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Valid Inequalities and Branch-and-Cut for the Clique Pricing Problem

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Abstract. Motivated by an application in highway pricing, we consider the problem that consists in setting profit-maximizing tolls on a clique subset of a multicommodity transportation network. Following a proof that clique pricing is NP-hard, we propose strong valid inequalities, some of which define facets of the 2-commodity polyhedron. The numerical efficiency of these inequalities is assessed by embedding them within a branch-and-cut framework.

Keywords. Network pricing, mixed-integer programming, combinatorial optimization, clique.

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1 Introduction

The paradigm of pricing, either for improving the performance of infrastructures, or for maximizing the revenue of a private firm, pervades the economics literature. In the present paper, we consider the problem faced by a highway manager that seeks to maximize the revenue raised from tolls set on a network, while anticipating that users will travel on paths that maximize their individual utilities. This situation is closely related to the problem known as 'product line pricing' (see Green and Krieger [11], Dobson and Kalish [9, 10]), which was proved to be challenging from both the theoretical and computational points of view. Some years ago, Labbé et al. [19] recognized that the network pricing problem fits the framework of bilevel programming, a branch of optimization concerned with the solution of nonconvex programs involving two noncooperative agents, and that is akin to a leader-follower, or Stackelberg, game. This approach led to studies that focused on the combinatorial nature of network pricing, either in its original formulation or variants thereof. Representative of this approach are the works of Bouhtou et al. [1], van Hoesel et al. [23], Grigoriev et al. [12], Heilporn et al. [15], Kohli and Krishnamurti [17] and Roch et al. [20].

In the present paper, we consider a variant of the problem where all roads controlled by an authority are connected and form a path, as occurs in toll highways. Assuming that tolls are levied with respect to all possible combinations of entry and exit points on the highway, one may focus on networks where a virtual arc is created for each entry-exit combination, and thus form an 'inner' clique. Shortest paths that do not go through the highway are represented by arcs linking the various origins and destinations, and form an 'outer' clique (see Figure 1). The aim of this paper is to provide a better understanding of the Clique Pricing Problem and to develop algorithmic tools that can be transposed to situations arising in the field of revenue management (see Côté et al. [5]). More precisely, we are interested in the polyhedral structure of a specific Network Pricing Problem. Note that preliminary results were obtained by Heilporn et al. [16], who provided a theoretical study of the single commodity Clique Pricing Problem.

The structure of the paper is as follows: Section 2 introduces the Clique Pricing Problem, together with its formulation; Section 3 deals with strong valid inequalities derived from the underlying network structure of the model; Section 4 provides proofs that the inequalities, as well as several constraints of the initial model, define facets of the two-commodity problem; finally, numerical results (Section 5) show that several of the valid inequalities are efficient, in the sense that their integration within a branch-and-cut scheme decreases the integrality gap, CPU time and number of nodes explored in the resulting implicit enumeration process.

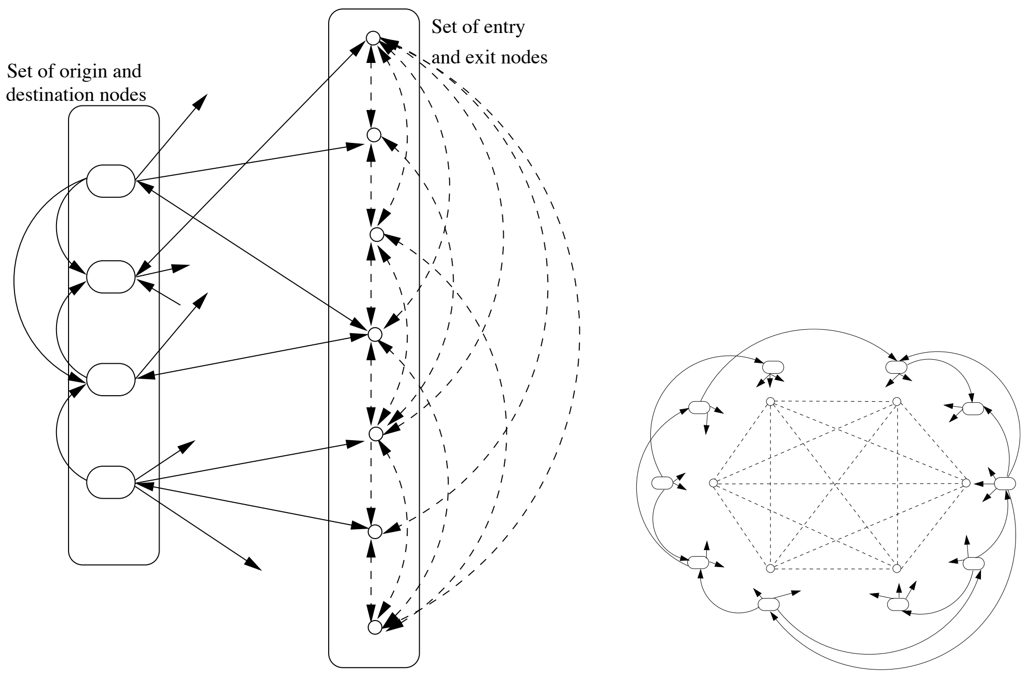

Figure 1: Topology of the Clique Pricing Problem, where toll arcs are dashed and toll-free arcs are solid. The highway network (left) is represented by two cliques (right). Nodes of the inner clique are the entry-exit nodes on the highway, while nodes of the outer clique represent various origins and destinations.

2 Mathematical formulation of the Clique Pricing Problem

Let us consider a linear highway composed of n entry-exit nodes, over which may transit m commodities, each of them associated with an origin-destination pair $k \in \mathcal{K}$ and a demand η^k. To each entry-exit pair correspond an arc $a \in \mathcal{A}$ and a commodity-specific cost $c_a^k + t_a$, where t_a is a toll. Commodities can either transit through the toll network, at cost $c_a^k + t_a$, or use alternative direct paths at cost u^k. Assuming that all combinations of origin-destination and entry-exit nodes are present, the topology of the network is that of two cliques linked by 'access nodes' (see Figure 1). The 'outer' clique is related to demand, while the 'inner' clique is a respresentation of the linear highway network.

Following Dewez [7], the Clique Pricing Problem can be formulated as the bilevel program:

$$\mathcal{CPP}: \quad \max_{t,x} \sum_{k\in\mathcal{K}} \sum_{a\in\mathcal{A}} \eta^k t_a x_a^k \tag{1}$$

subject to:

$$t_a \geq 0 \qquad \forall a \in \mathcal{A} \tag{2}$$

$$(x,y) \in \arg\min_{\bar{x},\bar{y}} \sum_{k\in\mathcal{K}} \left(\sum_{a\in\mathcal{A}} (c_a^k + t_a)\bar{x}_a^k + u^k \bar{y}^k \right) \tag{3}$$

subject to:

$$\sum_{a\in\mathcal{A}} x_a^k + y^k = 1 \qquad \forall k \in \mathcal{K} \tag{4}$$

$$x_a^k \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}. \tag{5}$$

At the upper level, the authority seeks to maximize the profits earned by imposing tolls t_a on the inner clique arcs. At the lower level, commodities are assigned to shortest paths with respect to the sum of fixed costs and tolls. The flow constraints (4) ensure that each commodity $k \in \mathcal{K}$ is assigned either to a toll path a of the inner clique ($x_a^k = 1$), or to a toll-free path of the outer clique ($y^k = 1$). Note that lower level solutions represent origin-destination paths carrying either no flow or the total origin-destination flow. Since the matrix of constraints is totally unimodular, flow proportions x_a^k and y^k can be assumed either discrete or continuous. It has been proved that the Clique Pricing Problem is $\mathcal{NP}$-hard (see Heilporn [14]), although particular cases are polynomially solvable (see Dewez [7]).

Recently, Heilporn [14] proposed a linear MIP (Mixed Integer Programming) formulation of the Clique Pricing Problem, that makes use of the 'revenue' variables p_a^k defined as

$$p_a^k = \begin{cases} t_a & \text{if commodity } k \text{ uses arc } a \in \mathcal{A}, \\ 0 & \text{otherwise} \end{cases}$$

and dispenses with the variables y^k:

$$\mathcal{CP}: \quad \max_p \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \eta^k p_a^k \tag{6}$$

subject to:

$$\sum_{a \in \mathcal{A}} x_a^k \leq 1 \qquad \forall k \in \mathcal{K} \tag{7}$$

$$\sum_{b \in \mathcal{A}} \left(p_b^k + c_b^k x_b^k \right) + u^k (1 - \sum_{b \in \mathcal{A}} x_b^k) \leq t_a + c_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{8}$$

$$p_a^k \leq M_a^k x_a^k \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{9}$$

$$t_a - p_a^k \leq N_a (1 - x_a^k) \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{10}$$

$$p_a^k \leq t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A} \tag{11}$$

$$p_a^k \geq 0 \qquad \forall a \in \mathcal{A} \tag{12}$$

$$x_a^k \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \tag{13}$$

where M_a^k and N_a denote 'big-M' constants that can be set to $M_a^k = \max\{0, u^k - c_a^k\}$ and $N_a = \max_{k \in \mathcal{K}} M_a^k$ for all $k \in \mathcal{K}, a \in \mathcal{A}$.

Let one identify $\mathcal{A}$ with a set of products, $\mathcal{K}$ with a set of purchaser segments, and $u^k - c_a^k \stackrel{\text{def}}{=} r_a^k$ with 'reservation prices' that represent the maximal price that purchaser k is willing to spend for product a. Then, if the 'utility' of purchaser k towards product a is set to the difference between the reservation price r_a^k and the actual product price p_a^k, the Clique Pricing Problem can be cast within the framework of product pricing problems, which have been studied in the economics literature (see Green and Krieger [11], Dobson and Kalish [9, 10], Kohli et al. [17, 18] or Shioda et al. [22]). For one, Shioda et al. [22] proposed a MIP formulation which coincides with $\mathcal{CP}$, modulo the substitution of the Shortest Path constraints (8) by

$$\sum_{b \in \mathcal{A}: b \neq a} \left((u^k - c_b^k) x_b^k - p_b^k \right) \geq (u^k - c_a^k) \sum_{b \in \mathcal{A}: b \neq a} x_b^k - t_a \qquad \forall k \in \mathcal{K}, \forall a \in \mathcal{A}, \tag{14}$$

which, after adding the term $(u^k - c_a^k) x_a^k - p_a^k$ to both sides of the inequality, can be rewritten as

$$\sum_{b\in\mathcal{A}}(p_b^k + c_b^k x_b^k) \quad \leq \quad t_a + c_a^k \sum_{b\in\mathcal{A}} x_b^k + p_a^k. \tag{15}$$

If one expresses constraints (8) as

$$\sum_{b\in\mathcal{A}}(p_b^k + c_b^k x_b^k) \quad \leq \quad t_a + c_a^k - u^k(1 - \sum_{b\in\mathcal{A}} x_b^k), \tag{16}$$

it can be readily verified that the right-hand-side of (16) is smaller than that of (15), whenever c_a^k is less than u^k. Note that, in a pre-processing step, one could have set $x_a^k = 0$ for all $a \in \mathcal{A}$ and $k \in \mathcal{K}$ such that $c_a^k > u^k$, since such an arc does not contribute positively to the objective function. Hence one may conclude that the Shortest Path constraints are stronger than (14). In particular, if a consumer $k \in \mathcal{K}$ refrains from buying, constraints (14) are redundant for this k, while constraints (8) impose $u^k \leq c_a^k + t_a$ for all toll arcs $a \in \mathcal{A}$. In a related paper, Shioda et al. [21] described the purchaser's behavior by probabilistic choice models. From the latter, they derived several mixed integer programming programs that they compare in terms of optimal solutions and computational times.

