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Robust Inventory Routing under Demand Uncertainty

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Abstract. This paper introduces a robust inventory routing problem where a supplier distributes a single product to multiple customers facing dynamic uncertain demands over a finite discrete time horizon. The probability distribution of the uncertain demand at each customer is not fully specified. The only available information is that these demands are independent and symmetric random variables which can take some value from their support interval. The supplier is responsible for the inventory management of its customers, has sufficient inventory to replenish the customers, and distributes the product using a capacitated vehicle. Backlogging of the demand at customers is allowed. The problem is to determine the delivery quantities as well as the times and routes to the customers while ensuring feasibility regardless of the realized demands and minimizing the total cost composed of transportation, inventory holding and shortage costs. Using a robust optimization approach, we propose two robust mixed integer programming (MIP) formulations for the problem. We also propose a new MIP formulation for the deterministic (nominal) case of the problem. We implement these formulations within a branch-and-cut algorithm and report results on a set of instances adapted from the literature.

Keywords. Inventory routing problem, lot sizing, robust optimization, integer programming, branch-and-cut.

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1. Introduction

The inventory routing problem (IRP) can be defined as the problem of determining the delivery times, delivery quantities and delivery routes to a set of geographically dispersed customers served by a supplier. The IRP is an important problem arising in vendor-managed inventory (VMI) systems where the supplier (vendor) is responsible for the management of inventories at the customers. For a description of benefits and industrial applications of VMI systems, one can refer to Çetinkaya and Lee (2000) and Campbell et al. (2002).

There exist several variants of the IRP depending mainly on the nature of the demand at customers (deterministic vs. stochastic) and on the length of the planning horizon (finite vs. infinite). For example, some studies consider single period IRPs with stochastic demand (e.g. Federgruen and Zipkin 1984), with deterministic demand (e.g. Chien et al. 1989), multi-period finite horizon IRPs with constant demand (e.g. Dror and Ball 1987, Campbell and Savelsbergh 2004) and with dynamic deterministic demand (e.g. Bertazzi et al. 2002, Abdelmaguid and Dessouky 2006, Archetti et al. 2007, Yugang et al. 2008, Abdelmaguid et al. 2009), infinite horizon IRPs with constant deterministic demand (e.g. Anily and Federgruen 1990, Viswanathan and Mathur 1997, Chan et al. 1998), and infinite horizon IRPs with stochastic demand (e.g. Kleywegt et al. 2002, 2004, Adelman 2004, Hvattum and Løkketangen 2009, Hvattum et al. 2009). For recent reviews on the IRP, one can refer to Moin and Salhi (2007) and Andersson et al. (2010).

Demand is widely accepted to be dynamic and stochastic in real life inventory routing problems (Campbell et al. 1998). Many studies consider the IRP with dynamic deterministic demand, which leads to more tractable yet less realistic models compared to those with stochastic demand. On the other hand, stochastic IRP (SIRP) models are intractable in that only very small instances can be solved to optimality (Hvattum and Løkketangen 2009) and therefore several heuristics have been proposed for their resolution. Studies on SIRPs also assume full knowledge of the probability distribution of demand, which may be unavailable or difficult to obtain. There is clearly a need to consider the IRP with dynamic stochastic demand in a tractable way, where no information for the probability distribution of demand is required.

In this paper, we introduce a single product multi-period finite horizon IRP with dynamic stochastic demands at customers, where a polyhedral (interval) demand uncertainty structure with no specific probability distribution is considered. The supplier holds an unlimited amount of the product to replenish the customers, and backlogging (i.e. not meeting demand on time) of demand at customers is allowed. Although most distribution management problems involve multiple vehicles, for simplicity and because this study is the first to address such a complex problem, we consider a single vehicle for the distribution of the product. The vehicle can visit all customers in a period, provided that the total amount shipped to the visited customers does not exceed its capacity. We make use of recently developed robust optimization approaches in addressing demand uncertainty in a tractable way. The problem, referred to as the robust inventory routing problem (RIRP), is to decide on the delivery times and quantities to customers as well as delivery routes such that whatever value the demands take within their supports, the solution remains feasible and the total cost is minimized. The RIRP is obviously NP-hard since it includes the classical traveling salesman problem (TSP) as a special case.

To the best of our knowledge, the paper by Aghezzaf (2008) is the only study that incorporates robustness into an IRP. In contrast to the dynamic demands with an ambiguous probability distribution considered in this paper, Aghezzaf (2008) assumes normally distributed stationary demands at the customers, and travel times with constant averages and bounded standard deviations. He only considers a cyclic distribution strategy, which is theoretically not optimal but claimed to be a good approximation, and he develops a nonlinear mixed integer programming (MIP) formulation to find a minimum cost solution that is feasible for all possible realizations of demands and travel times within their supports.

Robust optimization has emerged as a powerful methodology for problems involving uncertain parameters with no information on their probability distributions. This is achieved by finding the best solution (often called minimax solution) which ensures feasibility regardless of the realized values of uncertain parameters. The first study in robust optimization is that of Soyster (1973) which considers linear programming (LP) problems with uncertain parameters and sets the uncertain parameters to their worst-case values in the uncertainty set. This approach, however, usually results in overconservative solutions. To cope with the overconservativeness, El-Ghaoui and Lebret (1997), El-Ghaoui et al. (1998) and Ben-Tal and Nemirovski (1998, 1999, 2000) consider uncertain convex optimization problems under ellipsoidal uncertainty sets. However, this approach increases the complexity of the nominal problem (i.e. problem without uncertainty) and cannot easily be extended to discrete optimization problems. For instance, if the nominal problem is an LP problem (a second order cone program), its robust counterpart becomes a second order cone program (a semidefinite program). Bertsimas and Sim (2003, 2004) develop a robustness approach, called the "budget of uncertainty" approach, which controls the level of conservativeness by allowing only some of the uncertain parameters to deviate from their nominal values simultaneously. An important feature of this approach is that the robust counterpart preserves the complexity of its nominal problem (e.g. if the nominal problem is an LP problem, the robust counterpart is also an LP problem), and thus can easily be extended to discrete optimization problems. All studies on robust optimization mentioned until now were designed to obtain robust counterparts of static decision making problems in that all the decision variables are determined a priori ("here and now" decisions). Ben-Tal et al. (2004) introduce the adjustable robust counterpart of the multistage uncertain LP problems in which some of the decisions can be made after some of the uncertain parameters become known ("wait and see" decisions), thus enabling these decisions to adjust themselves according to the realized data. The adjustable robust counterpart provides a better optimal objective value than the robust counterpart if uncertainty affecting a constraint has an effect on the other constraints (Ben-Tal et al. 2004). However, the adjustable robust counterpart is intractable. Therefore, Ben-Tal et al. (2004) propose an affinely adjustable robust counterpart as a tractable approximation in which "wait and see" decision variables are rewritten as affine functions of the uncertain data.

Because the RIRP involves inventory management decisions, studies on robust inventory management under demand uncertainty are of interest. Ben-Tal et al. (2004, 2005) consider two different inventory management problems formulated as uncertain LP programs and show the value of the affinely adjustable robust counterpart over the robust counterpart. Bertsimas and Thiele (2006) apply the "budget of uncertainty" robustness approach described in Bertsimas and Sim (2003, 2004) to single echelon and multi-echelon (with distribution network structure) inventory management problems. They formulate the robust counterpart of the nominal LP (MIP) problem in the absence (presence) of fixed ordering costs, which is an LP (MIP) problem. Bienstock and Ozbay (2008) propose a Benders decomposition algorithm to find the robust basestock policy for a single echelon inventory management problem. Ben-Tal et al. (2009) apply an extension of the affinely adjustable robust counterpart, called globalized robust counterpart, to a multi-echelon inventory management problem with a serial structure. See and Sim (2009) consider a single echelon inventory management problem with nonzero lead times where uncertain demand is characterized by the mean, support, covariance and directional deviations. They formulate the problem as a stochastic optimization model, and approximate it using robust optimization which leads to a second order cone program. Except Bertsimas and Thiele (2006), none of the mentioned studies considers fixed ordering costs (i.e. all decision variables are continuous).

