- $\mathbf{2n \in \bar{U}}$

$$\begin{aligned} c_{jn} &= (0, n, j, 2n, \theta^p, \theta^d, 1, \sigma^p, \tau^p, \tau^d, n+1, n+j, \sigma^d, 2n+1) \\ &- (0, j, n, 2n, \theta^p, \theta^d, 1, \sigma^p, \tau^p, \tau^d, n+1, n+j, \sigma^d, 2n+1) \\ &= (j, 2n) - (n, 2n) + (0, n) - (0, j). \end{aligned}$$

In both cases, the leading edge of c_{jn} is (j, n).

4-4b. $\mathbf{j \in \bar{U}}$, which implies $n + j \in \bar{U}$.

$$\begin{aligned} c_{jn} &= (0, n, 2n, j, n+j, \theta^p, \theta^d, 1, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, 2n+1) \\ &- (0, j, n+j, \theta^p, \theta^d, 1, \sigma^p, \tau^p, \tau^d, n+1, n, 2n, \sigma^d, 2n+1) \\ &= \begin{cases} (j, 2n) + (n+1, \sigma^d_s) - (n, n+1) - (\sigma^d_s, 2n) + (0, n) - (0, j), & \text{if } \sigma^p \neq \emptyset \\ (j, 2n) + (n+1, 2n+1) - (n, n+1) - (2n, 2n+1) + (0, n) - (0, j), & \text{if } \sigma^p = \emptyset \end{cases} \end{aligned}$$

Since $j < n$, the leading edge of c_{jn} is $(j, 2n)$.

We define the set T_4 as the set of all vectors c_{ij}, i.e., $T_4 = \{c_{jk} : j = 2, \ldots, n-1, k = 1, \ldots, n\}$.

5. Leading edges $(\mathbf{n}, \mathbf{n+j}), \mathbf{j = 2}, \ldots, \mathbf{n-1}$:
We now construct the vectors d_j as linear combinations of feasible tours that satisfy (9) at equality. Their leading edges will be $(n, n+j)$.

5a. $\mathbf{n + j \in U}$

$$\begin{aligned} d_j &= (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+1, j, n, n+j, 2n, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+1, j, n, 2n, n+j, 2n+1) \\ &= (n, n+j) - (n, 2n) + (2n, 2n+1) - (n+j, 2n+1). \end{aligned}$$

Since $j < n$, $(n, n+j) \prec (n, 2n)$. Therefore, the leading edge of d_j is $(n, n+j)$.

5b. $\mathbf{n+j \in \bar{U}}$

- $\mathbf{j \in U}$
  We consider two cases:
  - $2n \in \bar{U}$

$$\begin{aligned} d_j &= (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+1, j, n, n+j, 2n, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+1, j, n, 2n, n+j, 2n+1) \\ &= (n, n+j) - (n, 2n) + (2n, 2n+1) - (n+j, 2n+1). \end{aligned}$$

  - $2n \in U$.

$$\begin{aligned} d_j &= (0, \sigma^p, \tau^p, \tau^d, j, n, n+j, \sigma^d, \theta^p, \theta^d, 1, n+1, 2n, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, j, n, 2n, n+j, \sigma^d, \theta^p, \theta^d, 1, n+1, 2n+1) \\ &= (n, n+j) - (n, 2n) + (n+1, 2n) - (n, 2n) + (2n, 2n+1) - (n+1, 2n+1). \end{aligned}$$

- $\mathbf{j \in \bar{U}}$. We define d_j as follows:

$$\begin{aligned} d_i &= (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, j, n+j, n, n+1, 2n, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, j, n+j, n+1, n, 2n, 2n+1) \\ &= (n, n+j) + (n+1, 2n) - (n+1, n+j) - (n, 2n). \end{aligned}$$

In all cases, since $j < n$, we have $(n, n+j) \prec (n, 2n)$ and therefore the leading edge of d_j is $(n, n+j)$.

We now define the set T_5 as the set of all vectors d_j, i.e. $T_5 = \{d_j : j = 2, \ldots, n-1\}$.

6. Leading edges $\mathbf{(n+j, n+k), j = 1, \ldots, n-2, k = j+1, \ldots, n-1}$**:**
We now construct the vectors e_{jk} that are linear combinations of feasible tours that satisfy (9) at equality and have the leading edges $(n+j, n+k)$.

6a. $\mathbf{n+j, n+k \in U}$ (implicitely, $2n \in U$).

- $\mathbf{j \neq 1}$.

$$\begin{aligned} e_{jk} &= (0, j, k, n, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 1, n+1, n+j, n+k, 2n, 2n+1) \\ &- (0, j, k, n, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 1, n+1, n+j, 2n, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1). \end{aligned}$$

- $\mathbf{j = 1}$.

$$\begin{aligned} e_{jk} &= (0, k, n, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 1, n+k, n+j, 2n, 2n+1) \\ &- (0, k, n, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 1, n+k, 2n, n+j, 2n+1) \\ &= (n+j, n+k) - (n+k, 2n) - (n+j, 2n+1) + (2n, 2n+1). \end{aligned}$$

In both cases, since $j < k$ and $k < n$, $(n+j, n+k) \prec (n+k, 2n)$. Therefore, the leading edge of e_{jk} is $(n+j, n+k)$.

6b. $\mathbf{n+j \in U, n+k \in \bar{U}}$

- $\mathbf{j \neq 1}$
  Since $n+j \in U$ and $j \neq 1$, it follows that $j \in U$ and $2n \in U$. We consider two cases:
  - $\mathbf{k \in U}$

$$\begin{aligned} e_{jk} &= (0, j, k, n, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+k, n+j, n+1, 2n, 2n+1) \\ &- (0, j, k, n, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+k, 2n, n+1, n+j, 2n+1) \\ &= (n+j, n+k) - (n+k, 2n) - (n+j, 2n+1) + (2n, 2n+1). \end{aligned}$$

  - $\mathbf{k \in \bar{U}}$

$$\begin{aligned} e_{jk} &= (0, j, n, \sigma^p, \tau^p, \tau^d, 1, k, \theta^p, \theta^d, \sigma^d, n+k, n+j, n+1, 2n, 2n+1) \\ &- (0, j, n, \sigma^p, \tau^p, \tau^d, 1, k, \theta^p, \theta^d, \sigma^d, n+k, 2n, n+j, n+1, 2n+1) \\ &= (n+j, n+k) - (n+k, 2n) - (n+j, 2n+1) + (2n, 2n+1). \end{aligned}$$

