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Abstract. Graph coloring is a well-known NP-hard problem with a number of practical applications including timetabling problems. Graph coloring is also difficult to solve because the natural expression of the cost function for this problem is characterized by large plateaus in which solutions “close” to each other have same cost (same number of colors). Plateaus are problematic to local search heuristics because of the lack of effective gradient in solutions neighborhood. The present work seeks to address this difficulty by adapting the multilevel cooperative search heuristic to solve the graph coloring problem. Multilevel cooperative search is a meta-heuristic based on information shared among concurrently executing search algorithms and different representation levels of the solution space. Information sharing and the multilevel representation combined with frequent re-starts of the local search algorithms can be used to compensate for the lack of gradient in cost functions. In the present work, re-starts are based of graph coloring solutions projected from a high level representation of the solution space to a lower level.

Keywords. Graph coloring problem, multilevel cooperative search, metaheuristic.

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1 Introduction

The graph coloring problem is a well-known NP-hard problem [12] with a number of practical applications including timetabling problems [4, 13]. Graph coloring is also difficult to solve because the natural expression of the cost function for this problem is characterized by large plateaus in which solutions "close" to each other have same cost (same number of colors). Plateaus are problematic to local search heuristics because of the lack of effective gradient in solutions neighborhood. The present work seeks to address this difficulty by adapting the multilevel cooperative search heuristic to solve the graph coloring problem.

As multilevel cooperative search is a parallel algorithm, the solution space is explored concurrently by different search algorithms, one algorithm (program) per level of representation. While search algorithms explore the solution space, they gather information which is then shared among levels and search algorithms. This information, which diffuses through local interactions, synthesizes a global strategy of the solution space exploration.

Keys to the multilevel cooperative search approach are the problem coarsening and re-coarsening operators which generate and modify the different "levels" of representations of the solution space. These operators need to be specifically adapted to each problem. Our approach to the graph coloring coarsening operator consists to aggregate vertices from an initial problem instance graph to generate a new (smaller) graph. The coarsening should be such that colorings of the coarsened graph can be expanded into feasible solutions of the initial uncoarsened graph instance. On the other hand, the re-coarsening operator takes elite solutions found by one search algorithm at level i to modify directly the solution space representation at level $i+1$. Our adaptation of the re-coarsening operator to the graph coloring problem uses good coloring at level i to obtain aggregated vertices at level $i+1$. Then, through similar re-coarsening operations at levels above i, information contained in elite colorings at level i propagates to all levels above i.

The purpose of the re-coarsening by elite solutions (best colorings) is to re-define the high level representations of the problem such to focus the exploration by search algorithms in more interesting regions of the solution space. Colorings of aggregated vertices at a coarsened level are projected to lower levels. They then provide a coloring of the lower (larger in terms of the number of vertices) representations of a graph coloring problem instance. In the present study, we want to use the projections to re-initialize the search algorithms at the lower levels each time the lack of gradient in the neighborhoods undermines the quality of the local search. It is expected that the re-start will also re-initialize the search in good regions of he solution space.

2 Multilevel algorithms

Multilevel was first a general iterative technique for solving numerical approximation problems [2, 9]. More recently, this technique has been adapted as a new heuristic for solving NP-hard combinatorial optimization problems (particularly graph related problems) [1, 3, 7]. The multilevel heuristic has three phases: coarsening, initial solving and projection/refinement. The coarsening phase transforms recursively a problem instance P_0 into a hierarchy $P_1, P_2, \ldots P_l$ of increasingly smaller sub-problem instances. The initial solving phase considers the smallest problem instance (P_l) and solves it using an exact solver, an approximation algorithm or a heuristic method. During the projection/refinement phase, starting at level l, the best solution s_i of problem P_i is projected to level $i-1$ where it is used as an initial solution of a move-based search heuristic for P_{i-1}. The move-based heuristic at level $i-1$ ”refines” the solution obtained from P_i by exploring the solution space of the problem instance P_{i-1}. Projection/refinement steps propagate the search downward among levels of problems until reaching level 0, the original problem instance.