The RIRP is also closely related to the IRP studied by Abdelmaguid and Dessouky (2006), and Abdelmaguid et al. (2009), where deterministic demands, storage capacity limits at the customers and multiple vehicles are considered unlike what is done in the RIRP. Abdelmaguid and Dessouky (2006) propose a genetic algorithm, while Abdelmaguid et al. (2009) develop construction and improvement heuristics for the problem. Both studies present an MIP formulation of the problem, which is a combination of standard inventory balance equations for the inventory replenishment decisions of the customers and a multi-commodity flow based formulation for the routing decisions. The upper and lower bounds obtained by solving the MIP formulation within a given time limit are used as benchmarks to measure the quality of the heuristics. The MIP formulation proposed could only solve very small instances to optimality using an off-the-shelf optimization solver. The first exact IRP algorithm, a branch-and-cut algorithm, is proposed by Archetti et al. (2007) for a related IRP in which deterministic demands, order-up-to level policy and no backlogging at the customers, and a single vehicle are considered. Solyalı and Süral (2008) improve the results of Archetti et al. (2007) by proposing a strong MIP formulation within a branch-and-cut algorithm. These authors also develop an effective MIP based heuristic for the problem. Both Archetti et al. (2007) and Solyalı and Süral (2008) use a computationally attractive two-index vehicle flow formulation for the routing decisions. They differ in the formulation used for the inventory replenishment decisions of the customers: the former uses the standard inventory balance equations whereas the latter uses a strong shortest path formulation.

In this paper, we first propose a new MIP formulation for the nominal case of the RIRP. This formulation combines a tight formulation for the inventory replenishment decisions of each customer and a two-index vehicle flow formulation for the routing decisions. We develop a branch-and-cut algorithm using the proposed formulation. Computational results on instances generated by Abdelmaguid et al. (2009) for the nominal case show the superiority of our formulation and of our branch-and-cut algorithm over the MIP formulation of Abdelmaguid and Dessouky (2006) and Abdelmaguid et al. (2009). Then, using the "budget of uncertainty" robustness approach of Bertsimas and Sim (2003, 2004), we formulate the RIRP as a tractable MIP formulation with slightly more constraints and variables than the formulation for the nominal case. This formulation of the RIRP makes decisions a priori (i.e. at time 0). Modifying the MIP formulation for the nominal case, we obtain a variation of that formulation which we use in developing another robust MIP formulation for the RIRP. We show that the new robust formulation is equivalent to its adjustable robust counterpart, and is indeed a nominal formulation with modified demands. Computational results on instances from the literature reveal that the "budget of uncertainty" based robust formulation generally yields slightly better objective values than the robust formulation with modified demands, whereas the latter is faster than the former.

The contributions of this study can be summarized as follows:

- We propose the first exact algorithm for a deterministic inventory routing problem with backlogging for which only heuristics are known to exist. Our MIP formulation, implemented through branch-and-cut, is far superior to the only other MIP formulation existing in the literature.
- This study is the first to consider a robust IRP under ambiguous demand distribution. We propose two different tractable robust MIP formulations for the problem. One of the robust formulations has slightly more variables and constraints than its nominal formulation, whereas the other robust formulation is basically the same as its nominal formulation, but with modified demand values. Both formulations are implemented within a branch-and-cut algorithm and yield robust solutions protecting against uncertainty in demand.

The remainder of this paper is organized as follows. In Section 2, we present a brief review on the robust optimization approach that we use. We give the formal description of the RIRP, present the proposed nominal and robust formulations, and describe the proposed branch-and-cut algorithm in Section 3. In Section 4, we present the computational results on an extensive set of test instances. Finally, we conclude the paper in Section 5.

2. The Robustness Approach

This section provides a brief review of relevant results by Bertsimas and Sim (2003, 2004). Let n be the number of decision variables indexed by j, and let m be the number of constraints indexed by i. An interval (or polyhedral) uncertainty structure is considered. Each objective coefficient c_j is an independent random variable which can take a value from the interval $[\bar{c}_j, \bar{c}_j + \hat{c}_j]$, where $\bar{c}_j$ is the nominal value and $\hat{c}_j$ is the maximum deviation from the nominal value. Each coefficient a_{ij} in the constraints of the formulation (i.e. the coefficient of the j^{th} variable in the i^{th} constraint) is an independent, symmetric and bounded random variable which can take a value from the interval $[\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$, where $\bar{a}_{ij}$ is the nominal value and $\hat{a}_{ij}$ is the maximum deviation from the nominal value. The probability distribution of the random variables is not known. Unlike Soyster (1973), who optimizes a problem in which each uncertain parameter is equal to its worst-case value in a set, Bertsimas and Sim (2003, 2004) optimize against the worst-case by allowing a degree of control that avoids overconservative solutions. This control is achieved by imposing a so-called "budget of uncertainty", Γ_i ($i = 0$ for the objective function, and $i \in [1, m]$ for the constraints), which ensures that only some uncertain parameters can simultaneously deviate from their nominal value. Thus, one can adjust the level of robustness by assigning a value to Γ_i ($i \in [0, m]$) from the interval $[0, |J_i|]$, where $J_0 = \{j | \hat{c}_j > 0\}$ and $J_i = \{j | \hat{a}_{ij} > 0\}$ for $i \in [1, m]$, with $\Gamma_i = 0$ meaning no protection against uncertainty (i.e. the nominal problem is considered), and $\Gamma_i = |J_i|$ meaning the most conservative protection against the uncertainty (i.e. the approach of Soyster 1973). Next we show how this approach translates into a mathematical program. Let the scaled deviation of c_j be denoted by $z_{0j} = (c_j - \bar{c}_j)/\hat{c}_j$ and the scaled deviation of a_{ij} be denoted by $z_{ij} = (a_{ij} - \bar{a}_{ij})/\hat{a}_{ij}$. Define b_i as the right-hand side constant for constraint i, and l_j and u_j as the lower and upper bounds on decision variable x_j, respectively. Then, given the nominal problem

$$\begin{aligned} \min \ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m \\ & l_j \le x_j \le u_j \quad 1 \le j \le n, \end{aligned} \tag{1}$$

where $c_j = \bar{c}_j$ and $a_{ij} = \bar{a}_{ij}$, its robust counterpart is as follows:

$$\begin{aligned} \min \ & \sum_{j=1}^{n} \bar{c}_j x_j + \max\{\sum_{j\in J_0} \hat{c}_j |x_j| z_{0j} : \sum_{j\in J_0} z_{0j} \le \Gamma_0, 0 \le z_{0j} \le 1, j \in J_0\} \\ \text{s.t.} \ & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \max\{\sum_{j\in J_i} \hat{a}_{ij} |x_j| z_{ij} : \sum_{j\in J_i} z_{ij} \le \Gamma_i, 0 \le z_{ij} \le 1, j \in J_i\} \le b_i \quad 1 \le i \le m \\ & l_j \le x_j \le u_j \quad 1 \le j \le n, \end{aligned} \tag{2}$$

where each c_j and each a_{ij} in (1) is replaced by $\bar{c}_j + \hat{c}_j z_{0j}$ and $\bar{a}_{ij} + \hat{a}_{ij} z_{ij}$, respectively, and the related expressions are maximized to optimize against the worst-case. Using the strong duality theorem, Bertsimas and Sim (2003, 2004) formulate the above nonlinear robust model equivalently as the following linear model:

$$\begin{aligned} \min \ & \sum_{j=1}^{n} \bar{c}_j x_j + \theta_0 \Gamma_0 + \sum_{j\in J_0} \alpha_{0j} \\ \text{s.t.} \ & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \theta_i \Gamma_i + \sum_{j\in J_i} \alpha_{ij} \le b_i \quad 1 \le i \le m \\ & \theta_0 + \alpha_{0j} \ge \hat{c}_j y_j \quad j \in J_0 \\ & \theta_i + \alpha_{ij} \ge \hat{a}_{ij} y_j \quad 1 \le i \le m, j \in J_i \\ & \theta_i \ge 0 \quad 0 \le i \le m \\ & \alpha_{ij} \ge 0 \quad 0 \le i \le m, j \in J_i \\ & y_j \ge 0 \quad 1 \le j \le n \\ & -y_j \le x_j \le y_j \quad 1 \le j \le n \\ & l_j \le x_j \le u_j \quad 1 \le j \le n, \end{aligned} \tag{3}$$

where θ_i and α_{ij} are the corresponding dual variables used to linearize the model. Note that Bertsimas and Sim (2003, 2004) present generic robust models involving integer and continuous variables. Since we consider nonnegative continuous (and integer) variables in this paper, we have modified the above robust models accordingly.