Shioda et al. [22] also introduced three sets of valid inequalities, the first set corresponding to optimality cuts and the next two to feasibility cuts:

$$p_a^{k_1} \geq \min_{k\in\mathcal{K}}\{u^k - c_a^k\} x_a^{k_1} \qquad \forall k_1 \in \mathcal{K}, \forall a \in \mathcal{A} \tag{17}$$

$$p_a^{k_1} \leq (u^{k_2} - c_a^{k_2}) x_a^{k_2} + (u^{k_1} - c_a^{k_1})(1 - x_a^{k_2}) \qquad \forall k_1, k_2 \in \mathcal{K}, \forall a \in \mathcal{A} \tag{18}$$

$$\begin{aligned} x_a^{k_2} \geq x_a^{k_1} \qquad & \forall k_1, k_2 \in \mathcal{K}, \forall a \in \mathcal{A} \text{ that satisfy the conditions :} \\ & u^{k_2} - c_a^{k_2} \geq u^{k_1} - c_a^{k_1} \ \forall a \in \mathcal{A}, \\ & c_a^{k_2} - c_a^{k_1} > c_b^{k_2} - c_b^{k_1} \ \forall b \in \mathcal{A} \setminus \{a\}. \end{aligned} \tag{19}$$

Inequalities (17) and (18) provide lower and upper bounds on the product price variables p_a^k, which depend on the reservation prices $u^k - c_a^k$. Inequalities (19) link the flow variables x_a^k associated with purchaser segments. We refer the reader to Shioda et al. [22] for further details concerning these inequalities.

3 Valid Inequalities

Inequalities (8), which ensure that only shortest paths are allowed to carry positive flow, can be strengthened by considering interrelationships between pairs of commodities.

PROPOSITION 1 *(SSP inequalities) For any subset $\mathcal{S}$ of $\mathcal{A}$ and for any $a \in \mathcal{A}$, the inequalities*

$$\sum_{b\in\mathcal{A}}\big(p_b^{k_1} + c_b^{k_1}x_b^{k_1}\big) + u^{k_1}(1-\sum_{b\in\mathcal{A}} x_b^{k_1}) \le t_a + c_a^{k_1} + \sum_{b\in\mathcal{A}\setminus(\mathcal{S}\cup\{a\})}\big(p_b^{k_2} + (c_b^{k_1} - c_a^{k_1})x_b^{k_2}\big) \tag{20}$$

$$\sum_{b\in\mathcal{A}}\big(p_b^{k_1} + c_b^{k_1}x_b^{k_1}\big) + u^{k_1}(1-\sum_{b\in\mathcal{A}} x_b^{k_1}) \le u^{k_1} + \sum_{b\in\mathcal{A}\setminus\mathcal{S}}\big(p_b^{k_2} + (c_b^{k_1} - u^{k_1})x_b^{k_2}\big) \tag{21}$$

are valid for $\mathcal{CP}$.

Proof

If $x_b^{k_1} = 0$ for all $b \in \mathcal{A}$, then:

(i) If there exists $b \in \mathcal{A}\setminus(\mathcal{S}\cup\{a\})$ such that $x_b^{k_2} = 1$, (20)–(21) yield $u^{k_1} \le t_a + p_b^{k_2} + c_b^{k_1}$ for all $a \in \mathcal{A}$ and $u^{k_1} \le p_b^{k_2} + c_b^{k_1}$, respectively. As $p_b^{k_2} = t_b$ by (10) and (11), the inequalities imply that the cost of the path containing $b \in \mathcal{A}$ must be larger than the cost of the toll free path for commodity k_1, and are valid by (8) and (12).

(ii) In all other cases, (20)–(21) yield $u^{k_1} \le t_a + c_a^{k_1}$ for all $a \in \mathcal{A}$ and $u^{k_1} \le u^{k_1}$, respectively, which are valid by (8).

Now assume that there exists $b \in \mathcal{A}$ such that $x_b^{k_1} = 1$.

(i) If there exists $d \in \mathcal{A}\setminus(\mathcal{S}\cup\{a\})$ such that $x_d^{k_2} = 1$, (20)–(21) yield $p_b^{k_1} + c_b^{k_1} \le t_a + p_d^{k_2} + c_d^{k_1}$ for all $a \in \mathcal{A}$ and $p_b^{k_1} + c_b^{k_1} \le p_d^{k_2} + c_d^{k_1}$ respectively. As $p_b^{k_1} = t_b$ and $p_d^{k_2} = t_d$ by (10) and (11), the inequalities state that the path containing $b \in \mathcal{A}$ must be cheaper than the path containing $d \in \mathcal{A}$ for commodity k_1, and are valid by (8) and (12).

(ii) In all other cases (i.e., if there does not exist any $d \in \mathcal{A}\setminus(\mathcal{S}\cup\{a\})$ such that $x_d^{k_2} = 1$), (20)–(21) become $p_b^{k_1} + c_b^{k_1} \le t_a + c_a^{k_1}$ for all $a \in \mathcal{A}$ and $p_b^{k_1} + c_b^{k_1} \le u^{k_1}$ respectively. Thus the path containing $b \in \mathcal{A}$ must be cheaper than any other path for commodity k_1, and the validity of the inequalities follows from (8). □

Note that, in the above theorem, the only relevant constraints are those that satisfy $c_b^{k_1} \le c_a^{k_1}$ for all $b \in \mathcal{A}\setminus(\mathcal{S}\cup\{a\})$, since the remaining ones are weaker than the Shortest Path constraints.

Although the possible number of subsets $\mathcal{S}$, and hence the number of constraints (20)–(21), is exponential, it is yet possible to determine the most violated constraint in polynomial time. To outline the separation procedure for a given commodity k_1, we observe that, for a distinct commodity k_2 and an arc a, the right-hand-side of (20) will be minimal if we insert into the set $\mathcal{A}\setminus(\mathcal{S}\cup\{a\})$ all arcs for which $p_b^{k_2} + (c_b^{k_1} - c_a^{k_1})x_b^{k_2}$ is negative, i.e., $(p_b^{k_2} + c_b^{k_1}x_b^{k_2})/x_b^{k_2} < c_a^{k_1}$. If the scalars $(p_b^{k_2} + c_b^{k_1} x_b^{k_2})/x_b^{k_2}$ and the costs $c_a^{k_1}$ are both sorted in increasing order (the latter operation can be performed off-line), then it becomes straightforward to update the optimal set $\mathcal{S}$ when switching from one candidate toll arc to its successor in the ordered list. The complexity of the separation procedure for a commodity $k_1 \in \mathcal{K}$ is thus dominated by the sort operation, and is in the order of $O(|\mathcal{K}||\mathcal{A}|\log|\mathcal{A}|)$.

The Profit Upper Bound inequalities (9) can also be strengthened by considering pairs of commodities. In this context, we say that two toll arcs a and b are *compatible* with respect to commodities k_1 and k_2 if there exists a feasible solution of $\mathcal{CP}$ where $x_b^{k_1} = x_a^{k_2} = 1$. In this case, we write $(b, k_1) \sim (a, k_2)$, and $(b, k_1) \not\sim (a, k_2)$ otherwise.

LEMMA 1 *Let* $c_b^{k_1} \leq u^{k_1}$, $c_b^{k_2} \leq u^{k_2}$, $c_a^{k_2} \leq u^{k_2}$ *and* $c_a^{k_1} \leq u^{k_1}$. *Then,* $(b, k_1) \sim (a, k_2)$ *if and only if* $c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}$.

Proof If $x_b^{k_1} = x_a^{k_2} = 1$, we must have that $t_b + c_b^{k_1} \leq t_a + c_a^{k_1}$ and $t_a + c_a^{k_2} \leq t_b + c_b^{k_2}$ by the Shortest Path constraints (8). This yields $c_b^{k_1} - c_a^{k_1} \leq t_a - t_b \leq c_b^{k_2} - c_a^{k_2}$.

Conversely, if $c_b^{k_2} - c_a^{k_2} \geq 0$, setting $x_b^{k_1} = x_a^{k_2} = 1$, $t_b = p_b^{k_1} = 0$, $t_a = p_a^{k_2} = c_b^{k_2} - c_a^{k_2}$ and $t_d = N_d$ for all $d \in \mathcal{A} \setminus \{a, b\}$ yields a feasible solution of $\mathcal{CP}$. Indeed, the Shortest Path constraints (8) imply that

$$\begin{aligned} p_b^{k_1} + c_b^{k_1} \leq t_a + c_a^{k_1} \qquad &\iff \quad c_b^{k_1} \leq c_b^{k_2} - c_a^{k_2} + c_a^{k_1} \\ p_a^{k_2} + c_a^{k_2} \leq t_b + c_b^{k_2} \qquad &\iff \quad c_b^{k_2} - c_a^{k_2} + c_a^{k_2} \leq c_b^{k_2}, \end{aligned}$$

which are valid since $c_a^{k_2} - c_a^{k_1} \leq c_b^{k_2} - c_b^{k_1}$. The remaining Shortest Path constraints hold since variables t_d have been set sufficiently large for all $d \in \mathcal{A} \setminus \{a, b\}$. Further, $c_b^{k_2} \leq u^{k_2}$ ensures that $p_a^{k_2} \leq M_a^{k_2}$, i.e., constraints (9) are satisfied. In the same way, if $c_b^{k_2} - c_a^{k_2} < 0$, the point $x_b^{k_1} = x_a^{k_2} = 1$, $t_a = 0$, $t_b = c_a^{k_2} - c_b^{k_2}$ and $t_d = N_d$ for all $d \in \mathcal{A} \setminus \{a, b\}$ is a feasible solution of $\mathcal{CP}$. □

LEMMA 2 *If* $M_a^{k_2} \geq M_a^{k_1}$ *and* $x_a^{k_1} = 1$, *there exists* $b \in \mathcal{A}$ *such that* $x_b^{k_2} = 1$ *and* $t_b + c_b^{k_2} \leq t_a + c_a^{k_2}$.

Proof Since $x_a^{k_1} = 1$, one has $t_a = p_a^{k_1} \leq M_a^{k_1}$ by (9), (10) and (11). Hence $t_a \leq M_a^{k_2}$, i.e., the path containing toll arc $a \in \mathcal{A}$ is cheaper than the toll free path for commodity k_2. □

To derive the next inequalities, we introduce the set $\mathcal{A}_a^{>} = \{b \in \mathcal{A} : c_b^{k_2} - c_b^{k_1} > c_a^{k_2} - c_a^{k_1}\}$ of toll arcs $b \in \mathcal{A}$ such that $(b, k_1) \sim (a, k_2)$ and $(b, k_2) \not\sim (a, k_1)$, together with its complement $\mathcal{A}_a^{<}$ (for the sake of readibility, we adopted this notation rather than $\mathcal{A}_a^{\leq}$).