- $\mathbf{j = 1}$
  We consider two cases:
  - $\mathbf{k \in U}$

$$\begin{aligned} e_{jk} &= (0, k, n, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+k, n+1, 2n, 2n+1) \\ &- (0, k, n, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+k, 2n, n+1, 2n+1) \\ &= (n+1, n+k) - (n+k, 2n) - (n+1, 2n+1) + (2n, 2n+1). \end{aligned}$$

  - $\mathbf{k \in \bar{U}}$

$$\begin{aligned} e_{jk} &= (0, n, \sigma^p, \tau^p, \tau^d, 1, k, \theta^p, \theta^d, \sigma^d, n+k, n+1, 2n, 2n+1) \\ &- (0, n, \sigma^p, \tau^p, \tau^d, 1, k, \theta^p, \theta^d, \sigma^d, n+k, 2n, n+1, 2n+1) \\ &= (n+1, n+k) - (n+k, 2n) - (n+1, 2n+1) + (2n, 2n+1). \end{aligned}$$

In all cases, the leading edge of e_{jk} is $(n+1, n+k)$.

6c. $\mathbf{n+j \in \bar{U}, n+k \in U}$

Since $k \neq 1, k \in U$, it follows that $2n \in U$. We also note that $j \neq 1$, since $n+j \in \bar{U}$, while $n+1 \in U$. We consider two cases:

- $\mathbf{j \in U}$

$$\begin{aligned} e_{jk} &= (0, j, k, n, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+j, n+k, n+1, 2n, 2n+1) \\ &- (0, j, k, n, \sigma^p, \tau^p, \tau^d, 1, \theta^p, \theta^d, \sigma^d, n+j, 2n, n+1, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1). \end{aligned}$$

- $\mathbf{j \in \bar{U}}$

$$\begin{aligned} e_{jk} &= (0, k, n, \sigma^p, \tau^p, \tau^d, 1, j, \theta^p, \theta^d, \sigma^d, n+j, n+k, n+1, 2n, 2n+1) \\ &- (0, k, n, \sigma^p, \tau^p, \tau^d, 1, j, \theta^p, \theta^d, \sigma^d, n+j, 2n, n+1, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1). \end{aligned}$$

In both cases, since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading edge of e_{jk} is $(n+j, n+k)$.

6d. $\mathbf{n+j, n+k \in \bar{U}}$

$$\begin{aligned} e_{jk} &= (0, n, \sigma^p, \tau^p, \tau^d, [j, k, 1], \theta^p, \theta^d, \sigma^d, n+j, n+k, n+1, 2n, 2n+1) \\ &- (0, n, \sigma^p, \tau^p, \tau^d, [j, k, 1], \theta^p, \theta^d, \sigma^d, n+j, 2n, n+1, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1). \end{aligned}$$

Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading element of e_{jk} corresponds to the edge $(n+j, n+k)$.

We recall that according to our notation we had

$$[j, k, 1] = \begin{cases} (j, k, 1), & \text{if } j \in U, k \in \bar{U} \text{ or } j, k \in U \text{ or } j, k \in \bar{U} \\ (k, j, 1), & \text{if } j \in \bar{U}, k \in U \end{cases}$$

We now define the set T_6 as the set of all vectors e_{jk}, i.e. $T_6 = \{e_{jk} : j = 1, \ldots, n-1, k = 2, \ldots, n-1\}$.

7. Leading edges $\mathbf{(n+j, 2n), j = 2, \ldots, n-2}$**:**

We now construct the vectors f_j that are linear combinations of feasible tours that satisfy (9) at equality and have the leading edges $(n+j, 2n)$.

7a. $\mathbf{n+j \in \bar{U}}$

- $\mathbf{2n-1 \in U}$ (Since $2n-1 \in U$, it follows that $2n \in U$.)

  We first construct the vectors

$$\begin{aligned} w_j &= (0, \sigma^p, \tau^p, \tau^d, n, n-1, j, \theta^p, \theta^d, \sigma^d, 1, 2n-1, n+1, 2n, n+j, 2n+1) \\ &- (0, \sigma^p, \tau^p, \tau^d, n, n-1, j, \theta^p, \theta^d, \sigma^d, 1, 2n-1, 2n, n+1, n+j, 2n+1) \\ &= (n+1, 2n-1) + (n+j, 2n) - (2n, 2n-1) - (n+1, 2n) + (2n, 2n+1) - (n+j, 2n+1). \end{aligned}$$

  We now define f_j as follows:

$$\begin{aligned} f_j &= w_j - e_{1,n-1} \\ &= (n+1, 2n-1) + (n+j, 2n) - (2n, 2n-1) - (n+1, 2n) + (2n, 2n+1) - (n+j, 2n+1) \\ &- (n+1, 2n-1) - (2n, 2n+1) + (2n-1, 2n) - (n+j, 2n+1) \\ &= (n+j, 2n) - (n+1, 2n) + (n+1, 2n+1) - (n+j, 2n+1). \end{aligned}$$

  We recall that $(n+j, 2n) \prec (n+1, 2n)$ (according to the order defined on the set of edges). Therefore, the leading edge of f_j is $(n+j, 2n)$.

- $\mathbf{2n-1 \in \bar{U}}$

$$\begin{aligned} f_j &= (0, n, \sigma^p, \tau^p, \tau^d, [n-1, j], \sigma^d, \theta^p, \theta^d, 1, n+1, 2n, n+j, 2n-1, 2n+1) \\ &- (0, n, \sigma^p, \tau^p, \tau^d, [n-1, j], \sigma^d, \theta^p, \theta^d, 1, n+1, 2n, 2n-1, n+j, 2n+1) \\ &= (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1). \end{aligned}$$

  The leading edge of f_j is $(n+j, 2n)$.
  We note that in this case the notation $[n-1, j]$ was used for:

$$[n-1, j] = \begin{cases} (n-1, j), & \text{if } n-1, j \in U \text{ or } n-1 \in U, j \in \bar{U} \text{ or } n-1, j \in \bar{U} \\ (j, n-1), & \text{if } j \in U, n-1 \in \bar{U} \end{cases}$$

7b. $\mathbf{n+j \in U}$

- $\mathbf{2n-1 \in U}$ (It follows that $n-1 \in U$.)

$$\begin{aligned} f_j &= (0, n, \sigma^p, \tau^p, \tau^d, n-1, j, \sigma^d, \theta^p, \theta^d, 1, n+1, 2n, n+j, 2n-1, 2n+1) \\ &- (0, n, \sigma^p, \tau^p, \tau^d, n-1, j, \sigma^d, \theta^p, \theta^d, 1, n+1, 2n, 2n-1, n+j, 2n+1) \\ &= (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1). \end{aligned}$$

  The leading element of f_j corresponds to the edge $(n+j, 2n)$.