The "budget of uncertainty" approach of Bertsimas and Sim (2003, 2004) ensures that the optimal solution, say x^*, is deterministically feasible if at most $\lfloor \Gamma_i \rfloor$ coefficients a_{ij} change in each constraint

i; otherwise x^* is feasible for constraint i with a high probability depending on the chosen Γ_i. The probability of x^* being infeasible for constraint i can be calculated as follows:

$$\Pr(\sum_{j=1}^{n} a_{ij}x^* > b_i) \leq B(n,\Gamma_i) = \frac{1}{2^n}\left\{(1-\mu)\sum_{l=\lfloor v\rfloor}^{n}\binom{n}{l} + \mu\sum_{l=\lfloor v\rfloor+1}^{n}\binom{n}{l}\right\}, \tag{4}$$

where $n = |J_i|$, $v = (\Gamma_i + n)/2$, $\mu = v - \lfloor v\rfloor$. The bound in (4) may be difficult to compute due to the combinations, but the following expression yields an easy to compute bound and a very good approximation of (4) (Bertsimas and Sim 2004):

$$B(n,\Gamma_i) \leq (1-\mu)C(n,\lfloor v\rfloor) + \sum_{l=\lfloor v\rfloor+1}^{n} C(n,l), \tag{5}$$

where

$$C(n,l) = \begin{cases} \frac{1}{2^n} & \text{if } l = 0 \text{ or } l = n \\ \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(n-l)l}}\exp\left(n\log\left(\frac{n}{2(n-l)}\right) + l\log\left(\frac{n-l}{l}\right)\right) & \text{otherwise.} \end{cases}$$

A good feature of the above probability bounds is that they are independent of x^* and a relatively small Γ_i (compared to $|J_i|$) gives a high probability for the feasibility of contraint i.

3. Problem Description and Formulations

We consider an inventory routing problem where a supplier 0 distributes a single product to N customers over a finite discrete time horizon T, using a single vehicle of capacity C. The supplier is responsible for the inventory management of the customers and has sufficient inventory to replenish the customers. Each customer $i \in \mathcal{M} = \{1, 2, ..., N\}$ faces a dynamic uncertain demand d_{it} in period $t \in \mathcal{T} = \{1, 2, ..., T\}$. The probability distribution of the random variable d_{it} is unknown. The only information available on the demands is that they are independent, symmetric and can take a value from the interval $[\bar{d}_{it} - \hat{d}_{it}, \bar{d}_{it} + \hat{d}_{it}]$, where $\bar{d}_{it}$ is the point estimate (nominal value) and $\hat{d}_{it}$ is the maximum deviation for the demand of i in period t. Customer i is first replenished in a period, and then its demand is deducted from the total amount available, which is the sum of inventory level of the previous period and the amount replenished. If the total amount available is not sufficient to satisfy the demand at customer i, unmet demand is backlogged to be either satisfied in the future or not satisfied at all, otherwise excess inventories are carried over at customer i to satisfy future demands. Each unit held in inventory at the end of period t incurs a unit holding cost of h_{it} at customer i, whereas each unit backlogged at the end of period t incurs a unit backlogging cost of g_{it}, where $g_{it} > h_{it}$. We assume the vehicle can make at most one trip in each period but can visit any subset of the customers provided that the total replenishment quantity to the customers does not exceed the vehicle capacity. Using the vehicle in period t incurs a fixed vehicle dispatching cost f_t. The vehicle visiting customer $j \in \mathcal{M}' = \{0\} \cup \mathcal{M}$ directly after customer $i \in \mathcal{M}'$ incurs a transportation cost c_{ij}. We consider symmetric transportation costs, i.e. $c_{ij} = c_{ji}$ $\forall i, j \in \mathcal{M}'$. We assume initial inventory level at the customers is zero and all the parameters are nonnegative. The RIRP is to determine delivery times, delivery quantities and routes to customers such that the total cost composed of inventory holding, backlogging, fixed vehicle dispatching and transportation costs is minimized. We call this problem a RIRP since we incorporate robustness to the classical inventory routing problem by ensuring that the capacity of the vehicle will not be exceeded regardless of the value the demands can take from their support and by minimizing the total cost against the worst possible case. To control the degree of robustness and conservativeness of the solution, we define the "budget of uncertainty" Γ_t $(0 \leq t \leq T)$ as described in Section 2 which allows only some of the demand figures to concurrently deviate from their nominal value.

Let alone the fact that there does not yet exist any good formulation, even for the nominal case of the RIRP, the incorporation of robustness into the MIP formulations yields weaker formulations, as observed in Bertsimas and Sim (2003). Thus, it is crucial to develop a strong formulation for the exact solution of the RIRP. To this end, considering the RIRP as a combination of the inventory replenishment problem of the customers and the routing problem of the vehicle, we use effective mathematical programming representations for both. All existing studies in the IRP literature, except Solyalı and Süral (2008), consider standard inventory balance equations to model the corresponding inventory replenishment problem (see e.g. Abdelmaguid and Dessouky 2006, Abdelmaguid et al. 2009) which provide a weak link between the replenishment and the routing problems, and thus a weak lower bound. The inventory replenishment problem of each customer can be seen as the uncertain version of the deterministic demand uncapacitated lot sizing problem with backlogging (ULSB), for which tight reformulations are known (Pochet and Wolsey 1988). We use the facility location reformulation which defines the convex hull of feasible solutions of the inventory replenishment problem of each customer in the case of deterministic demand. Note that our representation is also different from the one used (i.e. a form of standard inventory balance equations) in the studies on robust inventory management. For the routing problem of the vehicle, we use a two-index vehicle flow formulation which is one of the most computationally attractive formulations for the vehicle routing problem (Laporte 2007) and has been successfully applied to a deterministic IRP with order-up-to level policy (Archetti et al. 2007, Solyalı and Süral 2008).

3.1. The Nominal Formulation

We now present the formulation we propose for the RIRP. Define q_{itk} as the total inventory cost of replenishing customer i in period $t \in \mathcal{T}$ to satisfy its demand in period $k \in \mathcal{T}$, and $q_{i,T+1,k}$ as the total inventory cost of not meeting the demand of customer i in period $k \in \mathcal{T}$. Let w_{itk} be the fraction of the demand of customer i in period $k \in \mathcal{T}$ that is delivered in period $t \in \mathcal{T}$, and let $w_{i,T+1,k}$ be the fraction of the demand of customer i in period $k \in \mathcal{T}$ that is left unmet. Since all the demand has to be met in the ULSB given in Pochet and Wolsey (1988), we additionally define $q_{i,T+1,k}$ and $w_{i,T+1,k}$ to account for the unmet demand case. Let y_{it} be 1 if customer i is replenished in period $t \in \mathcal{T}$ and 0 otherwise, and y_{0t} be 1 if the vehicle is used in period $t \in \mathcal{T}$ to replenish some subset of customers and 0 otherwise. Let x_{ijt} $(i > j)$ be the number of times the edge between nodes i and j is traversed by the vehicle in period $t \in \mathcal{T}$; x_{ijt} is a binary variable when $j \in \mathcal{M}$, while it is an integer variable which can be at most 2, corresponding to a single customer trip when $j = 0$. Then, the RIRP can be formulated as follows:

$$UF : \min \sum_{t \in \mathcal{T}} f_t y_{0t} + \sum_{i \in \mathcal{M}'} \sum_{j \in \mathcal{M}', j<i} \sum_{t \in \mathcal{T}} c_{ij} x_{ijt} + \sum_{i \in \mathcal{M}} \sum_{t=1}^{T+1} \sum_{k=1}^{T} d_{ik} q_{itk} w_{itk} \tag{6}$$

$$\text{s.t.} \sum_{t=1}^{T+1} w_{itk} = 1 \quad i \in \mathcal{M}, k \in \mathcal{T} \tag{7}$$

$$w_{itk} \leq y_{it} \quad i \in \mathcal{M}, t \in \mathcal{T}, k \in \mathcal{T}, d_{ik} > 0 \tag{8}$$

$$\sum_{i \in \mathcal{M}} \sum_{k=1}^{T} d_{ik} w_{itk} \leq C y_{0t} \quad t \in \mathcal{T} \tag{9}$$

$$\sum_{j \in \mathcal{M}', j<i} x_{ijt} + \sum_{j \in \mathcal{M}', j>i} x_{jit} = 2y_{it} \qquad i \in \mathcal{M}', t \in \mathcal{T} \tag{10}$$