PROPOSITION 2 *(SPUB inequalities) If, for a given triple* (k_1, k_2, a) *such that* $M_a^{k_1} \leq M_a^{k_2}$, *there* *exist no toll arc b that satisfies $c_a^{k_2} - c_a^{k_1} = c_b^{k_2} - c_b^{k_1}$, then the following inequalities are valid*

$$p_a^{k_2} \leq M_a^{k_2} x_a^{k_2} + \Big(M_a^{k_2} - M_a^{k_1}\Big) \big(\sum_{b \in \mathcal{A}_a^< \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1} \big) \tag{22}$$

$$p_a^{k_2} \leq M_a^{k_2} x_a^{k_2} + \Big(M_a^{k_2} - M_a^{k_1}\Big) \big(\sum_{b \in \mathcal{A}_a^< \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1} \big) + \big(M_{b^*}^{k_2} - M_{b^*}^{k_1}\big) \sum_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}), \tag{23}$$

with $b^* = \arg\min_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} \{c_b^{k_1} - c_b^{k_2}\}$.

Note that, for given $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$, inequality (22) (resp. (23)) is not redundant if and only if $x_b^{k_1} = 1 = x_a^{k_2}$ for $b = a$ (resp. $b = a$ or $b \in \mathcal{A}_a^>$), and helps to restrain the upper bound on $p_a^{k_2}$ in this case.

Proof If $\sum_{b \in \mathcal{A}_a^< \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1}$ is non negative, then the corresponding inequality (22) is redundant by (9). Similarly, if $\sum_{b \in \mathcal{A}_a^< \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1}$ and $\sum_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1})$ are non negative, then (23) is redundant.

Assume that $\sum_{b \in \mathcal{A}_a^< \setminus \{a\}} (x_b^{k_2} - x_b^{k_1}) - x_a^{k_1} < 0$, i.e., (i) there exists $b \in \mathcal{A}_a^<$ such that $x_b^{k_1} = 1$, and (ii) $x_b^{k_2} = 0$ for all $b \in \mathcal{A}_a^< \setminus \{a\}$. By Lemma 2, (i) implies that there exists $d \in \mathcal{A}$ such that $x_d^{k_2} = 1$. Further, Lemma 1 and the definition of the set $\mathcal{A}_a^<$ yield $d \in \mathcal{A}_a^<$, thus $d = a$ by (ii), i.e., $x_a^{k_2} = 1$. Now, from Lemma 1 and the assumption that there does not exist any $b \in \mathcal{A}$ such that $c_b^{k_2} - c_b^{k_1} = c_a^{k_2} - c_a^{k_1}$, one obtains that $b = a$, i.e., $x_a^{k_1} = 1$. Hence inequalities (22)–(23) become $p_a^{k_2} \leq M_a^{k_1}$, whose validity follows from (9), (10) and (11).

Concerning inequality (23), it can occur that $\sum_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}) < 0$, which means that (i) there exists $b \in \mathcal{A}_a^>$ with $M_b^{k_2} \geq M_b^{k_1}$ such that $x_b^{k_1} = 1$ and (ii) $x_b^{k_2} = 0$ for all $b \in \mathcal{A}_a^>$ with $M_b^{k_2} \geq M_b^{k_1}$. In this situation, Lemma 2 implies that there must exist $d \in \mathcal{A}$ such that $x_d^{k_2} = 1$. By contradiction, assume that $d \in \mathcal{A}_a^>$ with $M_d^{k_2} < M_d^{k_1}$. Lemma 1 implies that $c_d^{k_2} - c_d^{k_1} \leq c_b^{k_2} - c_b^{k_1}$, which cannot occur since $u^{k_2} - u^{k_1} < c_d^{k_2} - c_d^{k_1}$ and $c_b^{k_2} - c_b^{k_1} \leq u^{k_2} - u^{k_1}$. As (ii) also holds, one concludes that $d \in \mathcal{A}_a^<$.

If $d \in \mathcal{A}_a^< \setminus \{a\}$, inequality (23) becomes $0 \leq (M_a^{k_2} - M_a^{k_1}) - (M_{b^*}^{k_2} - M_{b^*}^{k_1})$, which is true since $b \in \mathcal{A}_a^>$. Otherwise $d = a$, i.e., $x_a^{k_2} = 1$, and (23) yields $p_a^{k_2} \leq M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_a^{k_2}$, which is also valid. Indeed, constraint (8) imposes that $p_a^{k_2} + c_a^{k_2} \leq t_b + c_b^{k_2}$. Further, $p_b^{k_1} \leq M_b^{k_1}$ by (9). As $t_b = p_b^{k_1}$ by (10) and (11), one has $p_a^{k_2} \leq M_b^{k_1} + c_b^{k_2} - c_a^{k_2}$. The result follows from the definition of b^*. □

Proposition 3 *(SPUB inequalities) Under the assumptions of Theorem 2 the following in-*

equalities are valid:

$$p_a^{k_2} - p_a^{k_1} \leq M_a^{k_2} \sum_{b \in \mathcal{A}_a^<} (x_b^{k_2} - x_b^{k_1}) \tag{24}$$

$$p_a^{k_2} - p_a^{k_1} \leq M_a^{k_2} \sum_{b \in \mathcal{A}_a^<} (x_b^{k_2} - x_b^{k_1}) + \left(M_{b^*}^{k_2} - M_{b^*}^{k_1}\right) \sum_{b \in \mathcal{A}_a^> : M_b^{k_2} \geq M_b^{k_1}} (x_b^{k_2} - x_b^{k_1}). \tag{25}$$

Note that, for given $k_1, k_2 \in \mathcal{K}$ and $a \in \mathcal{A}$, (24)–(25) are not redundant either if $x_a^{k_1} = x_a^{k_2} = 1$ or if $x_b^{k_1} = 1 = x_a^{k_2}$ for $b \in \mathcal{A}_a^>$ such that $M_b^{k_2} \geq M_b^{k_1}$. Since the proof is similar to that of Theorem 2, it will be omitted.

4 Assessing the valid inequalities

The valid inequalities introduced in Section 3 are strong. Indeed, we will show that they define facets of the convex hull $\mathcal{P}$ of feasible solutions to a two-commodity Clique Pricing Problem, defined as

$$\mathcal{P} = \text{conv}\left\{(\mathbf{t}; \mathbf{p}^{k_1}; \mathbf{p}^{k_2}; \mathbf{x}^{k_1}; \mathbf{x}^{k_2}) \in \mathbb{R}_+^n \times \mathbb{R}_+^{2n} \times \{0,1\}^{2n} : (7) - (13)\right\},$$

where boldface letters denote real vectors.

Our results are dependent on the choice of big-M constants in the MIP formulation of clique pricing. While we let $M_a^k = \max\{0, u^k - c_a^k\}$ as before, we set $N_a = \max_{k \in \mathcal{K}}\{M_a^k\} + \epsilon$, for some arbitrarily small number ϵ. This latter choice, which differs from that in Section 2, is motivated by the fact that some flexibility with respect to the toll variables t_a is required whenever we encounter the degenerate situation where some toll arc carries no flow. For sufficiently small ϵ, this strategy leaves the set of optimal solutions unchanged.

In the sequel, $\mathbf{e}_a$ will denote a unit vector in the direction a, and the following technical assumptions will hold.

Assumption 1 *$M_a^k > 0$ for every $k \in \mathcal{K}$ and $a \in \mathcal{A}$.*

Assumption 2 *For all $b \in \mathcal{A}$, either $M_b^{k_1} \neq M_b^{k_2}$, or there exists $d \in \mathcal{A} \backslash \{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$.*

Both assumptions ensure that the convex hull is not contained in some hyperplane, either $p_a^k = 0$ in the former case, or

$$\sum_{b \in \mathcal{A}} (p_b^{k_1} + c_b^{k_1} x_b^{k_1}) + u^{k_1}(1 - \sum_{b \in \mathcal{A}} x_b^{k_1}) + K = \sum_{b \in \mathcal{A}} (p_b^{k_2} + c_b^{k_2} x_b^{k_2}) + u^{k_2}(1 - \sum_{b \in \mathcal{A}} x_b^{k_2})$$

in the latter, i.e, the cost structure is identical for commodities k_1 and k_2, and it becomes a single-commodity problem.

It can be proved that the polyhedron $\mathcal{P}$ is full dimensional and that most of the constraints in our formulation are tight, i.e., they induce facets of the convex hull of feasible solutions, under mild conditions. For the sake of completeness, these results, whose proofs can be found in Heilporn [14], are listed below.

PROPOSITION 4 *The polyhedron $\mathcal{P}$ has full dimension, i.e., $Dim(\mathcal{P}) = 5n$.*

PROPOSITION 5 *The constraint $\sum_{b\in\mathcal{A}} x_b^{k_2} \leq 1$ defines a facet of $\mathcal{P}$ if and only if, for each $b \in \mathcal{A}$ such that $M_b^{k_1} > M_b^{k_2}$, there exists $d \in \mathcal{A} \setminus \{b\}$ such that $M_d^{k_1} \leq M_d^{k_2}$ and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$.*

PROPOSITION 6 *The constraint $p_{\tilde{a}}^{k_2} \leq M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2}$ defines a facet of $\mathcal{P}$ if and only if either $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$ or there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(\tilde{a}, k_1) \sim (b, k_2)$.*

PROPOSITION 7 *The constraint $t_{\tilde{a}} - p_{\tilde{a}}^{k_2} \leq N_{\tilde{a}}(1 - x_{\tilde{a}}^{k_2})$ defines a facet of $\mathcal{P}$ if and only if either $M_{\tilde{a}}^{k_1} < M_{\tilde{a}}^{k_2}$ or there exists $b \in \mathcal{A} \setminus \{\tilde{a}\}$ such that $(b, k_1) \sim (\tilde{a}, k_2)$.*

PROPOSITION 8 *Constraints (11) are never facet defining for $\mathcal{P}$.*

PROPOSITION 9 *The constraint $p_{\tilde{a}}^{k_2} \geq 0$ defines a facet of $\mathcal{P}$ if and only if one of the following conditions holds:*

(i) $M_{\tilde{a}}^{k_2} < M_{\tilde{a}}^{k_1}$;

(ii) $M_{\tilde{a}}^{k_2} > M_{\tilde{a}}^{k_1}$ *and there exists* $b \in \mathcal{A} \setminus \{\tilde{a}\}$ *such that* $(\tilde{a}, k_1) \sim (b, k_2)$;

(iii) $M_{\tilde{a}}^{k_2} = M_{\tilde{a}}^{k_1}$ *and either there exists* $b \in \mathcal{A} \setminus \{\tilde{a}\}$ *such that* $(\tilde{a}, k_1) \sim (b, k_2)$*, or there exists* $b \in \mathcal{A} \setminus \{\tilde{a}\}$*,* $v \in \mathbb{R}$ *such that* $(b, k_1) \sim (\tilde{a}, k_2)$*,* $0 \leq v \leq M_b^{k_1}$ *and* $c_{\tilde{a}}^{k_2} - c_b^{k_2} \leq v \leq c_{\tilde{a}}^{k_1} - c_b^{k_1}$.

One can also show that most inequalities introduced in Section 3 define facets of $\mathcal{P}$. To prove such results, let $(\mathbf{t}; \mathbf{p}^{k_1}; \mathbf{p}^{k_2}; \mathbf{x}^{k_1}; \mathbf{x}^{k_2}) \in \mathcal{P}$ and $\mathcal{H}$ the hyperplane induced by a given inequality. We need to show that $\mathcal{H}$ is the sole hyperplane that contains $\mathcal{P} \cap \mathcal{H}$

Let $\mathcal{H} = \left\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \mu\mathbf{t} + \nu^{k_1}\mathbf{p}^{k_1} + \nu^{k_2}\mathbf{p}^{k_2} + \xi^{k_1}\mathbf{x}^{k_1} + \xi^{k_2}\mathbf{x}^{k_2} = 0\right\}$ and assume that all points of $\mathcal{P} \cap \mathcal{H}$ lie on a generic hyperplane $\mathcal{G} = \left\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : \alpha\mathbf{t} + \beta^{k_1}\mathbf{p}^{k_1} + \beta^{k_2}\mathbf{p}^{k_2} + \gamma^{k_1}\mathbf{x}^{k_1} + \gamma^{k_2}\mathbf{x}^{k_2} = \delta\right\}$. The following lemmas relate the coefficients of $\mathcal{H}$ and $\mathcal{G}$.