3 Cooperative & multilevel cooperative search

The multilevel cooperative search approach combines cooperative search with the multilevel approach for combinatrorial optimization. Cooperative search [14] is a complex history-based parallel heuristic method running on top of a set of independent meta-heuristics. The independent meta-heuristics gather information when they explore in parallel the solution space. The cooperation heuristic defines how this information can be shared such to improve the performance of cooperating search programs. Which information is gathered, how information is shared among programs, and how programs use this information to guide their own exploration of the solution space, are some of the design issues that must be addressed by any cooperative algorithm. Cooperation also induces correlations among locally interacting programs as well as global information diffusion. These can be used to improve convergence, but is unknown how to model the impact of correlated interactions on the performance of cooperating search heuristics, consequently the design of cooperative search algorithms remains mostly intuitive and highly empirical.

Multilevel cooperative search is a parallel multilevel algorithm in the cooperative search paradigm, i.e. it is a set of independent meta-heuristics sharing information. There are $l+1$ independent meta-heuristics, one for each problem instance P_i. Each meta-heuristic is a loop that executes a move-based heuristic to explore the search space of problem instance P_i, performs asynchronous information requests either with levels $i-1$ or $i+1$, and executes two cooperation operations: projection and re-coarsening. Projection consists to re-start the move-based heuristic at level i based on an elite solution obtained from level $i+1$. Re-coarsening consists to change the coarsening of level i using elite solutions obtained from level $i-1$. The multilevel cooperative search approach has been tested on several problems [5, 10, 11].

4 Multilevel cooperation for graph coloring

Graph coloring is a problem where the vertices of a graph $G=(V,E)$ are partitioned into sets $C_1, C_2, \dots, C_n$ of non-adjacent vertices such that $C_i \cap C_j = \emptyset$ if $i \neq j$ and $\cup_{i=1}^{n} C_i = V$. The graph coloring problem seeks the minimum number of subsets $C_1, C_2, \dots, C_n$ that partitions V and satisfies the above constraints. The graph coloring problem is a well-known NP-hard problem [12] and has a number of practical applications including timetabling problems [4, 13], i.e. a basic timetabling problem with only clashing constraints between exams can be modeled as the problem of finding the minimum number of time periods (colors) to accommodate all the exams (vertices).

4.1 Coarsening operators

We consider two coarsening strategies which compute the initial hierarchy of coarsened graph coloring problem instances. Assume $G_0 = (V_0, E_0)$ is the graph coloring problem instance for which we seek a coloring of its vertices. Let $G_i = (V_i, E_i)$ be the graph at level i. The first coarsening procedure is described in Figure 1. Each new vertex $uv \in G_{i+1}$ is adjacent to vertex jk in V_{i+1} if either

while (unmarked vertices in graph G_i)
  randomly select a pair of non-adjacent vertices $u, v \in V_i$
  mark u and v
  merge u and v into a single vertex uv in V_{i+1}
for $u \in V_i$ unmarked and there exists no non-adjacent vertex $v \in V_i$
  add u to V_{i+1}

Figure 1: Coarsening Operator

$\{u, j\}, \{u, k\} \in E_i$ or $\{v, j\}, \{v, k\} \in E_i$.

The second coarsening strategy is the "neighbours of neighbours" coarsening procedure [15]. The neighbours of neighbours algorithm merges a pair of vertices u, v provide they are not adjacent while both are adjacent to a same third vertex k.

4.2 The local search heuristic

The coarsening phase is completed once l coarsened problem instances have been generated. Then, a graph coloring heuristic is initiated in parallel for each level. As in [15], our move-based heuristic for solving the graph coloring problem at each level is the tabu search algorithm of Hertz & deWerra [8]. We will use a modified version of Culberson's implementation of Hertz & deWerra tabu search heuristic [6]. This code will be modified such that elite solutions can be requested asynchronously from levels $i+1$ and $i-1$. This code also need to be extended with a projection operator and a multilevel re-coarsening operator.