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}, j<i} x_{ijt} \leq \sum_{i \in \mathcal{S}} y_{it} - y_{kt} \qquad \mathcal{S} \subseteq \mathcal{M}, t \in \mathcal{T}, k \in \mathcal{S} \tag{11}$$

$$y_{it} \leq y_{0t} \qquad i \in \mathcal{M}, t \in \mathcal{T} \tag{12}$$

$$x_{ijt} \in \{0,1\} \qquad i \in \mathcal{M}, j \in \mathcal{M}, j < i, t \in \mathcal{T} \tag{13}$$

$$x_{i0t} \in \{0,1,2\} \qquad i \in \mathcal{M}, t \in \mathcal{T} \tag{14}$$

$$y_{it} \in \{0,1\} \qquad i \in \mathcal{M}', t \in \mathcal{T} \tag{15}$$

$$w_{itk} \geq 0 \qquad i \in \mathcal{M}, k \in \mathcal{T}, 1 \leq t \leq T+1, \tag{16}$$

where $q_{itk} = \sum_{l=t}^{k-1} h_{il}$ if $t \leq k$ and $q_{itk} = \sum_{l=k}^{t-1} g_{il}$ if $t > k$.

The objective function (6) is the total of fixed vehicle dispatching, transportation, inventory holding and shortage costs. Constraints (7) stipulate that either the demand of customer i in period k is met from period 1 through T or is left unmet. Constraints (8) ensure that the vehicle visits customer i in period t if any replenishment to customer i occurs in period t. Constraints (9) guarantee that the capacity of the vehicle is not exceeded. Constraints (10) are the degree constraints ensuring two edges are incident to node i (customer or supplier) if i is visited in period t. Constraints (11) are generalized subtour elimination constraints. Constraints (12), which are used to strengthen the routing part of the model, force the vehicle to depart from the supplier if any customer i is visited. Constraints (13)–(15) are integrality constraints, and constraints (16) are nonnegativity constraints.

The nominal formulation, referred to as the formulation NF, is obtained when $d_{it} = \bar{d}_{it}$ for $i \in \mathcal{M}, t \in \mathcal{T}$ in the UF.

3.2. The Robust Formulation

Since the uncertain parameter d_{ik} appears both in the objective function and in constraints of UF, we apply the robust optimization methodology described in Section 2 to both (6) and (9), and obtain the following nonlinear robust formulation for the RIRP:

$$\begin{aligned}
\min \quad & \sum_{t\in\mathcal{T}} f_t y_{0t} + \sum_{i\in\mathcal{M}'} \sum_{j\in\mathcal{M}', j<i} \sum_{t\in\mathcal{T}} c_{ij} x_{ijt} + \sum_{i\in\mathcal{M}} \sum_{t=1}^{T+1} \sum_{k=1}^{T} \bar{d}_{ik} q_{itk} w_{itk} + \\
& \max\{\sum_{i\in\mathcal{M}} \sum_{t=1}^{T+1} \sum_{k=1}^{T} \hat{d}_{ik} z_{ik}^0 q_{itk} w_{itk} : \sum_{i\in\mathcal{M}} \sum_{k=1}^{T} z_{ik}^0 \le \Gamma_0, 0 \le z_{ik}^0 \le 1, i \in \mathcal{M}, k \in \mathcal{T}\} \quad (17) \\
\text{s.t.} \quad & (7), (8), (10) - (16), \\
& \sum_{i\in\mathcal{M}} \sum_{k=1}^{T} \bar{d}_{ik} w_{itk} + \max\{\sum_{i\in\mathcal{M}} \sum_{k=1}^{T} \hat{d}_{ik} z_{ik}^t w_{itk} : \sum_{i\in\mathcal{M}} \sum_{k=1}^{T} z_{ik}^t \le \Gamma_t, \\
& \qquad 0 \le z_{ik}^t \le 1, i \in \mathcal{M}, k \in \mathcal{T}\} \le C y_{0t} \quad t \in \mathcal{T}. \quad (18)
\end{aligned}$$

The objective function (17) is the total of fixed vehicle dispatching costs, transportation costs, nominal inventory holding and shortage costs, as well as the worst possible inventory holding and shortage costs depending on the replenishments to the customers. Constraints (18) guarantee that regardless of the realization of demand (within the specified support intervals) the capacity of the vehicle is not exceeded.

Using the strong duality theorem as in Section 2, we obtain the following linear robust formulation.

$$\begin{aligned}
RF1 : \min \quad & \sum_{t\in\mathcal{T}} f_t y_{0t} + \sum_{i\in\mathcal{M}'} \sum_{j\in\mathcal{M}', j<i} \sum_{t\in\mathcal{T}} c_{ij} x_{ijt} + \sum_{i\in\mathcal{M}} \sum_{t=1}^{T+1} \sum_{k=1}^{T} \bar{d}_{ik} q_{itk} w_{itk} + \Gamma_0 \theta_0 + \sum_{i\in\mathcal{M}} \sum_{k\in\mathcal{T}} \alpha_{ik}^0 \quad (19) \\
\text{s.t.} \quad & (7), (8), (10) - (16), \\
& \sum_{i\in\mathcal{M}} \sum_{k=1}^{T} \bar{d}_{ik} w_{itk} + \Gamma_t \theta_t + \sum_{i\in\mathcal{M}} \sum_{k=1}^{T} \alpha_{ik}^t \le C y_{0t} \quad t \in \mathcal{T} \quad (20) \\
& \theta_0 + \alpha_{ik}^0 \ge \hat{d}_{ik} \sum_{t=1}^{T+1} q_{itk} w_{itk} \quad i \in \mathcal{M}, k \in \mathcal{T} \quad (21) \\
& \theta_t + \alpha_{ik}^t \ge \hat{d}_{ik} w_{itk} \quad i \in \mathcal{M}, t \in \mathcal{T}, k \in \mathcal{T} \quad (22) \\
& \theta_j \ge 0 \quad 0 \le j \le T \quad (23) \\
& \alpha_{ik}^j \ge 0 \quad 0 \le j \le T, i \in \mathcal{M}, k \in \mathcal{T}. \quad (24)
\end{aligned}$$

The objective function (19) and constraints (20) are the linearized versions of constraints (17) and (18), respectively. Constraints (21) and (22) originate from the implementation of the strong duality theorem. That is, they are the constraints of the duals of the maximization problems in (17) and (18).