LEMMA 3 *Let $\mathcal{P} \cap \mathcal{H} \subseteq \mathcal{G}$. We have:*

(i) *If $\mu = 0$, then $\alpha = 0$ and $\delta = 0$.*

(ii) *If (i) holds and there exists* $b \in \mathcal{A}$ *such that* $\xi_b^{k_1} = -\xi_b^{k_2}$*, then* $\gamma_b^{k_1} = -\gamma_b^{k_2}$*.*

(iii) *If (i), (ii) hold and* b *is such that* $\nu_b^{k_1} = -\nu_b^{k_2}$*, then* $\beta_b^{k_1} = -\beta_b^{k_2}$*.*

Proof If $\mu = 0$, then $\left(\sum_{a\in\mathcal{A}} N_a\mathbf{e}_a; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}\right)$ and $\left(\sum_{a\in\mathcal{A}} N_a\mathbf{e}_a - \epsilon\mathbf{e}_b; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}\right)$ belong to $\mathcal{P}\cap\mathcal{H}$ for all $b \in \mathcal{A}$. As these points also belong to the generic hyperplane $\mathcal{G}$, it follows that

$$\begin{aligned} \sum_{a\in\mathcal{A}} N_a\alpha_a &= \delta \\ \sum_{a\in\mathcal{A}} N_a\alpha_a - \epsilon\alpha_b &= \delta, \end{aligned}$$

thus $\alpha = 0$ and $\delta = 0$. Further, if there exists $b \in \mathcal{A}$ with $\xi_b^{k_1} = -\xi_b^{k_2}$, the point $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a; \mathbf{0}; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_b\right)$ belongs to $\mathcal{P}\cap\mathcal{H}$, and one obtains $\gamma_b^{k_1} + \gamma_b^{k_2} = 0$.

Next, if $\nu_b^{k_1} = -\nu_b^{k_2}$, then $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \epsilon\mathbf{e}_b; \mathbf{e}_b; \mathbf{e}_b\right)$ also belongs to $\mathcal{P}\cap\mathcal{H}$, which yields $\beta_b^{k_1} = -\beta_b^{k_2}$. □

Lemma 4 *Let* $\mathcal{G}$ *be a hyperplane containing* $\mathcal{P}\cap\mathcal{H}$ *and* $M_b^{k_1} < M_b^{k_2}$ *for some* $b \in \mathcal{A}$*. If* $\mu = 0$ *and* $\nu_b^{k_2} = 0 = \xi_b^{k_2}$*, then* $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$*.*

Proof The points

$$\begin{aligned} &\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + M_b^{k_1}\mathbf{e}_b; \mathbf{0}; M_b^{k_1}\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\Big) \\ &\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + (M_b^{k_1}+\epsilon)\mathbf{e}_b; \mathbf{0}; (M_b^{k_1}+\epsilon)\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\Big) \end{aligned}$$

belong to $\mathcal{P}\cap\mathcal{H}$. By Lemma 3, one obtains $\alpha = 0$ and $\delta = 0$, hence

$$\begin{aligned} &M_b^{k_1}\beta_b^{k_2} + \gamma_b^{k_2} = 0 \\ &(M_b^{k_1}+\epsilon)\beta_b^{k_2} + \gamma_b^{k_2} = 0, \end{aligned}$$

and $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. □

Lemma 5 *Let* $\mathcal{G}$ *be a hyperplane containing* $\mathcal{P}\cap\mathcal{H}$*, and* $b, d \in \mathcal{A}$ *be such that* $c_d^{k_2} - c_d^{k_1} \le c_b^{k_2} - c_b^{k_1}$ *and* $M_d^{k_1} \le M_d^{k_2}$ *(resp.* $M_d^{k_1} \ge M_d^{k_2}$*).*

(i) *If* $\mu = 0$*,* $\xi_b^{k_1} = -\xi_d^{k_2}$ *and* $\nu_b^{k_1} = \nu_d^{k_2} = 0$*, then* $\beta_b^{k_1} = -\beta_d^{k_2}$ *and* $(M_b^{k_1} - M_d^{k_1})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$ *(resp.* $(M_b^{k_2} - M_d^{k_2})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$*).*

(ii) Further, if $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$*, then* $\beta_b^{k_1} = 0 = \beta_d^{k_2}$ *and* $\gamma_b^{k_1} = -\gamma_d^{k_2}$.

Proof If $M_d^{k_1} \leq M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_1}\mathbf{e_b} + M_d^{k_1}\mathbf{e}_d; M_b^{k_1}\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_1}-\epsilon)\mathbf{e}_b + (M_d^{k_1}-\epsilon)\mathbf{e}_d; (M_b^{k_1}-\epsilon)\mathbf{e}_b; (M_d^{k_1}-\epsilon)\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

belong to $\mathcal{P}\cap\mathcal{H}$. If $M_d^{k_1} \geq M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e_b} + M_d^{k_2}\mathbf{e}_d; M_b^{k_2}\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_2}-\epsilon)\mathbf{e}_b + (M_d^{k_2}-\epsilon)\mathbf{e}_d; (M_b^{k_2}-\epsilon)\mathbf{e}_b; (M_d^{k_2}-\epsilon)\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

belong to $\mathcal{P}\cap\mathcal{H}$. If $M_d^{k_1} \leq M_d^{k_2}$ (the case $M_d^{k_1} \geq M_d^{k_2}$ is similar), Lemma 3 implies that $\alpha = 0$ and $\delta = 0$, and one obtains

$$M_b^{k_1}\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$$
$$(M_b^{k_1}-\epsilon)\beta_b^{k_1} + (M_d^{k_1}-\epsilon)\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = -\beta_d^{k_2}$ and $(M_b^{k_1} - M_d^{k_1})\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$. Further, if $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$, the point

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_1}-\epsilon)\mathbf{e}_b + M_d^{k_1}\mathbf{e}_d; (M_b^{k_1}-\epsilon)\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

or

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_2}+\epsilon)\mathbf{e}_b + M_d^{k_2}\mathbf{e}_d; (M_b^{k_2}+\epsilon)\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

is in $\mathcal{P}\cap\mathcal{H}$ (for $M_d^{k_1} \leq M_d^{k_2}$ or $M_d^{k_1} > M_d^{k_2}$ respectively). If we assume $M_d^{k_1} \leq M_d^{k_2}$ (the case $M_d^{k_1} \geq M_d^{k_2}$ is similar), one obtains

$$(M_b^{k_1} - M_d^{k_1} - \epsilon)\beta_b^{k_1} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = 0 = \beta_d^{k_2}$ and $\gamma_b^{k_1} = -\gamma_d^{k_2}$. □

Lemma 6 *Let $\mathcal{G}$ be a hyperplane containing $\mathcal{P} \cap \mathcal{H}$ and let $b \in \mathcal{A}$ be such that $M_b^{k_1} \leq M_b^{k_2}$. If $\mu = 0$ and $\xi_b^{k_2} = -M_b^{k_2}\nu_b^{k_2}$, then $\gamma_b^{k_2} = -M_b^{k_2}\beta_b^{k_2}$.*

Proof The points $\left(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e}_b; \mathbf{0}; M_b^{k_2}\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b\right)$ belong to $\mathcal{P} \cap \mathcal{H}$. As $\mu = 0$, we have that $\alpha = 0$ and $\delta = 0$ by Lemma 3. It follows that $\gamma_b^{k_2} = -M_b^{k_2}\beta_b^{k_2}$. □

Lemma 7 *Let $\mathcal{G}$ be a hyperplane containing $\mathcal{P} \cap \mathcal{H}$ and let $b \in \mathcal{A}$ such that $M_b^{k_1} > M_b^{k_2}$. If the coefficients of $\mathcal{H}$ are such that $\mu = 0$, $\xi_b^{k_1} = -M_b^{k_2}\nu_b^{k_1}$ and $\xi_b^{k_2} = -M_b^{k_2}\nu_b^{k_2}$, then $\gamma_b^{k_1} = -M_b^{k_2}\beta_b^{k_1}$ and $\gamma_b^{k_2} = -M_b^{k_2}\beta_b^{k_2}$.*

Proof As previously, we have that $\alpha = 0$ and $\delta = 0$ by Lemma 3. Since the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e}_b; M_b^{k_2}\mathbf{e}_b; \mathbf{0}; \mathbf{e}_b; \mathbf{0}\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e}_b; M_b^{k_2}\mathbf{e}_b; M_b^{k_2}\mathbf{e}_b; \mathbf{e}_b; \mathbf{e}_b\Big),$$

belong to $\mathcal{P} \cap \mathcal{H}$, one obtains

$$M_b^{k_2}\beta_b^{k_1} + \gamma_b^{k_1} = 0$$
$$M_b^{k_2}\beta_b^{k_1} + M_b^{k_2}\beta_b^{k_2} + \gamma_b^{k_1} + \gamma_b^{k_2} = 0.$$

The result follows. □

Based on the previous lemmas, we are in position to prove that most SSP and SPUB inequalities presented in Section 3 define facets of $\mathcal{P}$.

Proposition 10 *If $M_b^{k_1} < M_b^{k_2}$ for all $b \in \mathcal{S} \subseteq \mathcal{A}$, then the SSP inequalities (21) define facets of $\mathcal{P}$.*

Proof First note that inequality (21) can be rewritten as

$$\sum_{b\in\mathcal{A}} \left(p_b^{k_1} - M_b^{k_1}x_b^{k_1}\right) - \sum_{b\in\mathcal{A}\setminus\mathcal{S}} \left(p_b^{k_2} - M_b^{k_1}x_b^{k_2}\right) \leq 0.$$

Let $\mathcal{H} = \Big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : \sum_{b\in\mathcal{A}} \left(p_b^{k_1} - M_b^{k_1}x_b^{k_1}\right) - \sum_{b\in\mathcal{A}\setminus\mathcal{S}} \left(p_b^{k_2} - M_b^{k_1}x_b^{k_2}\right) = 0\Big\}$. We have that $\alpha = 0$ and $\delta = 0$ by Lemma 3. Further, for any $b \in \mathcal{S}$, and provided that $M_b^{k_1} < M_b^{k_2}$, one obtains $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$ by Lemma 4. For all $b \in \mathcal{A} \setminus \mathcal{S}$, Lemma 3 yields $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$.