- $\mathbf{2n-1} \in \bar{\mathbf{U}}$ We first construct the vectors

$$
\begin{aligned}
w_j &= (0, \sigma^p, \tau^p, \tau^d, n, [n-1, j], \theta^p, \theta^d, \sigma^d, 1, 2n-1, n+1, 2n, n+j, 2n+1) \\
&\quad - (0, \sigma^p, \tau^p, \tau^d, n, [n-1, j], \theta^p, \theta^d, \sigma^d, 1, 2n-1, 2n, n+1, n+j, 2n+1) \\
&= (n+1, 2n-1) + (n+j, 2n) - (2n, 2n-1) - (n+1, 2n) + (2n, 2n+1) - (n+j, 2n+1).
\end{aligned}
$$

We now define f_j as follows:

$$
\begin{aligned}
f_j &= w_j - e_{1,n-1} \\
&= (n+1, 2n-1) + (n+j, 2n) - (2n, 2n-1) - (n+1, 2n) + (2n, 2n+1) - (n+j, 2n+1) \\
&\quad - (n+1, 2n-1) - (2n, 2n+1) + (2n-1, 2n) - (n+j, 2n+1) \\
&= (n+j, 2n) - (n+1, 2n) + (n+1, 2n+1) - (n+j, 2n+1).
\end{aligned}
$$

We recall that $(n+j, 2n) \prec (n+1, 2n)$ (according to the order defined on the set of edges). Therefore, the leading element of f_j corresponds to the edge $(n+j, 2n)$.

We note that in this case the notation $[n-1, j]$ was used for:

$$
[n-1, j] = \begin{cases} (n-1, j), & \text{if } n-1, j \in U \text{ or } n-1 \in U, j \in \bar{U} \text{ or } n-1, j \in \bar{U} \\ (j, n-1), & \text{if } j \in U, n-1 \in \bar{U} \end{cases}
$$

We now define the set T_7 as the set of all vectors f_j, i.e. $T_7 = \{f_j : j = 2, \ldots, n-2\}$.

The vectors in $T = T_0 \cup T_1 \cup \cdots \cup T_7$ are the vectors needed. ■

5.3 Subtour elimination constraints

Next, we prove that inequality (11) is facet of the PDTSP polytope, when S contains pickup vertices for which their corresponding destination is not in S and pairs of pickup/destination vertices.

Assumption 5.9 *The vertex i in S mentioned in Proposition 4.10 can be assumed to be vertex 1. Therefore $1, n+1 \in S$.*

It is clear that by a simple relabelling of vertices, this assumption is always true.

Notation 5.10 *In what follows we introduce some notation.*

- *Let $S^p = \{i : i \in V^p \cap S\}$, $S_p = \{i : i \in S^p, n+i \notin S\}$, and $S_{pd} = \{i : i \in S^p, i \neq 1, n+i \in S\}$. We note that $S^p = S_p \cup S_{pd} \cup \{1\}$.*

  Similarly, let $\bar{S}^p = \{i : i \in V^p \cap \bar{S}\}$, and $\bar{S}_{pd} = \{i : i \in \bar{S}^p, n+i \in \bar{S}\}$.

  We denote by σ^p any permutation on S_p or subsets of S_p, by τ^p any permutation on S_{pd} or subsets of S_{pd}, and by θ^p any permutation on $\bar{S}_{pd}$ or subsets of $\bar{S}_{pd}$.

- *We use the notation $[i, j, k]$ inside a tour to denote a sequence of vertices that contains i, j, and k, ordered such that (11) is satisfied at equality ($[i, j, k] \in \Pi(i, j, k)$).*

We observe that if $S_p = \emptyset$, then the constraint (11) is simply a subtour elimination constraint for the TSP and does not define a facet for the PDTSP polytope. Therefore a necessary condition for (11) being facet is $S_p \neq \emptyset$.

We also note that if there is a pickup vertex i, $i \notin S^p$, such that $n + i \in S$, then (11) is not facet for the PDTSP polytope. This is easy to check for small values of n, using Porta.

To summarise, we will prove that the lifted subtour elimination constraints are facets for the PDTSP under the following assumptions:

Assumption 5.11 *Necessary conditions for (11) being facet of the PDTSP polytope:*

- *There is no $i \in V^p$ such that $i \notin S$ and $n + i \in S$.*
- $S_p \neq \emptyset$.

Assumption 5.12 *Without loss of generality we make the following assumptions:*

- *The vertices in S^p satisfy the following condition: $\forall i \in S_p, j \in S_{pd}, i < j$.*
- *Every pickup vertex in $\bar{S}$ is smaller than every pickup vertex in S (i.e., $\forall i \in S^p, j \in \bar{S}^p, i > j$).*
- *Every pickup vertex from S_{pd} is smaller than any pickup vertex from S_p. We note that since $S_p \neq \emptyset$, we have $n \in S$ and $2n \notin S$.*

Figure 2 illustrates our notation and the most general structure of the sets S and $\bar{S}$, taking into account our assumptions.

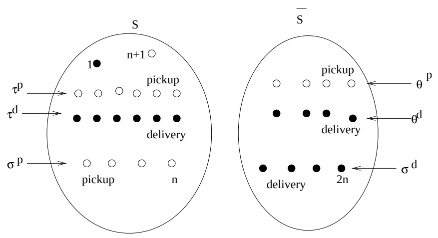

Figure 2: Illustration of our assumptions and notation for the sets S and $\bar{S}$.