3.3. The Adjustable Robust Formulation

Because each uncertain parameter d_{ik} appears more than once in UF, an affinely adjustable robust formulation may provide a better optimal objective value than its pure robust formulation (Ben-Tal et al. 2004). All decisions associated with x, y, and w variables are "wait and see" decisions (i.e. these variables are adjustable). However, UF is not a fixed recourse model due to the uncertain coefficients of adjustable w variables in (6) and (9), which means that an affinely adjustable robust counterpart of UF is intractable (Ben-Tal et al. 2004). Therefore, we consider a variation of UF in which we set $w'_{itk} = d_{ik} w_{itk}$:

$$
\begin{aligned}
UF' : \min \quad & \sum_{t \in \mathcal{T}} f_t y_{0t} + \sum_{i \in \mathcal{M}'} \sum_{j \in \mathcal{M}', j<i} \sum_{t \in \mathcal{T}} c_{ij} x_{ijt} + \sum_{i \in \mathcal{M}} \sum_{t=1}^{T+1} \sum_{k=1}^{T} q_{itk} w'_{itk} && (25) \\
\text{s.t.} \quad & (10) - (15) \\
& \sum_{t=1}^{T+1} w'_{itk} \geq d_{ik} \quad i \in \mathcal{M}, k \in \mathcal{T} && (26) \\
& w'_{itk} \leq d_{ik} y_{it} \quad i \in \mathcal{M}, t \in \mathcal{T}, k \in \mathcal{T} && (27) \\
& \sum_{i \in \mathcal{M}} \sum_{k=1}^{T} w'_{itk} \leq C y_{0t} \quad t \in \mathcal{T} && (28) \\
& w'_{itk} \geq 0 \qquad i \in \mathcal{M}, k \in \mathcal{T}, 1 \leq t \leq T+1. && (29)
\end{aligned}
$$

The nominal counterpart of UF', equivalent to NF, is obtained by replacing each d_{ik} with $\bar{d}_{ik}$ for $i \in \mathcal{M}, k \in \mathcal{T}$. Applying the "budget of uncertainty" robustness approach of Bertsimas and Sim (2003, 2004) does not make sense for UF' because there is a single uncertain parameter in each constraint. A pure robust formulation ensuring feasibility for any $d_{ik} \in [\bar{d}_{ik} - \hat{d}_{ik}, \bar{d}_{ik} + \hat{d}_{ik}]$ is obtained as follows:

$$
\begin{aligned}
RF2 : \min \quad & \sum_{t \in \mathcal{T}} f_t y_{0t} + \sum_{i \in \mathcal{M}'} \sum_{j \in \mathcal{M}', j<i} \sum_{t \in \mathcal{T}} c_{ij} x_{ijt} + \sum_{i \in \mathcal{M}} \sum_{t=1}^{T+1} \sum_{k=1}^{T} q_{itk} w'_{itk} \\
\text{s.t.} \quad & (10) - (15), (28), (29), \\
& \sum_{t=1}^{T+1} w'_{itk} \geq \bar{d}_{ik} + \hat{d}_{ik} \quad i \in \mathcal{M}, k \in \mathcal{T} && (30) \\
& w'_{itk} \leq (\bar{d}_{ik} + \hat{d}_{ik}) y_{it} \quad i \in \mathcal{M}, t \in \mathcal{T}, k \in \mathcal{T}. && (31)
\end{aligned}
$$

Instead of (31), one should normally write (27) as $w'_{itk} \leq (\bar{d}_{ik} - \hat{d}_{ik}) y_{it}$ for $i \in \mathcal{M}, t \in \mathcal{T}, k \in \mathcal{T}$ in deriving a robust counterpart if d_{ik} in (27) was treated as an uncertain parameter. Here, however, d_{ik} in (27) should not be seen as an uncertain parameter, but is rather used as a big-M value to force the y_{it} variable to be 1 if w'_{itk} variable takes a nonzero value. This is because the realized value of d_{ik} in (27) should not be a limit on the value of w'_{itk}. Next, an interesting result regarding the adjustable robust counterpart of the the UF' is given.

THEOREM 1. *The adjustable robust counterpart of* UF' *is equivalent to* $RF2$.

Proof. Since each uncertain parameter d_{ik} appears only once in UF' as discussed above, the thesis follows from Theorem 2.1 in Ben-Tal et al. (2004). □

A direct implication of Theorem 1 is that there is no need to consider an affinely adjustable robust counterpart of UF' since this would only be an approximation of the adjustable one, whereas $RF2$ is equivalent to the latter.

3.4. Branch-and-Cut Algorithm

Since all the proposed formulations (i.e. NF, $RF1$, $RF2$) involve subtour elimination constraints (11) which are exponential in number, it is not practical to solve those formulations to optimality by adding (11) a priori to the formulations. Therefore, we implement these formulations within a branch-and-cut framework in which constraints (11) are added dynamically as they are found to be violated. Let BC(.) denote the branch-and-cut algorithm using the (.) formulation. The generic BC(.) we use is as follows:

Step1. Set the best upper bound (UB*) to infinity and add the formulation (.) without integrality constraints and (11) to the node list.

Step2. If the node list is nonempty, select a node from it using the best-bound first strategy. Otherwise, stop since the optimality has been proved.

Step3. Solve the current formulation associated with the selected node and set the current lower bound (LB) to the objective value. If $UB^* < LB$, fathom the current node and go to Step 2.

Step4. If the current solution violates any of the constraints (11), add the violated constraints (11) to the current formulation and go to Step 3.

Step5. If the current solution satisfies all the integrality constraints, fathom the current node and update the best upper bound. Otherwise, select a fractional variable for branching, which adds new nodes to the node list. Go to Step 2.

For branching, we give a higher priority to y variables than to x variables. To check whether any of the constraints (11) are violated or not, we use the exact separation algorithm of Padberg and Rinaldi (1991) proposed for the TSP. However, we add only violated constraints (11) for $k = \arg\max_{i\in S} y_{it}$ instead of adding them for all $k \in S$. The structure of BC(.) is similar to those in Archetti et al. (2007) and Solyalı and Süral (2008), the only difference being the formulations used and the absence of an initial upper bound in BC(.).

4. Computational Results

We have performed computational experiments on the test instances of Abdelmaguid et al. (2009) and on newly generated instances in order to assess the effectiveness of the formulation given in Abdelmaguid and Dessouky (2006), and Abdelmaguid et al. (2009), referred to as *ADO*, and of our branch-and-cut algorithms, BC(NF), BC($RF1$) and BC($RF2$). We have also analyzed the price of robustness with regard to its effect on CPU times and objective values. We present the computational results in two sections. In Section 4.1, we give the results for the nominal case of the RIRP, and we present our results for the RIRP in Section 4.2. All formulations and algorithms have been coded in C++ using the Concert Technology of CPLEX 10.11 and solved by CPLEX 10.11. All computational experiments were performed on a workstation with lx24-amd computer architecture and 2 GB memory running under Linux. A time limit of two hours was imposed for the solution of any instance.

4.1. Results for the Nominal Case

We have conducted extensive computational experiments on the instances generated by Abdelmaguid et al. (2009) which include the instances generated by Abdelmaguid and Dessouky (2006). Abdelmaguid et al. (2009) consider three scenarios. The first and second scenarios involve instances with five, 10 and 15 customers, five and seven periods, and one and two vehicles. The second scenario instances have transportation costs twice as large as in the first scenario, and a tighter average daily demand over vehicle capacity ratio than in the first scenario instances. The aim is to make backlogging decisions economical. Third scenario instances involve 20, 25 and 30 customers, seven periods, and two vehicles, and have a medium average daily demand over vehicle capacity ratio which is in the middle of the first two scenarios. For each parameter combination, five instances were generated. For detailed information on the instances, see Abdelmaguid et al. (2009). We follow the naming convention used in Abdelmaguid et al. (2009), i.e. $S\#$-NTV-R, where $S\#$ is the scenario number, N is the number of customers, T is the length of planning horizon, V is the number of vehicles and R is the instance number. Because a single vehicle is considered in this paper, we use only the single vehicle instances in the first two scenarios, and adapt the instances of the third scenario to the single vehicle case. Thus, we have 75 instances in total. Since Abdelmaguid and Dessouky (2006) and Abdelmaguid et al. (2009) consider a storage capacity S_i at each customer i, we add constraints $\sum_{k=1}^{t}(\sum_{l=1}^{T} \bar{d}_{il} w_{ikl} - \bar{d}_{ik}) \leq S_i$ for $i \in \mathcal{M}, t \in \mathcal{T}$ to NF so as to compare ADO and BC(NF) under the same conditions. Note that left-hand side of these constraints denote the inventory level of customer i at the end of period t.