Next, for all $b \in \mathcal{A}$ such that $M_b^{k_1} \geq M_b^{k_2}$ (resp. $M_b^{k_1} < M_b^{k_2}$), switching the commodity indices k_1 and k_2 in Lemma 6 (resp. Lemma 7) yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. Now, if there exist

$d \in \mathcal{A} \setminus \mathcal{S}, b \in \mathcal{A}$ such that $(b, k_1) \sim (d, k_2)$ and $c_b^{k_1} < c_d^{k_1}$, the points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b,d\}} N_a \mathbf{e}_a + (c_d^{k_1} - c_b^{k_1}) \mathbf{e}_b; (c_d^{k_1} - c_b^{k_1}) \mathbf{e}_b; \mathbf{0}; \mathbf{e}_b; \mathbf{e}_d \Big)$$

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b,d\}} N_a \mathbf{e}_a + (c_d^{k_1} - c_b^{k_1} + \epsilon) \mathbf{e}_b + \epsilon \mathbf{e}_d; (c_d^{k_1} - c_b^{k_1} + \epsilon) \mathbf{e}_b; \epsilon \mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d \Big)$$

also belong to $\mathcal{P} \cap \mathcal{H}$. This yields $\beta_b^{k_1} = -\beta_d^{k_2}$. On the other hand, if $c_b^{k_1} \geq c_d^{k_1}$, the points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b,d\}} N_a \mathbf{e}_a + (c_b^{k_1} - c_d^{k_1}) \mathbf{e}_d; \mathbf{0}; (c_b^{k_1} - c_d^{k_1}) \mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d \Big)$$

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b,d\}} N_a \mathbf{e}_a + \epsilon \mathbf{e_b} + (c_b^{k_1} - c_d^{k_1} + \epsilon) \mathbf{e}_d; \epsilon \mathbf{e}_b; (c_b^{k_1} - c_d^{k_1} + \epsilon) \mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_b \Big)$$

belong to $\mathcal{P} \cap \mathcal{H}$. Hence one obtains $\beta_b^{k_1} = -\beta_d^{k_2}$ for all $b \in \mathcal{A}$ and $d \in \mathcal{A} \setminus \mathcal{S}$. □

The SSP inequalities (20) can also define facets of $\mathcal{P}$. However, since this only occurs under restrictive conditions, it will not be mentioned any further in this paper. Next, we turn our attention to the SPUB inequalities, which also define facets of $\mathcal{P}$.

Proposition 11 *Under the assumption that, for all $b \in \mathcal{A}_a^{>}$ such that $M_b^{k_1} = M_b^{k_2}$, there exists $d \in \mathcal{A}_a^{>}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$, then the SPUB inequalities (22) define facets of $\mathcal{P}$.*

Proof Let $\tilde{a} \in \mathcal{A}$ and

$$\mathcal{H} = \Big\{ (\mathbf{t}; \mathbf{p}; \mathbf{x}) : p_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_2} x_{\tilde{a}}^{k_2} - (M_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1}) \big(\sum_{b \in \mathcal{A}_{\tilde{a}}^{<} \setminus \{\tilde{a}\}} (x_b^{k_2} - x_b^{k_1}) - x_{\tilde{a}}^{k_1} \big) = 0 \Big\}.$$

By Lemma 3, $\alpha = 0$ and $\delta = 0$, and $\beta_b^{k_1} = -\beta_b^{k_2}$, $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$. Further, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 4 yields $\beta_b^{k_2} = 0 = \gamma_b^{k_2}$. If $M_b^{k_2} < M_b^{k_1}$, switching the commodity indices k_1 and k_2 in Lemma 4 yields $\beta_b^{k_1} = 0 = \gamma_b^{k_1}$. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ and $\gamma_b^{k_1} = \gamma_b^{k_2} = 0$ for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} \neq M_b^{k_2}$.

For all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} = M_b^{k_2}$, there exists $d \in \mathcal{A} \setminus \{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$ by Assumption 2. Then, provided there exists such a toll arc d in $\mathcal{A}_{\tilde{a}}^{>}$, one obtains $\beta_b^{k_1} = \beta_b^{k_2} = 0$ by Lemma 5, thus also $\gamma_b^{k_2} = 0 = \gamma_b^{k_1}$ by Lemma 6.

Next, for all $b \in \mathcal{A}_{\tilde{a}}^{<} \setminus \{\tilde{a}\}$, recall that, by assumption, one has $c_b^{k_2} - c_b^{k_1} \neq c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1}$. Hence, setting $b = \tilde{a}$ and $d = b$ in Lemma 5 yields $\beta_{\tilde{a}}^{k_1} = 0 = \beta_b^{k_2} = \beta_b^{k_1}$ and $\gamma_{\tilde{a}}^{k_1} = -\gamma_b^{k_2}$.

Finally, setting $b = \tilde{a}$ in Lemma 6 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$. As the point

$$\Big(\sum_{a\in\mathcal{A}\setminus\{\tilde{a}\}} N_a\mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}\Big)$$

also belongs to $\mathcal{P}\cap\mathcal{H}$, one obtains $\gamma_{\tilde{a}}^{k_1} = \big(M_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1}\big)\beta_{\tilde{a}}^{k_2}$ and the result follows. □

Note that the conditions imposed in the previous proposition imply that either $M_b^{k_1} \neq M_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ or there exist at least two toll arcs $b, d \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. The proof of the next result is deferred to the appendix.

Proposition 12 *If, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_b^{k_1} > M_b^{k_2}$), there exists $d \in \mathcal{A}_{\tilde{a}}^{\geq}\setminus\{b\}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$, then the SPUB inequalities (23) define facets of $\mathcal{P}$.*

Next, we address the case of the SPUB inequalities (24) and (25).

Proposition 13 *If the following conditions hold:*

(i) for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} = M_b^{k_2}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\geq}\setminus\{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;

(ii) for all $b \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$;

(iii) there exists $b \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_1} - c_b^{k_1} \leq v \leq c_{\tilde{a}}^{k_2} - c_b^{k_2}$ and $0 \leq v \leq M_b^{k_2}$,

then the SPUB inequalities (24) define facets of $\mathcal{P}$.

Proof Let $\mathcal{H} = \Big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} - M_{\tilde{a}}^{k_2}\sum_{b\in\mathcal{A}_{\tilde{a}}^{\leq}}(x_b^{k_2} - x_b^{k_1}) = 0\Big\}$. Lemma 3 yields $\alpha = 0$, $\delta = 0$, $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$.

For any $b \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$, the assumptions ensure that there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$ with $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. One can check that $M_d^{k_1} \leq M_d^{k_2}$ for all $d \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$, which implies, by Lemma 5, that $\beta_b^{k_1} = \beta_d^{k_2} = 0$ and $\gamma_b^{k_1} = -\gamma_d^{k_2}$ for all $b, d \in \mathcal{A}_{\tilde{a}}^{\leq}\setminus\{\tilde{a}\}$.

Now, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} < M_b^{k_2}$, Lemma 4 yields $\beta_b^{k_2} = \gamma_b^{k_2} = 0$. If $M_b^{k_2} < M_b^{k_1}$, one obtains $\beta_b^{k_1} = \gamma_b^{k_1} = 0$ by switching the commodity indices k_1 and k_2 in Lemma 4. Hence $\beta_b^{k_1} = \beta_b^{k_2} = 0$ and $\gamma_b^{k_1} = \gamma_b^{k_2} = 0$ for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} \neq M_b^{k_2}$.

On the other hand, for all $b \in \mathcal{A}_{\tilde{a}}^{\geq}$ such that $M_b^{k_1} = M_b^{k_2}$, there exists $d \in \mathcal{A}\setminus\{b\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$ by Assumption 2. Then, provided there exists such a toll arc d in $\mathcal{A}_{\tilde{a}}^{\geq}$, one obtains $\beta_b^{k_1} = \beta_b^{k_2} = 0$ by Lemma 5 and $\gamma_b^{k_2} = \gamma_b^{k_1} = 0$ by Lemma 6.

Next, provided there exists $b \in \mathcal{A}^{\leq}_{\tilde{a}} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c^{k_1}_{\tilde{a}} - c^{k_1}_{b} \leq v \leq c^{k_2}_{\tilde{a}} - c^{k_2}_{b}$ and $0 \leq v \leq M^{k_2}_b$, the point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b,\tilde{a}\}} N_a \mathbf{e}_a + v\mathbf{e}_b; \mathbf{0}; v\mathbf{e}_b; \mathbf{e}_{\tilde{a}}; \mathbf{e}_b\Big)$$

belongs to $\mathcal{P} \cap \mathcal{H}$. One can verify that the existence of $v \in \mathbb{R}$ is required since $x^{k_1}_{\tilde{a}} = x^{k_2}_{b} = 1$ (with $b \in \mathcal{A}^{\leq}_{\tilde{a}} \setminus \{\tilde{a}\}$) implies that $p^{k_1}_{\tilde{a}} = 0$ for points in $\mathcal{H}$. Hence $\gamma^{k_2}_b = -\gamma^{k_1}_{\tilde{a}}$ for all $b \in \mathcal{A}^{\leq}_{\tilde{a}} \setminus \{\tilde{a}\}$.

Finally, setting $b = \tilde{a}$ in Lemma 6 yields $\gamma^{k_2}_{\tilde{a}} = -M^{k_2}_{\tilde{a}}\beta^{k_2}_{\tilde{a}}$. As $\gamma^{k_1}_b = -\gamma^{k_2}_b$ for all $b \in \mathcal{A}$, one obtains $\gamma^{k_2}_b = -M^{k_2}_{\tilde{a}}\beta^{k_2}_{\tilde{a}} = -\gamma^{k_1}_b$ for all $b \in \mathcal{A}$, and the result follows. □

The final result is stated without proof.

PROPOSITION 14 *If the following conditions hold:*

(i) for all $b \in \mathcal{A}^{\geq}_{\tilde{a}}$ such that $M^{k_1}_b \leq M^{k_2}_b$ (resp. $M^{k_1}_b > M^{k_2}_b$), there exists $d \in \mathcal{A}^{\geq}_{\tilde{a}} \setminus \{b\}$ such that $M^{k_1}_d \leq M^{k_2}_d$ (resp. $M^{k_1}_d > M^{k_2}_d$) and $c^{k_2}_b - c^{k_1}_b \neq c^{k_2}_d - c^{k_1}_d$;

(ii) for all $b \in \mathcal{A}^{\leq}_{\tilde{a}} \setminus \{\tilde{a}\}$, there exists $d \in \mathcal{A}^{\leq}_{\tilde{a}} \setminus \{\tilde{a}\}$ such that $c^{k_2}_b - c^{k_1}_b \neq c^{k_2}_d - c^{k_1}_d$;

(iii) there exists $b \in \mathcal{A}^{\leq}_{\tilde{a}} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c^{k_1}_{\tilde{a}} - c^{k_1}_{b} \leq v \leq c^{k_2}_{\tilde{a}} - c^{k_2}_{b}$ and $0 \leq v \leq M^{k_2}_b$,

then the SPUB inequalities (25) define facets of $\mathcal{P}$.

Note that the conditions imposed so that inequalities (24)–(25) define facets of $\mathcal{P}$ are similar to the ones imposed for inequalities (22)–(23). Since most inequalities considered define facets of the two-commodity case, we can expect that they provide deep cuts for the general case, and help in the numerical solution of the Clique Pricing Problem. This will be the topic of the remainder of this work.

5 Numerical results for a branch-and-cut algorithm

In this section, we show that the SSP and SPUB inequalities introduced previously are not only tight for 2-commodity problems, but are also efficient, from an algorithmic point of view, for both the Clique Pricing Problem and an interesting variant thereof.

5.1 Problem generation

In order to test our valid inequalities, we generated scenarios built around the topology of Highway 10 in Québec (Canada). These involve a complete network based on 5 to 10 cities and

10 to 15 entry-exit nodes on the highway, i.e., 20 to 90 commodities and 90 to 210 toll arcs. For each scenario, replicated 5 times, demands for city pairs were set to random values between 10 and 100. The set-up of the cost structure is as follows:

- Consecutive toll roads on the highway are assigned random fixed costs between 20 and 70;
- Arcs in opposite directions are assigned identical fixed costs;
- Fixed costs on shortest toll-free roads linking the highway with cities are set to random numbers between 15 and 120;
- Fixed costs are additive, i.e., the cost of a 'virtual' arc in the inner clique composed of a sequence of 'physical' roads is set to the the sum of the latter fixed costs;
- Fixed costs between cities are set to random numbers between 150 and 1000.
- Toll arcs $a \in \mathcal{A}$ such that $c_a^k > u^k$ are clearly irrelevant, and are removed from the network.