The order defined on the set of edges is given by the following definition:

Definition 5.13 *We define the sets $E^0 = \{(0, 2n + 1)\}$ and $E^1 = E \setminus (E^0 \cup E^2)$, where $E^2 = (\delta(0) \cup \delta(2n + 1) \cup \{(n, n + 1), (n, 2n), (2n - 1, 2n)\}) \setminus E^0$. Let $\prec_{E^1}$ be the lexicographic order on the set E^1 and $\prec_{E^2}$ the lexicographic order on the set E^2. We define a relation of total order $\prec$ on the set of edges E as follows:*

i. *for any* $(i,j) \in E \setminus E^0$, $(0, 2n+1) \prec (i,j)$;

ii. *the restriction of* $\prec$ *to* E^1 *is* $\prec_{E^1}$;

iii. *the restriction of* $\prec$ *to* E^2 *is* $\prec_{E^2}$;

iv. *for any* $(i,j) \in E^1$ *and* $(k,l) \in E^2$, $(i,j) \prec (k,l)$.

Theorem 5.14 *For any set* $S \subseteq V^p \cup V^d$ *such that* $1 \in S$ *and* $n+1 \in S$ *and under our standing assumptions, the inequality (11) is facet defining for the PDTSP polytope for* $n \geq 3$.

Proof. It is easy to see that, under our assumptions, the face defined by (11) is proper. In order to prove that (11) is facet defining for the PDTSP polytope, we need to show that the dimension of the convex hull of the feasible tours that satisfy (11) at equality is one less than the dimension of the PDTSP polytope. Therefore, by Theorem 3.5, we need to prove that there are $2n^2 - n - 2$ affinely independent elements of the PDTSP polytope that satisfy (11) at equality.

We will prove this this by taking each tour in the PDTSP polytope that satisfies (11) at equality and considering it as a row in a matrix. We will demonstrate that this matrix has rank $2n^2 - n - 2$, and therefore there are $2n^2 - n - 2$ linearly independent rows of the matrix, i.e. $2n^2 - n - 2$ linearly independent tours of the PDTSP polytope that satisfy (11) at equality. Since linearly independence implies affinely independence, the result we needed is proved.

We will show that the rank of the matrix is $2n^2-n-2$ by using row operations. We will find $2n^2-n-2$ linear combinations of rows that will form an upper triangular matrix. We will explicitely describe eight sets of vectors, $T_0, T_1, \ldots, T_7$, that contain linear combinations of feasible tours of the PDTSP polytope satisfying 11 at equality. The construction of these sets ensures that they are disjoint. Their union is a set of $2n^2 - n - 2$ linearly independent vectors. Each vector from the union of the sets T_i, $i = 0, \ldots, 7$, will have a leading element corresponding to a different edge from the first $2n^2 - n - 2$ edges, according to the order defined in Definition 5.13.

We briefly describe the sets of edges that will correspond to leading elements of the vectors in the sets T_i, $i = 0, \ldots, 7$. We will denote the sets of edges by B_m, where B_i is associated with T_i, for any $i = 0, \ldots, 7$.

$B_0 = \{(0, 2n+1)\}$.

$B_1 = \{(1,j) : j = 2, \ldots, n\}$.

$B_2 = \{(1, n+j) : j = 2, \ldots, n\}$.

$B_3 = \{(j,k) : j = 2, \ldots, n-2, k = j+1, \ldots, n-1\} \cup \{(j,n) : j = 2, \ldots, n-1\}$.

$B_4 = \{(j, n+k) : j = 2, \ldots, n-1, k = 1, \ldots, n-1\} \cup \{(j, 2n) : j = 2, \ldots, n-1\}$.

$B_5 = \{(n, n+j) : j = 1, \ldots, n-1\} \setminus \{(n, n+1)\}$.

$B_6 = \{(n+j, n+k) : j = 1, \ldots, n-2, k = j+1, \ldots, n-1\}$.

$B_7 = \{(n+j, 2n) : j = 1, \dots, n-2\}$.

0. Leading edge $(0, 2n+1)$**:**

We first define the set T_0 that contains only one vector. The leading edge of this vector, with respect to the order on the set of edges defined, is $(0, 2n+1)$.

We define $T_0 = \{(0, 1, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, 2n+1)\}$.

1. Leading edges $(\mathbf{1}, \mathbf{j}), \mathbf{j} = \mathbf{2}, \dots, \mathbf{n}$**:**

We construct the vectors m_j as linear combinations of feasible vectors in the PDTSP polytope that satisfy (11) at equality. Their leading edges will be $(1, j)$. We define m_j as:

$$\begin{aligned} m_j &= (0, j, 1, \tau^p, n, n+1, \sigma^p, \tau^d, n+j, 2n, \sigma^d, \theta^o, \theta^d, 2n+1) \\ &- (0, j, n, \overleftarrow{\tau}^p, 1, n+1, \sigma^p, \tau^d, n+j, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &= (1, j) - (j, n) + (n, n+1) - (1, n+1). \end{aligned}$$

The leading edge of m_j is $(1, j)$.

We define the set T_1 as the set of all vectors m_j, i.e., $T_1 = \{m_j : j = 2, \dots, n\}$.

2. Leading edges $(\mathbf{1}, \mathbf{n}+\mathbf{j}), \mathbf{j} = \mathbf{1}, \dots, \mathbf{n}$**:**

We construct the vectors a_j that are linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be $(1, n+j)$.

2a. $\mathbf{j} \in \mathbf{S^p}$

We consider three cases:

- $\mathbf{j} = \mathbf{1}$.

  In this case $n+j \in S$.

  $$\begin{aligned} a_1 &= (0, n, 1, n+1, \sigma^p, \tau^p, \tau^d, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &- (0, 1, n, n+1, \sigma^p, \tau^p, \tau^d, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &= (1, n+1) - (n, n+1) + (0, n) - (0, 1). \end{aligned}$$

  The leading element of a_1 corresponds to the edge $(1, n+1)$.