4.3 Projection operator

The projection operation consists to re-start the local search at level i (using an elite solution from level $i+1$) whenever this procedure has failed to improve its best solutions for a pre-defined number of iterations. Given a coarsened graph $G_{i+1} = \{V_{i+1}, E_{i+1})$ at level $i+1$, an elite coloring of G_{i+1} is a solution vector $s = C_{i+1,1}, C_{i+1,2}, \ldots, C_{i+1,n}$ optimizing cost and satisfying the graph coloring conditions. By construction of the coarsening procedures, any vertex $w \in V_{i+1}$ is either an aggregate of two vertices $u, v \in V_i$ or a single vertex $u \in V_i$. Figure 2 describes the projection in which a solution $C_{i+1,1}, C_{i+1,2}, \ldots, C_{i+1,n}$ from level $i+1$ is projected on the graph G_i of level i.

input: V_i, V_{i+1}, $C_{i+1,1}, C_{i+1,2}, \ldots, C_{i+1,n}$
for $(u \in V_i)$
  if $(u = w$ or $u \in w)$
    for $(w \in C_{i+1})$
      assign color $C_{i+1,j}$ to u
return $C_{i,1}, C_{i,2}, \ldots, C_{i,m}$ /* $m > n$*/

Figure 2: Projection Operator

This operation is very helpful to address problems with large plateaus. Whenever the local search procedure at level i is lost following a sequence of moves without gradient, the projection re-initializes the search in a different region of the solution space. The re-initialization is based on a local optimum solution (elite solution) from level $i+1$, and this solution is transposed to a less coarsened representation of level i (the size of neighborhoods is larger). Therefore, it is likely that the local search algorithm at level i will improve (refine) the elite solution received from level $i+1$, thus making moves guided by the useful gradients of the cost function.

4.4 Re-coarsening operators

The re-coarsening operation at level i works as followed: The graph coloring procedure at level i asynchronously requests an elite solution vector s from level $i-1$. Given the partition $C_{i-1,1}, C_{i-1,2}, \ldots, C_{i-1,n}$ contained in s, the re-coarsening procedure randomly merges a pair of vertices u, v in the elite solution vector s if they are not adjacent in G_{i-1} and they belong to a same partition in s. The new vertex uv is then added to the graph G_i. The number of new vertices generated in each re-coarsening operation is an input parameter of the multi-level cooperative search algorithm, the value of this parameter is usually tuned empirically.

We also consider a second approach where re-coarsening at level i is based on vertices that belong to partitions C_j at level $i-1$ that have the lowest cardinality. We will also investigate other re-coarsening avenues based on different matching algorithms.

4.5 Convergence of the multilevel cooperative search algorithm

Each level of representation of the solution space is a subset of feasible solutions. In general, through re-coarsening and upward information diffusion, elite solutions of the lower levels increase the overlap of the top most representations with good regions of the solution space. This is a key element of the multilevel cooperative search approach. Repeated re-coarsening operations construct representations that overlap with more interesting regions of the solution space. In turn, through projection and downward information diffusion, the search performance of the local search algorithms at the lower levels improve because the projection operator re-initialize the exploration of these algorithms in better regions of the solution space. We expect this dynamic will play a similar role in solving the graph coloring problem.

Tests will be conducted on the test suite of problem instances that have been used in [15].

5 Conclusion

The natural expression of the graph coloring cost function induces large plateaus in the solution space, which makes difficult for a local search heuristic to guide the search toward interesting regions of the solution space. In this short paper, we have described an adaptation of the multilevel cooperative search heuristic to the graph coloring problem. Through information sharing and operators such re-coarsening and projection, the multilevel cooperative search heuristic synthesize a global guidance strategy of the solution space exploration based on the best moments (elite solutions) of the concurrent local search algorithms. It is expected that under re-initializations by the projection operator, local search algorithms will overcome the large plateaus of the graph coloring cost function and consequently improve their search performance.

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