Table 1 provides the minimum, maximum, average and standard deviation for the network size, in terms of path number per commodity.

size	min	max	μ	σ
(5,10)	1	64	22,6	17,3
(5,12)	1	69	29,7	19,0
(5,15)	1	140	41,9	27,1
(8,10)	1	63	17,6	15,9
(8,12)	1	92	28,3	21,2

Table 1: Number of feasible paths per commodity. Each entry shows the number of cities and highway nodes, respectively.

5.2 Implementation

Computational experiments were performed under scenarios involving the valid inequalities introduced in this work. These inequalities were computed at the root node of the branch-and-cut tree, and appended to the model, whenever they were violated. At each iteration, violation tests were conducted, and the number of SSP inequalities appended was limited to half the number of commodities. Finally, we set a computational time upper bound of 5 hours (18000 seconds), after which the solution process was halted.

In order to assess the efficiency of the valid inequalities, the related number of nodes, CPU times and the gap (in percentage) between the linear relaxation optimal solution Z_{lp} and the true optimal solution Z_{opt}, defined as

$$\text{gap} = 100 \times \frac{Z_{lp} - Z_{opt}}{Z_{opt}},$$

are reported. Note that Z_{lp} is computed after appending the valid inequalities, at the root of the branch-and-cut algorithm.

The models were implemented under Mosel of Xpress-MP, Optimizer version 18. All automated preprocessing features were switched on, although Xpress' automated heuristics were switched off, being unable to handle our manual cuts. Numerical experiments were carried out on a Pentium 4.3 GHz processor equipped with 2Gb of RAM and running under Linux Kernel version 2.6.4.

5.3 Numerical results for the Clique Pricing Problem

In this section, we assess the influence of the valid inequalities when integrated within a branch-and-cut scheme. The results are displayed in Tables 2–5. For the instances that could be solved to optimality within the allotted CPU time, we observe a sharp decrease in both the gap and number of nodes, when the SSP inequalities were incorporated. For all but the smallest instances, we also observed a sharp drop in the CPU time. However, merging both classes of valid inequalities (i.e., SSP and SPUB) did not yield a significant improvement in any of the performance measures. The typical evolution of the objective's lower and upper bounds is illustrated in Figures 2 and 3.

size	gap		time (sec)		nodes	
	μ	σ	μ	σ	μ	σ
(5,10)	11.85	7.03	20	31	1893	3315
(5,12)	16.82	5.83	188	283	4067	4899
(5,15)	13.10	3.68	405	497	9797	12446
2(8,10)	15.98	6.98	3520	4934	159015	222837
2(8,12)	19.19	4.6	5272	7004	147793	200928

Table 2: The base model $\mathcal{CP}$. The symbols μ and σ denote the average and standard deviation with respect to the instances solved to optimality, while 'nodes' refers to the number of nodes in the branch-and-bound process. Leftmost numbers (between stars) provide the number of instances that could not be solved to proven optimality within the time limit.

Next, we compare the efficiency of our formulation with that of Shioda et al.'s $\mathcal{SHPP}$, under valid inequalities (17), (18) and (19). We base our tests on the randomly generated instances described in Section 5.1. Inequalities (17) and (18) are appended to the initial model, while inequalities (19) are generated at the root of the branch-and-cut tree but only appended to the model when violated. The results obtained are presented in Table 6. Comparing Table 6

size	gap		time (sec)		nodes	
	μ	σ	μ	σ	μ	σ
(5,10)	1.52	1.45	21	25	388	683
(5,12)	1.84	2.09	52	64	103	84
(5,15)	1.86	1.54	241	256	719	591
2 (8,10)	3.56	1.96	3038	4262	9722	13173
2 (8,12)	1.59	1.48	1313	1454	3974	5350

Table 3: Appending the SSP inequalities (20)–(21).

size	gap		time (sec)		nodes	
	μ	σ	μ	σ	μ	σ
(5,10)	10.75	7.05	40	64	3150	5935
(5,12)	15.00	4.82	222	282	12410	14762
(5,15)	12.09	3.77	1050	1809	23965	41766
1 (8,10)	17.01	7.47	3430	3452	93890	100592
3 (8,12)	13.86	1.37	1128	41	31276	8104

Table 4: Appending the SPUB inequalities (22)–(23) and (24)–(25).

size	gap		time (sec)		nodes	
	μ	σ	μ	σ	μ	σ
(5,10)	1.50	1.44	18	21	342	603
(5,12)	1.84	2.09	57	67	114	100
(5,15)	1.83	1.54	249	249	1112	1313
2 (8,10)	3.41	1.95	1959	2728	5845	7692
2 (8,12)	1.55	1.47	1582	1938	4011	5584

Table 5: Appending the SSP inequalities (20)–(21) and the SPUB inequalities (22)–(23) and (24)–(25).

to Tables 2–5, we observe that model $\mathcal{CP}$ with or even without the valid inequalities clearly outperforms Shioda et al.'s formulation. This confirms that constraints (14) of $\mathcal{SHPP}$ are weaker, both theoretically and numerically, than constraints (8).

It is worth noting that Shioda et al.'s study was concerned with a product pricing problem unrelated to a specific network. For the sake of fairness, we now present numerical results obtained on a set of product pricing problems, generated as described in [22]. Precisely, we consider a number of purchaser segments ('commodities' in the network model) and a number of products ('toll arcs'), randomly generated within the intervals [40,80] and [10,60], respectively.

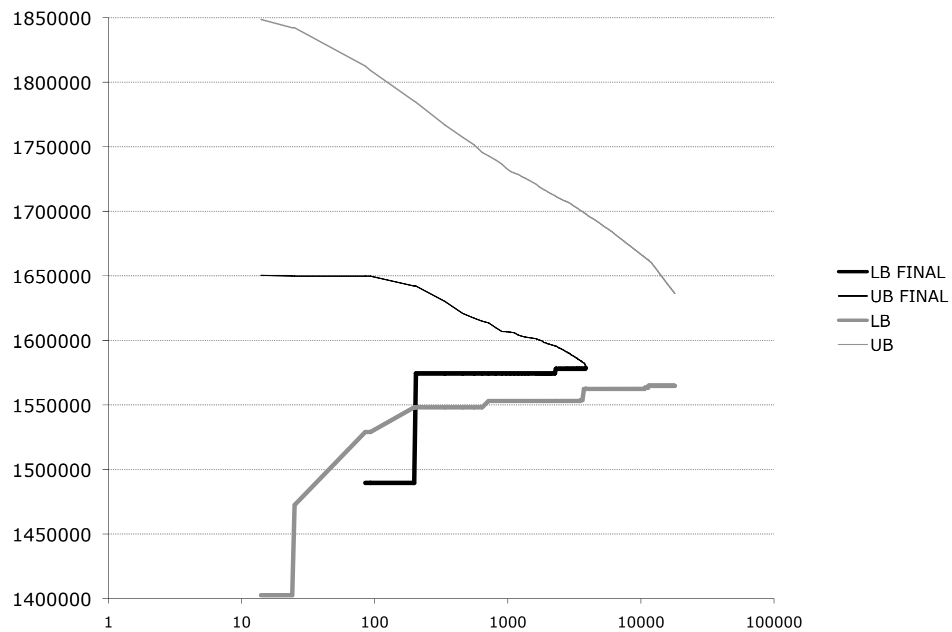

Figure 2: Evolution of the objective function with respect to the CPU time for an instance of class $(8, 10)$. The lower and upper bounds for the initial model $\mathcal{CP}$ are denoted 'LB' and 'UB', while the lower and upper bounds for model $\mathcal{CP}$ with both classes of valid inequalities are denoted 'LB Final' and 'UB Final', respectively.

size	gap		time (sec)		nodes	
	μ	σ	μ	σ	μ	σ
(5,10)	18.75	6.79	3134	5880	289522	553763
1 (5,12)	19.35	11.74	1796	2663	98679	146442
2 (5,15)	16.57	1.77	514	509	21763	17504
4 (8,10)	10.53	0	114.23	0	5662	0
5 (8,12)	*	*	*	*	*	*

Table 6: Model $\mathcal{SHPP}$ with inequalities (17), (18) and (19).

Each customer k is associated with a random demand $\eta^k \in [500, 799]$ and a reservation price $r_a^k \in [512, 1023]$ for product a, the latter corresponding to shortest toll-free arcs in the network model. The results are displayed in Tables 7 and 8, where the behaviour observed earlier gets even more pronounced.

5.4 A variant of the Clique Pricing Problem

In the original formulation of the Clique Pricing Problem, an optimal solution might be such that the sum of the tolls on consecutive arcs between two nodes i and j is less than the toll on

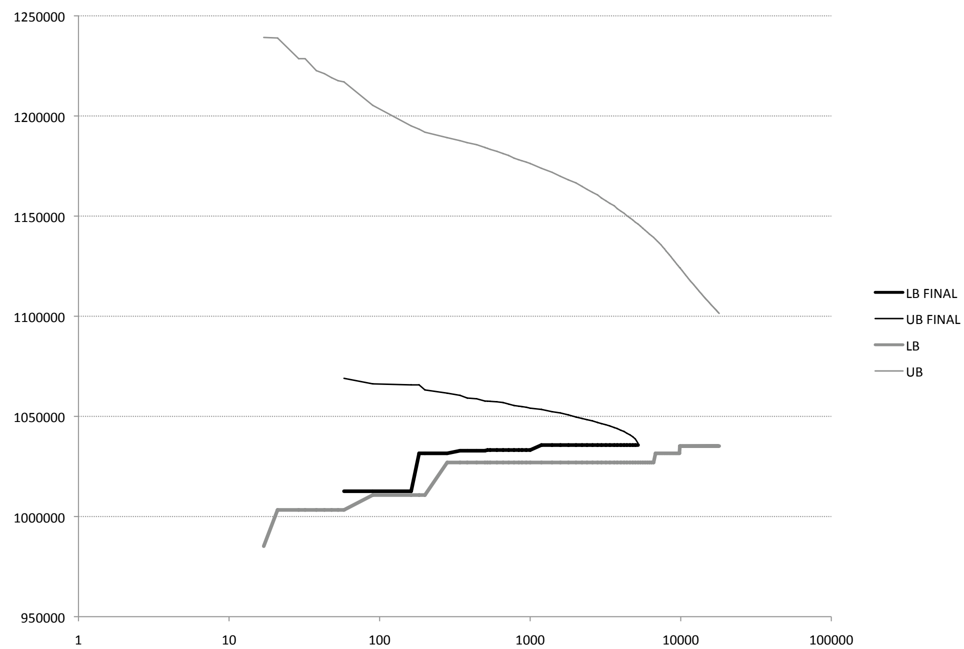

Figure 3: Evolution of the objective function with respect to the CPU time for an instance of class $(8,12)$.

	size	gap(%)		time (sec)		nodes	
		μ	σ	μ	σ	μ	σ
$*^{5}*$	(40,20)	*	*	*	*	*	*
$*^{1}*$	(40,40)	0.67	0.14	3657	3315	59420	46890
	(40,60)	0.33	0.07	658	744	3077	4183
$*^{5}*$	(60,20)	*	*	*	*	*	*
$*^{5}*$	(60,40)	*	*	*	*	*	*
$*^{5}*$	(60,60)	*	*	*	*	*	*