- $\mathbf{j} \neq \mathbf{1}$

  In this case $n+j \in S$ or $n+j \notin S$.

  $$\begin{aligned} a_j &= (0, \sigma^p, \tau^p, j, 1, n+j, n+1, \tau^d, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &- (0, 1, j, \overleftarrow{\tau}^p, \overleftarrow{\sigma}^p, n+1, \overleftarrow{\tau}^d, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &= (1, n+j) + (n+1, n+j) - (\sigma^p_s, n+1) + (\tau^d_e, \sigma^d_s) - (n+j, \tau^d_s) - (n+j, \sigma^d_s) + (0, \sigma^p_s) - (0, 1). \end{aligned}$$

We recall that $\sigma^p \neq \emptyset$. We note that if $\tau^p = \tau^d = \emptyset$, we have:
$a_j = (1, n+j) - \sigma^p_s, n+1) + (n+1, \sigma^d_s) - (n+j, \sigma^d_s) + (0, \sigma^p_s) - (0, 1)$.
It is clear that the leading element of a_j is the edge $(1, n+j)$.

2b. $\mathbf{j \notin S^p}$

In this case we have $n + j \notin S$.

$$\begin{aligned} a_j &= (0, j, n+j, 1, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &- (0, 1, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, j, n+j, 2n+1) \\ &= (1, n+j) - (j, n+\theta^p_e) - (n+j, 2n+1) + (n+\theta^p_e, 2n+1) + (0, j) - (0, 1). \end{aligned}$$

We recall that $\sigma^p \neq \emptyset$, since $n \in \sigma^p$. We note that if $\theta^p = \theta^d = \emptyset$, we have:
$a_j = (1, n+j) - (j, n+\sigma^p_e) - (n+j, 2n+1) + (n+\sigma^p_e, 2n+1) + (0, j) - (0, 1)$.

The leading edge of a_j is $(1, n+j)$.

We now define the set T_2 to be the set of all vectors a_j, i.e., $T_2 = \{a_j : j = 1, \ldots, n\}$.

3. Leading edges $\mathbf{(j, k), j = 2, \ldots, n-1, k = j+1, \ldots, n}$:

We construct the vectors b_{jk} as linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be (j, k).

3a. $\mathbf{j, k \notin S^p}$

In this case $n + j, n + k \notin S$.

$$\begin{aligned} b_{jk} &= (0, j, k, n+j, n+k, 1, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &- (0, j, n+j, k, n+k, 1, \sigma^p, \tau^p, \tau^d, n+1, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &= (j, k) + (n+j, n+k) - (j, n+j) - (k, n+k). \end{aligned}$$

The leading element of b_{jk} corresponds to the edge (j, k).

3b. $\mathbf{j \in S^p, k \notin S^p}$

In this case $n + k \notin S$ and $n + j \in S$ or $n + j \notin S$. According to Assumption 5.12 we also have $j > t$.

$$\begin{aligned} b_{jk} &= (0, k, j, \sigma^p, \tau^p, \tau^d, 1, n+1, n+j, n+k, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &- (0, j, \sigma^p, \tau^p, \tau^d, 1, n+1, n+j, k, n+k, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &= (j, k) + (n+j, n+k) - (k, n+j) - (k, n+k) + (0, k) - (0, j). \end{aligned} \tag{23}$$

The leading element of b_{jk} is (j, k).

3c. $\mathbf{j \notin S^p, k \in S^p}$

We note that in this case we have $n+j \notin S$, while $n+k \in S$ or $n+k \notin S$. Assumption 5.12 implies that $j < k$.

$$\begin{aligned} b_{jk} &= (0, j, k, \sigma^p, \tau^p, \tau^d, 1, n+k, n+1, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, k, \sigma^p, \tau^p, \tau^d, 1, n+k, n+1, j, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad = (j,k) + (n+1, n+j) - (j, n+1) - (j, n+j) + (0, j) - (0, k). \end{aligned}$$

The leading element of b_{jk} corresponds to the edge (j,k).

3d. $\mathbf{j \in S^p, k \in S^p}$

In this case $n+j \in S$ or $n+j \notin S$. Also, $n+k \in S$ or $n+k \notin S$.

$$\begin{aligned} b_{ij} &= (0, j, k, \sigma^p, \tau^p, \tau^d, 1, n+1, [n+k, n+j], \sigma^d, \theta^p, \theta^d, 2n+1) \\ &- (0, k, \sigma^p, \tau^p, \tau^d, 1, n+1, j, [n+k, n+j], \sigma^d, \theta^p, \theta^d, 2n+1) \\ &= \begin{cases} (j,k) + (n+1, n+j) - (j, n+1) - (j, n+j) + (0,j) - (0,k), & \text{if } [n+k, n+j] = (n+j, n+k) \\ (j,k) + (n+1, n+k) - (j, n+1) - (j, n+k) + (0,j) - (0,k), & \text{otherwise} \end{cases} \end{aligned}$$

We recall that according to our notation,

$$[n+k, n+j] = \begin{cases} (n+j, n+k), & \text{if } n+j \in S \text{ and } n+k \notin S \\ (n+k, n+j), & \text{otherwise} \end{cases}$$

It is clear that the leading element of b_{jk} is (j,k).

We define the set T_3 as the set of all vectors b_{jk}, i.e. $T_3 = \{b_{jk} : j = 2, \ldots, n-1, k = j+1, \ldots, n\}$.

4. Leading edges $\mathbf{(j, n+k), j = 2, \ldots, n-1, k = 1, \ldots, n}$**:**

We construct the vectors c_{jk} as linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be $(j, n+k)$.

4-1. We first consider the case when $j \neq k$ and $k = 1, \ldots, n-1$.

4-1a. $\mathbf{j, k \notin S^p}$

In this case, we know that $n+j, n+k \notin S$.

$$\begin{aligned} c_{jk} &= (0, \theta^p, \theta^d, 1, \sigma^p, \tau^p, \tau^d, n, n+1, \sigma^d, k, n+k, j, n+j, 2n, 2n+1) \\ &\quad - (0, \theta^p, \theta^d, 1, \sigma^p, \tau^p, \tau^d, n, n+1, \sigma^d, k, n+k, 2n, j, n+j, 2n+1) \\ &\quad = (j, n+k) + (n+j, 2n) + (2n, 2n+1) - (n+k, 2n) - (j, 2n) - (n+j, 2n+1). \end{aligned}$$

We have $n+k < 2n$, therefore $(j, n+k) \prec (j, 2n)$. It follows that the leading element of c_{jk} corresponds to the edge $(j, n+k)$.