Computational results for scenarios 1, 2 and 3 are presented in Tables 1–3, respectively. In Tables 1 and 2, column 1 indicates the name of the instance, column 2 the best upper bound (i.e. UB) obtained by *ADO*, column 3 the best lower bound (i.e. LB) obtained by *ADO*, and column 4 the percentage gap between UB and LB (i.e. %Gap = 100(UB–LB)/LB). Note that UB = LB means that optimality has been proved. Columns 5 and 7 show the CPU time in seconds needed to solve the instances to optimality by *ADO* and BC(NF) respectively unless the imposed two-hour time limit has been reached. Columns 6 and 8 give the number of nodes explored by *ADO* and BC(NF), respectively. Finally, columns 9–12 show the optimal objective value, the total transportation cost including the fixed vehicle dispatching costs, the total inventory holding cost and the total inventory shortage cost incurred in the optimal solution, respectively.

Results presented in Tables 1 and 2 reveal that our branch-and-cut algorithm BC(NF) is far superior to *ADO*. While *ADO* cannot solve instances with more than five customers to optimality in two-hour time limit, BC(NF) solves all of the first scenario instances within a second and all of the second scenario instances within six minutes. Since it is clear from Tables 1 and 2 that *ADO* cannot solve instances involving more than five customers, we did not try to solve the third scenario instances by *ADO*.

Table 3 gives the results for the third scenario instances solved by BC(NF). All column descriptions are the same as in Tables 1 and 2, except that the last three columns in Table 3 indicate the cost components of the best upper bound if optimality cannot be proved. Results show that BC(NF) is able to solve all but one of the third scenario instances to optimality within the time limit. Although a single instance could not be solved to optimality, the remaining percentage gap between the UB and LB for that instance is 0.36%, which is quite small.

4.2. Results for the RIRP

We adapt the instances described in Section 4.1 for the RIRP by defining three different levels for the maximum deviation of demand from its nominal value. Specifically, we first generate new instances by halving the vehicle capacity (C) in the first scenario instances and by doubling it in the second and third scenario instances, and then for each instance described in Section 4.1 we generate three instances with $\hat{d}_{it}$ set to 1%, 2.5% and 5% of $\bar{d}_{it}$, respectively. Our aim in generating new instances by changing the capacity level is to have a wide range for the ratio of average daily demand over vehicle capacity. Thus, we have 450 RIRP instances besides the 150 instances for the nominal case. In $RF1$, we set the "budget of uncertainty" parameter Γ_t equal to Γ for each $t \in \{0\} \cup \mathcal{T}$ since the same expression is constrained by Γ_t in the maximization problems of (17)

Table 1 Results for the First Scenario Instances

	ADO					BC(NF)					
Problem	UB	LB	%Gap	Seconds	Nodes	Seconds	Nodes	Total	Transp	Hold	Short
1-0551-1	205.84	205.84	0.00	24.0	14191	0.1	24	205.84	134	71.84	0.00
1-0551-2	150.74	150.74	0.00	5.6	5109	0.0	0	150.74	105	45.74	0.00
1-0551-3	186.60	186.60	0.00	7.4	3215	0.0	0	186.60	138	48.60	0.00
1-0551-4	200.80	200.80	0.00	9.8	2969	0.1	34	200.80	120	80.80	0.00
1-0551-5	184.80	184.80	0.00	51.1	30057	0.1	17	184.80	136	48.80	0.00
1-0571-1	278.96	278.96	0.00	3540.8	1274380	0.2	40	278.96	186	92.96	0.00
1-0571-2	268.68	268.68	0.00	1442.3	237897	0.2	18	268.68	162	106.68	0.00
1-0571-3	273.07	273.07	0.00	1253.1	290063	0.1	1	273.07	198	75.07	0.00
1-0571-4	312.25	312.25	0.00	109.2	10119	0.2	11	312.25	201	111.25	0.00
1-0571-5	310.98	310.98	0.00	2004.0	168960	0.2	42	310.98	228	82.98	0.00
1-1051-1	323.65	304.77	6.20	7209.3	584901	0.0	0	323.65	223	100.65	0.00
1-1051-2	276.62	261.81	5.66	7211.8	938601	0.1	0	275.17	191	84.17	0.00
1-1051-3	300.69	291.26	3.24	7211.7	1196501	0.1	0	300.69	210	90.69	0.00
1-1051-4	282.09	247.88	13.80	7212.4	874201	0.1	0	280.13	192	88.13	0.00
1-1051-5	249.63	227.63	9.67	7214.2	1174301	0.2	4	249.63	180	69.63	0.00
1-1071-1	448.80	411.44	9.08	7230.4	251772	0.4	0	448.80	322	126.80	0.00
1-1071-2	417.07	382.32	9.09	7232.6	274001	0.4	1	416.27	288	128.27	0.00
1-1071-3	458.07	409.99	11.73	7241.1	278801	0.5	6	457.61	303	154.61	0.00
1-1071-4	461.40	424.38	8.72	7224.6	309001	0.1	0	461.40	319	142.40	0.00
1-1071-5	396.07	354.12	11.85	7212.8	385001	0.2	0	395.96	265	130.96	0.00
1-1551-1	392.11	337.37	16.23	7253.2	434701	0.3	0	392.11	245	147.11	0.00
1-1551-2	348.76	298.88	16.69	7243.9	568701	0.2	0	348.76	217	131.76	0.00
1-1551-3	384.30	332.91	15.44	7243.0	365998	0.5	3	384.30	229	155.30	0.00
1-1551-4	369.80	309.70	19.41	7225.5	470930	0.6	0	366.80	251	115.80	0.00
1-1551-5	368.16	320.21	14.97	7239.8	404701	0.5	1	366.16	236	130.16	0.00
1-1571-1	523.57	446.67	17.22	7246.9	262606	0.6	0	523.57	343	180.57	0.00
1-1571-2	529.01	447.84	18.12	7224.1	206101	0.9	0	525.15	346	179.15	0.00
1-1571-3	485.02	399.29	21.47	7229.1	536654	0.9	0	479.02	300	179.02	0.00
1-1571-4	542.65	464.30	16.87	7220.1	168231	1.0	0	529.95	346	183.95	0.00
1-1571-5	512.48	431.16	18.86	7234.1	144401	0.6	0	512.48	336	176.48	0.00
Average	**348.09**	**315.89**	**8.81**	**5100.2**	**395568.8**	**0.3**	**6.7**	**347.01**	**231.7**	**115.34**	**0.00**

and (18). We find the Γ value for each instance using (4) such that constraints (9) may be violated with probability of at most 1%.

Computational results for the RIRP instances are presented in Tables 4 and 5. In Table 4, columns 1–5 indicate the vehicle capacity level (L for low, H for high), the scenario number ($S\#$), the number of customers (N), the length of the planning horizon (T), and the percentage of average daily demand over vehicle capacity (Ratio) for the nominal case. Note that one can obtain the 'Ratio' for any maximum demand deviation percentage by multiplying one plus the corresponding percentage with the 'Ratio' for the nominal case. Columns '1%', '2.5%' and '5%' indicate the CPU time in seconds in the columns 'Seconds' or the percentage increase in total cost with respect to the nominal case in the columns 'Price of Robustness' for the corresponding branch-and-cut (BC($RF1$) or BC($RF2$)) when the maximum deviation percentage for each demand is 1%, 2.5% and 5%, respectively. In addition to the columns 'Seconds' and algorithms 'BC($RF1$)' and 'BC($RF2$)', Table