Table 7: Model $\mathcal{SHPP}$ with inequalities (17), (18) and (19), tested on Shioda et al. instances. Size is measured with respect to number of commodities (purchaser segments) and number of toll arcs (products). Each row corresponds to 5 instances of a given size.

the direct arc from i to j. Although this situation may occur in the airline industry, it seems reasonable to forbid it in the context of road networks, hence the introduction of inequalities that ensure that it is not beneficial to leave the highway upstream to reenter it downstream. These triangle inequalities are expressed as

$$t_a \leq t_b + t_c \quad \forall a, b, c \in \mathcal{A} : t(a) = t(b),\ h(b) = t(c),\ h(c) = h(a), \tag{26}$$

	size	gap(%)		time (sec)		nodes	
		μ	σ	μ	σ	μ	σ
	(40,20)	0.57	0.26	94	69	975	1419
	(40,40)	0.11	0.06	103	40	107	127
	(40,60)	0.04	0.02	185	48	18	15
1	(60,20)	0.52	0.34	3456	3449	17585	15701
1	(60,40)	0.16	0.1	539	295	2561	2557
	(60,60)	0.09	0.07	1687	1954	3811	6818

Table 8: Model $\mathcal{CP}$ with inequalities (20)–(21), (22)–(23) and (24)–(25), tested on Shioda et al. instances.

where $h(a)$ and $t(a)$ denote the head and tail of $a \in \mathcal{A}$, respectively. In a similar fashion, we introduce monotonicity inequalities that specify that the toll on a path cannot be less than the toll on any subpath, i.e., the inequality $t_a \geq t_b$ holds for any pair of arcs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that one of the following four conditions involving their indices holds:

$$\begin{aligned} &\text{(i)} \quad t(a) = t(b) < h(a) = h(b) + 1 \\ &\text{(ii)} \quad t(a) = t(b) - 1 < h(a) = h(b) \\ &\text{(iii)} \quad t(a) = t(b) > h(a) = h(b) - 1 \\ &\text{(iv)} \quad t(a) = t(b) + 1 > h(a) = h(b). \end{aligned}$$

The model involving triangle and monotonicity constraints is labelled $\mathcal{CP}^*$. Note that constants $N_a : a \in \mathcal{A}$ that appear in constraints (10) can now be set to $N_a = N = \max_{k,a}\{M_a^k\}$ for all $a \in \mathcal{A}$.

Triangle and monotonicity constraints are generated at every node of the branch-and-cut algorithm and appended to model $\mathcal{CP}^*$ when violated. We impose an upper bound on the number of constraints appended at a single iteration of the branch-and-cut algorithm. For a given commodity, this bound is set to half the maximal number of feasible paths for the triangle constraints, and to twice that number for the monotonicity constraints. Random instances were generated according to the rules set in Section 5.1. Based on monotonicity constraints, a variable x_a^k ($a \in \mathcal{A}$, $k \in \mathcal{K}$) was set to zero whenever there exists an arc b such that $c_b^k < c_a^k$ and b lies in-between $t(a)$ and $h(a)$. This allowed to significantly reduce the number of admissible paths. Table 9 provides the minimum, maximum, mean and standard deviation of the number of feasible paths per commodity.

The numerical results, displayed in Tables 10–12, again indicate that, while the SPUB con-

size	min	max	μ	σ
(5,10)	1	20	17,3	24,2
(5,12)	1	24	10,9	24,2
(5,15)	1	35	12,9	53,1
(8,10)	1	20	7,4	25,6
(8,12)	1	24	10,5	37,1
(8,15)	1	35	13,2	66,1

Table 9: Number of feasible paths per commodity.

straints slightly improve the gap, the number of nodes explored and the overall CPU time does not decrease significantly.

	size	gap(%)		time(sec)		nodes	
		μ	σ	μ	σ	μ	σ
	(5,10)	18.10	7.22	2	1	184	151
	(5,12)	20.13	6.03	18	25	1131	1692
	(5,15)	19.52	4.82	5	3	407	368
	(8,10)	30.09	8.42	262	425	27991	47056
	(8,12)	32.04	8.28	947	1118	263875	348671
4	(8,15)	36.94	0	3571	0	202361	0

Table 10: Model $\mathcal{CP}^*$

	size	gap(%)		time(sec)		nodes	
		μ	σ	μ	σ	μ	σ
	(5,10)	6.53	4.47	4	2	69	39
	(5,12)	3.98	3.06	10	9	127	140
	(5,15)	4.55	3.18	8	6	101	85
	(8,10)	12.64	5.05	682	1004	9205	15224
1	(8,12)	9.96	5.55	1099	1249	23473	23453
4	(8,15)	16.28	0	5796	0	102359	0

Table 11: Model $\mathcal{CP}^*$ with SSP inequalities (20)–(21).

We conclude that the best valid inequalities for this variant of the Clique Pricing Problem are still the SSP inequalities (20)–(21), as they provide a significant decrease of the gap and number of nodes in the branch-and-cut algorithm, at the expense of increasing the CPU time. To illustrate the results, Figures 4 and 5 depict the evolution of the lower and upper bounds on the objective function with respect to the CPU time for two specific instances.

	size	gap(%)		time(sec)		nodes	
		μ	σ	μ	σ	μ	σ
	(5,10)	6.47	4.53	6	4	88	70
	(5,12)	3.97	3.06	7	6	138	113
	(5,15)	4.46	3.14	11	7	106	75
	(8,10)	12.29	5.19	1195	1981	14625	24737
1	(8,12)	9.79	5.53	1713	2515	27777	38116
4	(8,15)	16.26	0	2918	0	71731	0

Table 12: Model $\mathcal{CP}^*$ with SSP and SPUB inequalities (20)–(21), (22)–(23) and (24)–(25).

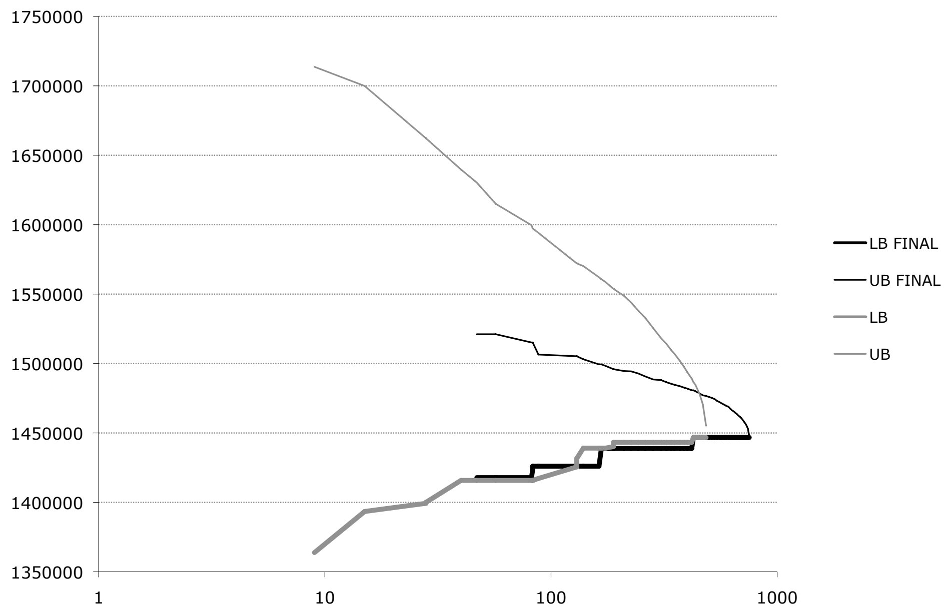

Figure 4: Evolution of the objective function with respect to the CPU time for an instance of class $(8, 10)$. The lower and upper bounds for the initial model $\mathcal{CP}^*$ are denoted 'LB' and 'UB', while the lower and upper bounds for model $\mathcal{CP}^*$ with inequalities (20)–(21) are denoted 'LB Final' and 'UB Final', respectively.

6 Conclusion

Together with its companion paper [16], the present work constitutes the first systematic study of the polyhedral properties of network pricing, or its equivalent in economics. In particular, we showed that a class of theoretically strong inequalities performed well numerically. As a follow-up, it would be interesting to investigate the polyhedral structure of a problem involving fixed costs on the toll arcs of the associated network design and pricing problem (see Brotcorne

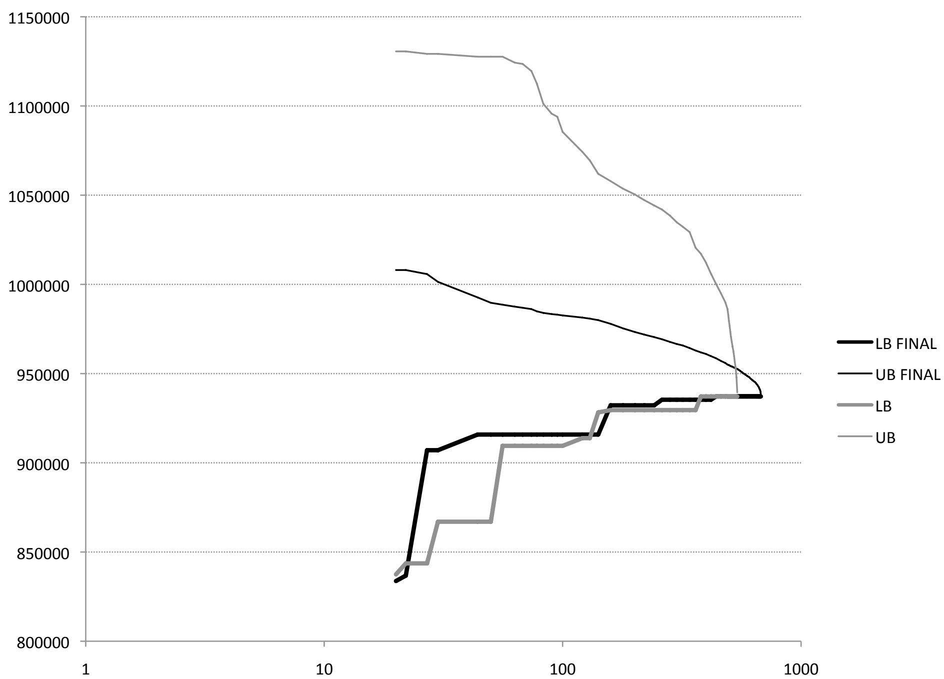

Figure 5: Evolution of the objective function with respect to the CPU time for an instance of class $(8, 12)$.

et al. [3]) akin to 'product line design' in the economics literature [9].

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Appendix. Proofs of Theorems 12 and 14

The proofs require the following lemma.