4-1b. $\mathbf{j} \in \mathbf{S^p}, \mathbf{k} \notin \mathbf{S^p}$

In this case $n+k \notin S$, but $n+j \in S$ or $n+j \notin S$. We define c_{jk} as follows:

$$\begin{aligned} c_{jk} &= (0, k, n+k, j, n, 2n, 1, n+1, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, k, n+k, n, j, 2n, 1, n+1, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad = (j, n+k) - (j, 2n) + (n, 2n) - (n, n+k). \end{aligned} \tag{24}$$

Since $(j, n+k) \prec (j, 2n)$, it follows that edge of c_{jk} is $(j, n+k)$.

4-1c. $\mathbf{j} \notin \mathbf{S^p}, \mathbf{k} \in \mathbf{S^p}$

We have that $n+j \notin S$ and $n+k \in S$ or $n+k \notin S$.

- $\mathbf{k = 1}$

$$\begin{aligned} c_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, 1, n+k, j, n+j, 2n, \sigma^d, 2n+1) \\ &\quad - (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, 1, n+k, 2n, j, n+j, \sigma^d, 2n+1) \\ &\quad = (j, n+k) + (n+j, 2n) + (2n, \sigma^d_s) - (n+k, 2n) - (j, 2n) - (n+j, \sigma^d_s). \end{aligned}$$

We note that if $\sigma^p = \sigma^d = \emptyset$ (possible, since n has already been used), we have $c_{jk} = (j, n+k) + (n+j, 2n) + (2n, 2n+1) - (n+k, 2n) - (j, 2n) - (n+j, 2n+1)$.

Since $k < n$, we have $(j, n+k) \prec (j, 2n)$. Therefore, the leading edge of c_{jk} is $(j, n+k)$.

- $\mathbf{k \neq 1}$

We note that $n+k \in S$ or $n+k \notin S$. We define c_{jk} as follows:

$$\begin{aligned} c_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, k, 1, n+1, n+k, j, n+j, 2n, \sigma^d, 2n+1) \\ &\quad - (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, k, 1, n+1, n+k, 2n, j, n+j, \sigma^d, 2n+1) \\ &\quad = (j, n+k) - (j, 2n) + (n+j, 2n) + (\sigma^d_s, 2n) - (n+k, 2n) - (n+j, \sigma^d_s). \end{aligned} \tag{25}$$

We note that if $\sigma^p = \sigma^d = \emptyset$ (possible since n is used separately), we have $c_{jk} = (j, n+k) - (j, 2n) + (n+j, 2n) + (2n, 2n+1) - (n+k, 2n) - (n+j, 2n+1)$.

In both cases the leading edge of c_{jk} is $(j, n+k)$.

4-1d. $\mathbf{j}, \mathbf{k} \in \mathbf{S^p}$

We note that $n+j \in S$ or $n+j \notin S$. We consider the following two cases:

- **k ≠ 1** In this case, $n+k \in S$ or $n+k \notin S$. We define c_{jk} as follows:

$$
\begin{aligned}
c_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, k, j, n+k, 2n, \sigma^d, 1, n+1, n+j, 2n+1) \\
&\quad - (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, k, j, 2n, \sigma^d, 1, n+1, [n+j, n+k], 2n+1) \\
&= \begin{cases} (j, n+k) - (j, 2n) + (n+k, 2n) + (n+1, n+j) - (n+1, n+k) - (n+j, n+k), \\ \qquad\qquad \text{if } [n+j, n+k] = (n+k, n+j) \\ \\ (j, n+k) - (j, 2n) + (n+k, 2n) + (n+j, 2n+1) - (n+j, n+k) - (n+k, 2n+1), \\ \qquad\qquad \text{otherwise} \end{cases}
\end{aligned}
$$

We recall that according to our notation,

$$
[n+j, n+k] = \begin{cases} (n+k, n+j), & \text{if } n+k \in S, n+j \notin S \\ (n+j, n+k), & \text{otherwise} \end{cases}
$$

The leading edge of c_{jk} is $(j, n+k)$.

- **k = 1**

$$
\begin{aligned}
c_{jk} &= (0, n, k, j, n+k, \sigma^p, \tau^p, \tau^d, n+j, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\
&\quad - (0, j, k, n, n+k, \sigma^p, \tau^p, \tau^d, n+j, 2n, \sigma^d, \theta^p, \theta^d, 2n+1) \\
&= (j, n+k) - (n, n+k) + (0, n) - (0, j).
\end{aligned}
$$

The leading edge of c_{jk} is $(j, n+k)$.

4-2. We now consider the case when $j = k$. In other words we will construct linear combinations of feasible tours that satisfy (11) at equality and have the leading element corresponding to the edges $(j, n+j)$.

We define the vectors c_{jj} as follows:

$$
\begin{aligned}
c_{jj} &= (0, n, j, n+j, 2n, 1, n+1, \sigma^p, \tau^p, \tau^d, \sigma^d, \theta^p, \theta^d, 2n+1) \\
&\quad - (0, j, n, n+j, 2n, 1, n+1, \sigma^p, \tau^p, \tau^d, \sigma^d, theta^p, \theta^d, 2n+1) \\
&= (j, n+j) - ((n, n+j) + (0, n) - (0, j).
\end{aligned}
$$

The leading edge of c_{jj} is $(j, n+j)$.

4-3. We now construct linear combinations of feasible tours that satisfy (11) at equality and have the leading element corresponding to the edges $(j, 2n)$, where $j = 2, \ldots, n-1$. In other words, this case corresponds to the situation when we want to obtain the edge $(j, n+k)$ with $k = n$.

We define c_{jn} as follows:

$$\begin{aligned} c_{jn} &= (0, n, j, 2n, 1, n+1, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad - (0, j, n, 2n, 1, n+1, \sigma^p, \tau^p, \tau^d, n+j, \sigma^d, \theta^p, \theta^d, 2n+1) \\ &\quad = (j, 2n) - (n, 2n) + (0, n) - (0, j). \end{aligned}$$

The leading edge of c_{jn} is $(j, 2n)$.

We define the set T_4 as the set of all vectors c_{ij}, i.e., $T_4 = \{c_{jk} : j = 2, \ldots, n-1, k = 1, \ldots, n\}$.