Table 2 Results for the Second Scenario Instances

	ADO					BC(NF)					
Problem	UB	LB	%Gap	Seconds	Nodes	Seconds	Nodes	Total	Transp	Hold	Short
2-0551-1	649.80	649.80	0.00	54.2	63490	0.1	81	649.80	478	6.27	165.53
2-0551-2	468.00	468.00	0.00	1.1	1103	0.1	1	468.00	459	9.00	0.00
2-0551-3	400.00	400.00	0.00	12.9	11110	0.1	26	400.00	339	17.91	43.09
2-0551-4	475.29	475.29	0.00	3.3	3538	0.1	22	475.29	445	5.11	25.18
2-0551-5	426.01	426.01	0.00	20.1	11903	0.1	49	426.01	374	18.35	33.66
2-0571-1	522.97	522.97	0.00	756.8	76649	1.0	330	522.97	463	59.97	0.00
2-0571-2	557.89	557.89	0.00	9.6	5359	0.2	11	557.89	406	22.57	129.32
2-0571-3	434.86	434.86	0.00	57.4	22299	0.3	56	434.86	368	29.80	37.06
2-0571-4	536.42	536.42	0.00	279.5	70951	0.4	97	536.42	481	50.38	5.04
2-0571-5	498.08	498.08	0.00	696.8	155111	0.5	115	498.08	439	39.64	19.44
2-1051-1	523.66	506.81	3.33	7211.7	617501	0.7	65	523.66	480	32.11	11.55
2-1051-2	485.84	433.15	12.16	7217.4	494701	2.2	232	480.70	418	54.34	8.36
2-1051-3	698.48	667.21	4.69	7221.8	2542901	0.3	16	698.48	541	3.60	153.88
2-1051-4	456.00	447.55	1.89	7212.0	1814801	1.7	222	456.00	431	25.00	0.00
2-1051-5	578.03	569.56	1.49	7205.2	746516	1.3	172	578.03	517	33.21	27.82
2-1071-1	771.52	729.90	5.70	7211.2	496601	13.3	979	771.52	707	31.22	33.30
2-1071-2	805.57	720.22	11.85	7212.4	915401	8.8	694	805.24	726	41.56	37.68
2-1071-3	719.42	667.01	7.86	7234.0	176062	9.5	591	717.84	635	82.84	0.00
2-1071-4	864.20	802.87	7.64	7215.9	274399	12.9	948	864.20	777	66.69	20.51
2-1071-5	752.69	713.32	5.52	7207.8	664920	5.3	345	752.69	683	43.61	26.08
2-1551-1	782.49	750.80	4.22	7250.8	510201	7.1	285	782.49	698	20.63	63.86
2-1551-2	757.69	724.76	4.54	7220.9	657301	3.2	89	757.69	742	15.69	0.00
2-1551-3	722.82	677.48	6.69	7224.3	914201	3.0	92	722.82	625	20.64	77.18
2-1551-4	681.32	604.95	12.63	7248.8	312080	16.1	712	680.22	626	42.88	11.34
2-1551-5	989.73	960.22	3.07	7224.5	880773	1.2	27	989.73	815	4.61	170.12
2-1571-1	861.73	754.60	14.20	7231.3	129401	43.0	963	848.18	719	129.18	0.00
2-1571-2	761.60	649.36	17.28	7231.4	84101	62.1	1035	749.66	606	143.66	0.00
2-1571-3	956.22	807.50	18.42	7242.7	99401	267.6	6680	924.91	820	104.91	0.00
2-1571-4	926.28	782.36	18.40	7234.3	130201	330.8	9240	922.56	833	89.56	0.00
2-1571-5	1206.18	1131.96	6.56	7247.2	298901	39.6	754	1206.18	937	41.25	227.93
Average	**675.69**	**635.70**	**5.60**	**4879.9**	**439395.9**	**27.8**	**831.0**	**673.40**	**586.3**	**42.87**	**44.26**

5 contains columns for 'BC(NF)', for '%Gap' and for 'R' which indicates the instance number. Note that Table 4 presents results averaged over five instances for each parameter combination, whereas Table 5 gives the detailed results for each difficult RIRP instance, in particular those that could not be solved to optimality within the two-hour time limit.

Results about CPU time show that BC($RF1$) is slower than both BC(NF) and BC($RF2$), which is expected since the incorporation of robustness using the "budget of uncertainty" approach weakens the strength of the nominal formulation, as discussed in Section 3, besides having a larger number of variables and constraints in $RF1$. BC($RF2$), on the other hand, is not negatively affected as much as BC($RF1$) by the incorporation of robustness since $RF2$ is actually a nominal formulation with modified demands. Even so, some instances that could not be solved to optimality by BC($RF1$) could be solved by BC($RF2$). In general, as the maximum deviation percentage increases, it becomes more difficult to solve the corresponding instances with BC($RF1$)

Table 3 Results for the Third Scenario Instances with BC(NF)

Problem	Seconds	Nodes	UB	LB	Transp	Hold	Short
3-2071-1	304.7	2530	481.81	481.81	384	97.81	0.00
3-2071-2	121.2	1142	450.37	450.37	342	108.37	0.00
3-2071-3	278.3	2317	481.04	481.04	377	104.04	0.00
3-2071-4	78.3	461	463.19	463.19	354	109.19	0.00
3-2071-5	996.1	9374	607.72	607.72	588	16.77	2.95
3-2571-1	1313.4	6159	557.81	557.81	490	67.81	0.00
3-2571-2	1357.7	5416	592.96	592.96	508	81.18	3.78
3-2571-3	781.9	3522	622.38	622.38	542	80.38	0.00
3-2571-4	639.4	1980	554.41	554.41	461	90.26	3.15
3-2571-5	1730.9	6892	566.54	566.54	479	87.54	0.00
3-3071-1	1377.0	2641	610.57	610.57	525	74.37	11.20
3-3071-2	1948.8	4373	590.37	590.37	513	71.19	6.18
3-3071-3	1708.3	4210	668.09	668.09	606	59.01	3.08
3-3071-4	5163.9	12828	664.30	664.30	606	55.15	3.15
3-3071-5	7206.8	14305	667.60	665.18	631	36.60	0.00
Average	**1667.1**	**5210.0**	**571.94**	**571.78**	**493.7**	**75.98**	**2.23**

and BC($RF2$). This is mainly due to having a more capacity constrained problem when the maximum deviation percentage increases. Although some instances could not be solved to optimality, the remaining percentage gap figures depicted by %Gap columns in Table 5 for those instances are quite small.

Results about the price of robustness reveal that both $RF1$ and $RF2$ yields uncertainty-immunized solutions at the expense of a small percent increase in total cost with respect to the nominal case when the percent average daily demand over vehicle capacity (Ratio) is not tight. The price one has to pay for robustness is higher, in particular, when the tranportation cost is higher (second scenario) and when the Ratio is tight ($C = H$). The explanation for this result is that ensuring feasibility with tight capacity constraints leads to higher cost routes amplified by a higher transportation cost and costly unavoidable backlogging of some of the demand. $RF1$ in general provides better results than $RF2$ in terms of the total cost except some instances that could not be solved to optimality by $RF1$. Note that the percentage increase in total cost with respect to the nominal case for the robust formulations is computed as $100(\text{UB}_{\text{BC}(.)}–\text{LB}_{\text{BC}(NF)})/\text{LB}_{\text{BC}(NF)}$, where $\text{UB}_{\text{BC}(.)}$ denotes the best upper bound found by BC($RF1$) or BC($RF2$), and $\text{LB}_{\text{BC}(NF)}$ denotes the best lower bound obtained by BC(NF). As expected, the price of robustness also increases when the maximum deviation percentage becomes larger.