Lemma 8 *Assume that $\mathcal{P}\cap\mathcal{H}$ is a subset of a the generic hyperplane $\mathcal{G}$, and let $b, d \in \mathcal{A}$ such that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. If the coefficients of $\mathcal{H}$ are such that $\mu = 0$, $\xi_b^{k_1} = -\xi_d^{k_2}$ and $\nu_b^{k_1} = 0 = \nu_d^{k_2}$, then $\beta_b^{k_1} = 0$ and $\min\{M_d^{k_1}, M_d^{k_2}\}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$.*

Proof If $M_d^{k_1} \leq M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_1}\mathbf{e_b} + M_d^{k_1}\mathbf{e}_d; M_b^{k_1}\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_1}-\epsilon)\mathbf{e}_b + M_d^{k_1}\mathbf{e}_d; (M_b^{k_1}-\epsilon)\mathbf{e}_b; M_d^{k_1}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$. Otherwise, i.e., if $M_d^{k_1} > M_d^{k_2}$, the points

$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + M_b^{k_2}\mathbf{e}_b + M_d^{k_2}\mathbf{e}_d; M_b^{k_2}\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$
$$\Big(\sum_{a\in\mathcal{A}\setminus\{b,d\}} N_a\mathbf{e}_a + (M_b^{k_2}+\epsilon)\mathbf{e}_b + M_d^{k_2}\mathbf{e}_d; (M_b^{k_2}+\epsilon)\mathbf{e}_b; M_d^{k_2}\mathbf{e}_d; \mathbf{e}_b; \mathbf{e}_d\Big)$$

are in $\mathcal{P}\cap\mathcal{H}$.

First, one knows that $\mu = 0$ implies $\alpha = 0$ and $\delta = 0$ by Lemma 3. Then, if $M_d^{k_1} \leq M_d^{k_2}$, one obtains

$$M_b^{k_1}\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$$
$$(M_b^{k_1}-\epsilon)\beta_b^{k_1} + M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = 0$ and $M_d^{k_1}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$. Otherwise, i.e., if $M_d^{k_1} > M_d^{k_2}$, one obtains

$$M_b^{k_2}\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$$
$$(M_b^{k_2}+\epsilon)\beta_b^{k_1} + M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0,$$

thus $\beta_b^{k_1} = 0$ and $M_d^{k_2}\beta_d^{k_2} + \gamma_b^{k_1} + \gamma_d^{k_2} = 0$. □

Proof of Theorem 12

Let $\mathcal{H} = \big\{(\mathbf{t};\mathbf{p};\mathbf{x}) : p_{\tilde a}^{k_2} - M_{\tilde a}^{k_2}x_{\tilde a}^{k_2} - \big(M_{\tilde a}^{k_2} - M_{\tilde a}^{k_1}\big)\big(\sum_{b\in\mathcal{A}_{\tilde a}^{\leq}\setminus\{\tilde a\}}(x_b^{k_2} - x_b^{k_1}) - x_{\tilde a}^{k_1}\big)$
$-\big(M_{b^*}^{k_2} - M_{b^*}^{k_1}\big)\sum_{b\in\mathcal{A}_{\tilde a}^{\geq}:M_b^{k_2}\geq M_b^{k_1}}(x_b^{k_2} - x_b^{k_1}) = 0\big\}$.

First of all, Lemma 3 yields $\alpha = 0$, $\delta = 0$ and $\beta_b^{k_1} = -\beta_b^{k_2}$, $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A} \setminus \{\tilde{a}\}$. Further, for any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, setting $b = \tilde{a}$ and $d = b$ in Lemma 5 yields $\beta_b^{k_2} = -\beta_{\tilde{a}}^{k_1}$, thus also $\beta_b^{k_1} = -\beta_{\tilde{a}}^{k_1}$.

For all $d \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$), the proposition hypothesis ensures that there exists $b \in \mathcal{A}_{\tilde{a}}^{>} \setminus \{d\}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As Lemma 5 yields $\beta_b^{k_1} = -\beta_d^{k_2}$, it follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b, d \in \mathcal{A}_{\tilde{a}}^{>}$ by Lemma 8. Now, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} > M_b^{k_2}$, switching the commodity indices k_1 and k_2 in Lemma 6 yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. As $\beta_b^{k_1} = 0$, one obtains $\gamma_b^{k_1} = 0$, thus also $\gamma_b^{k_2} = 0$.

Next, for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, recall that there does not exist any $b \in \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} = c_{\tilde{a}}^{k_2} - c_{\tilde{a}}^{k_1}$ by hypothesis. Hence, setting $b = \tilde{a}$ and $d = b$ in Lemma 8 yields $\beta_{\tilde{a}}^{k_1} = 0$ and $M_b^{k_1}\beta_b^{k_2} + \gamma_{\tilde{a}}^{k_1} + \gamma_b^{k_2} = 0$. As $\beta_b^{k_2} = -\beta_{\tilde{a}}^{k_1} = -\beta_b^{k_1}$, it follows that $\beta_b^{k_2} = 0 = \beta_b^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$. One also obtains $\gamma_b^{k_2} = -\gamma_{\tilde{a}}^{k_1}$.
Next, setting $b = \tilde{a}$ in Lemma 6 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$. As the point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}\}} N_a \mathbf{e}_a + M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; M_{\tilde{a}}^{k_1}\mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}; \mathbf{e}_{\tilde{a}}\Big)$$

also belongs to $\mathcal{P} \cap \mathcal{H}$, it follows that $\gamma_{\tilde{a}}^{k_1} = \big(M_{\tilde{a}}^{k_2} - M_{\tilde{a}}^{k_1}\big)\beta_{\tilde{a}}^{k_2}$.

Finally, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} \leq M_b^{k_2}$, the points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}, b\}} N_a \mathbf{e}_a + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\mathbf{e}_{\tilde{a}} + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2})\mathbf{e}_b;$$
$$(M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2})\mathbf{e}_b; (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\mathbf{e}_{\tilde{a}}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}}\Big)$$

are in $\mathcal{P} \cap \mathcal{H}$ since $x_b^1 = 1 = x_{\tilde{a}}^{k_2}$ $(b \in \mathcal{A}_{\tilde{a}}^{>} : M_b^{k_1} \leq M_b^{k_2})$ implies $p_{\tilde{a}}^{k_2} = M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}$ for points of $\mathcal{H}$, which yields

$$(M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\beta_{\tilde{a}}^{k_2} + \gamma_b^{k_1} + \gamma_{\tilde{a}}^{k_2} = 0.$$

As $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$, one obtains $\gamma_b^{k_1} = \big(M_{b^*}^{k_2} - M_{b^*}^{k_1}\big)\beta_{\tilde{a}}^{k_2}$ and the result follows. □

Proof of Theorem 14

Let $\mathcal{H} = \big\{(\mathbf{t}; \mathbf{p}; \mathbf{x}) : p_{\tilde{a}}^{k_2} - p_{\tilde{a}}^{k_1} - M_{\tilde{a}}^{k_2}\sum_{b \in \mathcal{A}_{\tilde{a}}^{\leq}}(x_b^{k_2} - x_b^{k_1})$
$-\big(M_{b^*}^{k_2} - M_{b^*}^{k_1}\big)\sum_{b \in \mathcal{A}_{\tilde{a}}^{\geq} : M_b^{k_2} \geq M_b^{k_1}}(x_b^{k_2} - x_b^{k_1}) = 0\big\}$.

Lemma 3 yields $\alpha = 0$, $\delta = 0$, $\beta_b^{k_1} = -\beta_b^{k_2}$ and $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$. Next, for any $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, there exists $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$ such that $c_b^{k_2} - c_b^{k_1} \neq c_d^{k_2} - c_d^{k_1}$ by the proposition hypothesis. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As $M_d^{k_1} \leq M_d^{k_2}$ for all $d \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, Lemmas 5 and 8 yield $\beta_b^{k_1} = -\beta_d^{k_2}$ and $\beta_b^{k_1} = 0$ respectively. Hence $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$.

For all $d \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_d^{k_1} \leq M_d^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$), the proposition hypothesis ensures that there exists $b \in \mathcal{A}_{\tilde{a}}^{>} \setminus \{d\}$ such that $M_b^{k_1} \leq M_b^{k_2}$ (resp. $M_d^{k_1} > M_d^{k_2}$) and $c_d^{k_2} - c_d^{k_1} \neq c_b^{k_2} - c_b^{k_1}$. Without loss of generality, let us assume that $c_d^{k_2} - c_d^{k_1} < c_b^{k_2} - c_b^{k_1}$. As Lemma 5 yields $\beta_b^{k_1} = -\beta_d^{k_2}$, it follows that $\beta_b^{k_1} = 0 = \beta_b^{k_2}$ for all $b, d \in \mathcal{A}_{\tilde{a}}^{>}$ by Lemma 8. Further, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} > M_b^{k_2}$, switching the commodity indices k_1 and k_2 in Lemma 6 yields $\gamma_b^{k_1} = -M_b^{k_1}\beta_b^{k_1}$. As $\beta_b^{k_1} = 0$, one obtains $\gamma_b^{k_1} = 0$, thus also $\gamma_b^{k_2} = 0$.

Next, provided there exists $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$, $v \in \mathbb{R}$ such that $c_{\tilde{a}}^{k_1} - c_b^{k_1} \leq v \leq c_{\tilde{a}}^{k_2} - c_b^{k_2}$ and $0 \leq v \leq M_b^{k_2}$, the point

$$\Big(\sum_{a \in \mathcal{A} \setminus \{b, \tilde{a}\}} N_a \mathbf{e}_a + v\mathbf{e}_b; \mathbf{0}; v\mathbf{e}_b; \mathbf{e}_{\tilde{a}}; \mathbf{e}_b \Big)$$

is in $\mathcal{P} \cap \mathcal{H}$. Note that the existence of $v \in \mathbb{R}$ is required since $x_{\tilde{a}}^{k_1} = 1 = x_b^{k_2}$ ($b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$) implies that $p_{\tilde{a}}^{k_1} = 0$ for points of $\mathcal{H}$. This yields $\gamma_b^{k_2} = -\gamma_{\tilde{a}}^{k_1}$ for all $b \in \mathcal{A}_{\tilde{a}}^{\leq} \setminus \{\tilde{a}\}$. Setting $b = \tilde{a}$ in Lemma 6 yields $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$. As $\gamma_b^{k_1} = -\gamma_b^{k_2}$ for all $b \in \mathcal{A}$, one obtains $\gamma_b^{k_2} = M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2} = -\gamma_b^{k_1}$ for all $b \in \mathcal{A}$.

Finally, for all $b \in \mathcal{A}_{\tilde{a}}^{>}$ such that $M_b^{k_1} \leq M_b^{k_2}$, the points

$$\Big(\sum_{a \in \mathcal{A} \setminus \{\tilde{a}, b\}} N_a \mathbf{e}_a + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\mathbf{e}_{\tilde{a}} + (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2})\mathbf{e}_b; \\ (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_b^{k_2})\mathbf{e}_b; (M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\mathbf{e}_{\tilde{a}}; \mathbf{e}_b; \mathbf{e}_{\tilde{a}} \Big)$$

are in $\mathcal{P} \cap \mathcal{H}$ since $x_b^1 = 1 = x_{\tilde{a}}^{k_2}$ ($b \in \mathcal{A}_{\tilde{a}}^{>} : M_b^{k_1} \leq M_b^{k_2}$) implies $p_{\tilde{a}}^{k_2} = M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2}$ for points of $\mathcal{H}$, which yields

$$(M_{b^*}^{k_1} + c_{b^*}^{k_2} - c_{\tilde{a}}^{k_2})\beta_{\tilde{a}}^{k_2} + \gamma_b^{k_1} + \gamma_{\tilde{a}}^{k_2} = 0.$$

As $\gamma_{\tilde{a}}^{k_2} = -M_{\tilde{a}}^{k_2}\beta_{\tilde{a}}^{k_2}$, one obtains $\gamma_b^{k_1} = \big(M_{b^*}^{k_2} - M_{b^*}^{k_1}\big)\beta_{\tilde{a}}^{k_2}$ and the result follows. □

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