5. Leading edges $(\mathbf{n}, \mathbf{n}+\mathbf{j}), \mathbf{j} = \mathbf{2}, \ldots, \mathbf{n}-\mathbf{1}$:
We construct the vectors d_j as linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be $(n, n+j)$.

We define d_j as follows:

$$\begin{aligned} d_j &= (0, \theta^p, \theta^d, j, \sigma^p, \tau^p, \tau^d, n, n+j, 2n, \sigma^d, 1, n+1, 2n+1) \\ &\quad - (0, \theta^p, \theta^d, j, \sigma^p, \tau^p, \tau^d, n, 2n, \sigma^d, 1, n+1, n+j, 2n+1) \\ &\quad = (n, n+j) - (n, 2n) + (n+j, 2n_{-}(n+1, n+j) + (n+1, 2n+1) - (n+j, 2n+1). \end{aligned}$$

Let T_5 be the set of all vectors d_j, i.e. $T_5 = \{d_j : j = 2, \ldots, n-1\}$.

6. Leading edges $(\mathbf{n}+\mathbf{j}, \mathbf{n}+\mathbf{k}), \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}-\mathbf{2}, \mathbf{k} = \mathbf{j}+\mathbf{1}, \ldots, \mathbf{n}-\mathbf{1}$:
We first note that if $n = 2$ this case does not exist. We construct the vectors e_{jk} as linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be $(n+j, n+k)$.

6a. $\mathbf{j}, \mathbf{k} \notin \mathbf{S^p}$

In this case $n+j, n+k \notin S$. We define e_{jk} as follows:

$$\begin{aligned} e_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, \sigma^d, j, k, n+j, n+k, 2n, 2n+1) \\ &\quad - (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, n, \sigma^d, j, k, n+j, 2n, n+k, 2n+1) \\ &\quad = (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

The leading edge of e_{jk} is $(n+j, n+k)$.

6b. $\mathbf{j} \in \mathbf{S^p}, \mathbf{k} \notin \mathbf{S^p}$

In this case we have $n+k \notin S$. We note in Case 6 $k = j+1, \ldots, n-1$, therefore $k < k$. However, from Assumption 5.12 we have $j > k$ unless $j = 1$. Therefore the only value that j can take is $j = 1$.

We define e_{jk} as follows:

$$\begin{aligned} e_{jk} &= (0, \theta^p, \theta^d, k, \sigma^p, \tau^p, \tau^d, j, n, n+j, n+k, \sigma^d, 2n, 2n+1) \\ &- (0, \theta^p, \theta^d, k, \sigma^p, \tau^p, \tau^d, j, n, n+j, 2n, \overleftarrow{\sigma}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1) \end{aligned}$$

Since $k < n$, $(n+j, n+k) \prec (n+j, 2n)$. Therefore, the leading edge of e_{jk} is $(n+j, n+k)$.

6c. $\mathbf{j} \notin \mathbf{S^p}, \mathbf{k} \in \mathbf{S^p}$

In this case $n+k \in S$ or $n+k \notin S$ and $k \neq 1$. We define e_{jk} as follows:

$$\begin{aligned} e_{jk} &= (0, \theta^p, \theta^d, j, \sigma^p, \tau^p, \tau^d, n, k, 1, n+1, n+k, n+j, \sigma^d, 2n+1) \\ &- (0, \theta^p, \theta^d, j, \sigma^p, \tau^p, \tau^d, n, k, 1, n+1, n+k, 2n, \overleftarrow{\sigma}^d, n+j, 2n+1) \\ &= (n+j, n+k) - (n+k, 2n) - (n+j, 2n+1) + (2n, 2n+1) \end{aligned}$$

The leading edge of e_{jk} is $(n+j, n+k)$.

6d. $\mathbf{j}, \mathbf{k} \in \mathbf{S^p}$

We not that $n+j$ and $n+k$ are "free" with respect to S; they can belong or not to the set S.

- **$\mathbf{j} = \mathbf{1}$**, which implies that $n+j \in S$.
  - $n+k \in S$

    We define e_{jk} as follows:

    $$\begin{aligned} e_{1k} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, 1, n+1, n+k, n, 2n, \sigma^d, 2n+1) \\ &- (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, 1, n+1, n, n+k, 2n, \sigma^d, 2n+1) \\ &= (n+1, n+k) + (n, 2n) - (n+k, 2n) - (n, n+1). \end{aligned} \quad (26)$$

    We note that according to the order defined on the set of edges, $(n+1, n+k) \prec (n, 2n)$ and $(n+1, n+k) \prec (n, n+1)$. Therefore, the leading edge of e_{1k} is $(n+j, n+k)$.
  - $n+k \notin S$

    We define e_{jk} as follows:

    $$\begin{aligned} e_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, j, n, n+j, n+k, \sigma^d, 2n, 2n+1) \\ &- (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, j, n, n+j, 2n, \overleftarrow{\sigma}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1). \end{aligned} \quad (27)$$

    The leading edge of e_{jk} is $(n+j, n+k)$.
- **$\mathbf{j} \neq \mathbf{1}$**

  We consider several cases:

- $n+j, n+k \notin S$ or $n+j \in S, n+k \notin S$. We define e_{jk} as follows:

$$\begin{aligned} e_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, 1, n, n+1, n+j, n+k, \sigma^d, 2n, 2n+1) \\ &- (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, 1, n, n+1, n+j, 2n, \overleftarrow{\sigma}^d, n+k, 2n+1) \\ &= (n+j, n+k) - (n+j, 2n) - (n+k, 2n+1) + (2n, 2n+1). \end{aligned}$$

  The leading edge of e_{jk} is $(n+j, n+k)$.
- $n+j \notin S, n+k \in S$. We define e_{jk} as follows:

$$\begin{aligned} e_{jk} &= (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, 1, n, n+1, n+k, n+j, \sigma^d, 2n, 2n+1) \\ &- (0, \theta^p, \theta^d, \sigma^p, \tau^p, \tau^d, k, 1, n, n+1, n+k, 2n, \overleftarrow{\sigma}^d, n+j, 2n+1) \\ &= (n+j, n+k) - (n+k, 2n) - (n+j, 2n+1) + (2n, 2n+1). \end{aligned}$$

  The leading edge of e_{jk} is $(n+j, n+k)$.
- $n+j, n+k \in S$
  We first define the vectors v_{jk}:

$$\begin{aligned} v_{jk} =&(0, \theta^p, \theta^d, j, k, n, \sigma^p, \tau^p, \tau^d, n+j, n+k, 2n, \sigma^d, 1, n+1, 2n+1) \\ &-(0, \theta^p, \theta^d, j, k, n, \sigma^p, \tau^p, \tau^d, n+j, 2n, \sigma^d, 1, n+1, n+k, 2n+1) \\ &=(n+j, n+k) + (n+k, 2n) + (n+1, 2n+1) - (n+j, 2n) - (n+1, n+k) \\ &-(n+k, 2n+1). \end{aligned}$$

  We see that we can obtain a vector that has the leading edge $(n+j, n+k)$ if we use v_{jk} and a feasible tour or a linear combination of tours so that the coefficient of $(n+1, n+k)$ becomes zero. We can do this by adding v_{jk} and e_{1k} given by (26). Indeed, in that case $k \in S$ and $n+k \in S$, which is our case here too.

$$\begin{aligned} e_{jk} &= v_{jk} + (n+1, n+k) + (n, 2n) - (n+k, 2n) - (n, n+1) \\ &= (n+j, n+k) - (n+j, 2n) + (n+1, 2n+1) - (n+k, 2n+1) + (n, 2n) - (n, n+1). \end{aligned}$$

  According to the order defined on the set of edges, we have $(n+j, n+k) \prec (n, 2n)$ and $(n+j, n+k) \prec (n, n+1)$. Therefore, the leading edge of e_{jk} is $(n+j, n+k)$.

We now define T_6 as the set of all vectors e_{jk}, i.e. $T_6 = \{e_{jk} : j = 1, \ldots, n-1, k = 2, \ldots, n-1\}$.

7. Leading edges $\mathbf{n+j, 2n), j = 1, \ldots, n-2}$:
We first note that if $n = 2$ this case does not exist. We now construct the vectors f_j as linear combinations of feasible tours that satisfy (11) at equality. Their leading edges will be $(n+j, 2n)$.

7a. $\mathbf{j \in S^p}$

- $\mathbf{j = 1}$, which implies that $n + j \in S$.
  We define f_j as follows:

$$
\begin{aligned}
f_j &= (0, \theta^p, \theta^d, n-1, n, \sigma^p, \tau^p, \tau^d, j, \sigma^d, 2n, n+j, 2n-1, 2n+1) \\
&\quad - (0, \theta^p, \theta^d, n-1, n, \sigma^p, \tau^p, \tau^d, j, \sigma^d, 2n, 2n-1, n+j, 2n+1) \\
&\quad = (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1)
\end{aligned} \tag{28}
$$

- $\mathbf{j \neq 1}$
  We define f_j as follows:

$$
\begin{aligned}
f_j &= (0, \theta^p, \theta^d, n-1, n, \sigma^p, \tau^p, \tau^d, j, 1, \sigma^d, 2n, n+j, n+1, 2n-1, 2n+1) \\
&\quad - (0, \theta^p, \theta^d, n-1, n, \sigma^p, \tau^p, \tau^d, j, 1, \sigma^d, 2n, 2n-1, n+1, n+j, 2n+1) \\
&\quad = (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1)
\end{aligned}
$$

  The leading edge of f_j is $(n+j, 2n)$.

7b. $\mathbf{j \notin S^p}$

Since $j \notin S^p$, we know that $j \neq 1$. We now consider two situations:

- $\mathbf{2n - 1 \notin S}$
  We define f_j as follows:

$$
\begin{aligned}
f_j &= (0, \theta^p, \theta^d, j, n-1, \sigma^p, \tau^p, \tau^d, 1, n, n+1, \sigma^d, 2n, n+j, 2n-1, 2n+1) \\
&\quad - (0, \theta^p, \theta^d, j, n-1, \sigma^p, \tau^p, \tau^d, 1, n, n+1, \sigma^d, 2n, 2n-1, n+j, 2n+1) \\
&\quad = (n+j, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+j, 2n+1)
\end{aligned}
$$

  The leading element of f_j corresponds to the edge $(n+j, 2n)$.

- $\mathbf{2n - 1 \in S}$
  Implicitely $n - 1 \in S$. We define the vectors z_j as follows:

$$
\begin{aligned}
z_j &= (0, \theta^p, \theta^d, j, n-1, \sigma^p, \tau^p, \tau^d, 2n-1, 1, 2n, n+j, \sigma^d, 2n, n+1, 2n+1) \\
&\quad - (0, \theta^p, \theta^d, j, n-1, \sigma^p, \tau^p, \tau^d, 2n-1, 1, 2n, n+1, \overleftarrow{\sigma}^d, n+j, 2n+1) \\
&\quad = (n+j, 2n) + (n+1, 2n+1) - (n+1, 2n) - (n+j, 2n+1)
\end{aligned}
$$

  We see that we can obtain a vector that has the leading edge $(n+j, 2n)$ if we use z_j and a feasible tour or a linear combination of feasible tours so that the coefficient of $(n+1, 2n)$ becomes zero. We can do this by adding z_j and f_1 given by (28). We define f_j as follows:

$$f_j = z_j + (n+1, 2n) + (2n-1, 2n+1) - (2n-1, 2n) - (n+1, 2n+1)$$
$$= (n+j, 2n) - (n+j, 2n+1) + (2n-1, 2n+1) - (2n-1, 2n).$$

The leading edge of f_j is $(n+j, 2n)$.

We define T_7 as the set of all vectors f_j, i.e. $T7 = \{f_j : j = 1, \ldots, n-2\}$. The vectors in $T = T_0 \cup T_1 \cup \cdots \cup T_7$ are the vectors needed. ■

6 Conclusions and future work

In this paper we have conducted a polyhedral analysis of the pickup and delivery travelling salesman problem. We have determined the size of the PDTSP polytope, we reviewed valid inequalities existent in the literature and identified which ones are not facets of the PDTSP polytope. We proposed new valid inequalities and we proved that several classes of inequalities, both new and already mentioned in the literature, define facets of the PDTSP polytope. In future work we intend to use the polyhedral results obtained in this study to design a branch-and-cut algorithm for the PDTSP.

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