Table 4 Results for the RIRP Instances

C	$S\#$	N	T	Ratio	BC($RF1$)						BC($RF2$)					
					Seconds			Price of Robustness			Seconds			Price of Robustness		
					1%	2.5%	5%	1%	2.5%	5%	1%	2.5%	5%	1%	2.5%	5%
1	1	5	5	37.1	0.1	0.1	0.1	0.31	0.82	1.61	0.1	0.0	0.0	0.32	0.85	1.66
			7	38.0	0.3	0.3	0.4	0.45	1.09	1.97	0.2	0.2	0.2	0.51	1.25	2.20
		10	5	37.7	0.1	0.1	0.1	0.27	0.67	1.34	0.1	0.1	0.1	0.30	0.76	1.51
			7	38.5	0.5	0.6	0.7	0.23	0.57	1.14	0.2	0.2	0.2	0.31	0.78	1.57
		15	5	37.5	0.6	0.8	0.9	0.27	0.67	1.34	0.4	0.5	0.4	0.36	0.89	1.77
			7	37.3	1.2	1.6	1.9	0.23	0.58	1.15	0.8	0.7	0.7	0.35	0.88	1.75
	2	5	5	47.7	0.2	0.3	0.2	1.16	2.47	3.49	0.1	0.1	0.1	1.17	2.48	3.52
			7	40.3	0.6	0.7	0.9	0.67	2.39	4.32	0.3	0.4	0.4	0.71	2.50	4.48
		10	5	44.7	1.5	1.1	1.5	0.21	0.56	1.63	0.8	0.8	0.7	0.25	0.79	1.87
			7	45.2	7.1	6.2	6.0	0.21	0.68	1.36	3.1	2.8	2.6	0.26	0.82	1.66
		15	5	46.3	5.0	4.9	7.1	0.34	0.75	1.76	3.0	3.7	3.7	0.42	0.93	2.28
			7	45.6	12.8	10.8	11.4	0.35	0.65	1.25	6.6	5.7	6.6	0.45	0.89	1.85
	3	20	7	41.7	30.6	48.4	52.5	0.23	0.80	1.43	17.0	33.5	27.5	0.35	1.11	2.02
		25	7	44.3	142.3	161.2	134.5	0.18	0.49	0.98	80.0	90.4	97.7	0.29	0.78	1.61
		30	7	46.1	411.2	422.9	502.4	0.28	0.63	1.31	218.1	252.2	280.2	0.42	0.97	2.22
2	1	5	5	74.2	0.2	0.2	0.3	1.11	2.57	4.51	0.1	0.1	0.1	1.11	2.57	4.51
			7	76.0	0.7	0.7	0.6	0.45	1.05	2.94	0.4	0.3	0.2	0.46	1.07	2.97
		10	5	75.4	1.7	2.0	2.0	0.41	1.06	2.22	0.9	1.2	1.0	0.44	1.12	2.30
			7	77.0	8.0	9.5	12.3	0.49	1.18	2.64	4.1	4.9	7.1	0.55	1.31	2.91
		15	5	74.9	5.4	7.4	6.9	0.23	0.62	1.23	3.3	3.1	3.3	0.25	0.67	1.31
			7	74.6	51.0	49.3	89.2	0.27	0.66	1.33	29.3	24.0	25.1	0.33	0.79	1.52
	2	5	5	95.3	0.2	0.1	0.1	1.69	6.17	15.53	0.1	0.1	0.1	1.69	6.17	15.59
			7	80.7	0.9	0.9	1.0	2.16	6.64	14.15	0.5	0.5	0.4	2.16	6.65	14.18
		10	5	89.4	1.9	1.8	2.4	2.37	7.67	19.89	1.3	1.3	1.2	2.37	7.68	19.92
			7	90.4	14.9	15.2	9.9	2.16	6.38	13.95	7.3	9.6	4.1	2.16	6.38	13.96
		15	5	92.7	8.4	6.6	5.7	4.98	13.00	30.69	5.5	3.8	2.7	4.98	13.00	30.69
			7	91.2	166.8	224.9	348.9	1.47	4.58	12.48	94.0	148.1	187.0	1.48	4.61	12.53
	3	20	7	83.4	355.7	397.6	550.5	0.84	4.81	15.26	171.1	183.4	267.1	0.88	4.93	15.50
		25	7	88.6	1875.7	3043.5	3385.5	1.13	2.92	6.18	1085.8	1591.3	1952.8	1.16	2.98	6.30
		30	7	92.2	5879.4	6514.8	6560.6	2.75	7.53	23.95	4031.1	4252.2	4930.0	2.72	7.48	24.06
Average					**299.5**	**364.5**	**389.9**	**0.93**	**2.69**	**6.43**	**192.2**	**220.5**	**260.1**	**0.97**	**2.80**	**6.67**

Until now, we have set Γ such that probability of constraint violation is at most 1%. To observe the impact of the chosen Γ, thus the probability of constraint violation, on the price of robustness, we present computational results for the instances with $\hat{d}_{it}$ set to 2.5% of $\bar{d}_{it}$ in Table 6, obtained

Table 5 Detailed Results for the difficult RIRP Instances*

N	R	Seconds							%Gap						
		BC(NF)	BC($RF1$)			BC($RF2$)			BC(NF)	BC($RF1$)			BC($RF2$)		
			1%	2.5%	5%	1%	2.5%	5%		1%	2.5%	5%	1%	2.5%	5%
25	1	1308.4	1430.7	2128.4	2958.5	1258.6	1233.3	1723.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	2	1370.7	2494.5	2416.5	7213.6	1720.0	715.1	5180.3	0.00	0.00	0.00	0.25	0.00	0.00	0.00
	3	787.7	3724.3	7020.6	3008.9	1412.3	4069.8	1056.1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	4	643.4	952.3	2432.9	2821.1	415.1	1373.8	1199.7	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	5	1731.9	776.9	1219.3	925.7	622.8	564.2	604.9	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30	1	1365.8	4008.4	7209.2	4632.3	2077.8	3858.8	2144.9	0.00	0.00	0.27	0.00	0.00	0.00	0.00
	2	1939.3	3766.7	7210.2	7143.6	1668.0	2732.3	7204.9	0.00	0.00	0.32	0.00	0.00	0.00	0.23
	3	1706.3	7208.2	3740.0	7206.4	4093.6	1102.8	7204.8	0.00	0.02	0.00	0.52	0.00	0.00	0.38
	4	5205.6	7207.2	7208.1	7205.8	5646.3	7208.4	7204.4	0.00	0.58	1.56	0.85	0.00	0.81	0.16
	5	7209.5	7206.4	7206.6	6615.0	6669.9	6358.7	891.0	0.29	0.49	0.61	0.00	0.00	0.00	0.00
Average		**2326.8**	**3877.6**	**4779.2**	**4973.1**	**2558.4**	**2921.7**	**3441.4**	**0.03**	**0.11**	**0.27**	**0.16**	**0.00**	**0.08**	**0.08**

* Instances in the last two rows of Table 4.

using BC($RF1$). In Table 6, columns 1–5 are the same as in Table 4, and columns 6–10 indicate the percentage increase in total cost with respect to the nominal case for 0%, 1%, 5%, 10% and >50% chance of constraint violation, respectively. Note that setting the Γ to $N\text{x}T$ (i.e. worst case) and zero (i.e. nominal case) give 0% and >50% chance of constraint violation, respectively. Results presented in Table 6 show that the price of robustness slightly decreases as the probability of constraint violation increases from 0% to 10%. The results also reveal that the price of robustness is less than 3% on average for the range of 0%–10% chance of constraint violation, which is quite satistactory considering that the solutions obtained under the nominal case are highly likely to be infeasible (i.e. more than 50%), whereas the solutions obtained by means of BC($RF1$) are feasible with a high probability at the expense of having a slightly greater total cost.

5. Conclusions

We have considered, for the first time, a robust inventory routing problem under polyhedral demand uncertainty and proposed two robust MIP formulations which were implemented within a branch-and-cut algorithm. Computational results on instances adapted from the literature have revealed that our robust formulations can solve to optimality instances with up to 30 customers and seven periods within reasonable times. The robust solutions thus obtained provide immunization against uncertainty with a slight increase in total cost compared to the nominal case, especially when the average daily demand over vehicle capacity ratio is low, whereas the price of robustness is larger when the average daily demand over vehicle capacity ratio is high. Moreover, we have proposed a

Table 6 **Trade-off between Price of Robustness and Probability of Constraint Violation on Instances with 2.5% maximum demand deviation***

C	$S\#$	N	T	Ratio	Price of Robustness				
					0%	1%	5%	10%	>50%
1	1	5	5	37.1	0.85	0.82	0.67	0.58	0.00
			7	38.0	1.25	1.09	0.91	0.79	0.00
		10	5	37.7	0.76	0.67	0.54	0.45	0.00
			7	38.5	0.78	0.57	0.45	0.37	0.00
		15	5	37.5	0.89	0.67	0.53	0.44	0.00
			7	37.3	0.88	0.58	0.45	0.37	0.00
	2	5	5	47.7	2.48	2.47	2.40	2.35	0.00
			7	40.3	2.50	2.39	2.27	2.12	0.00
		10	5	44.7	0.79	0.56	0.45	0.39	0.00
			7	45.2	0.82	0.68	0.56	0.50	0.00
		15	5	46.3	0.93	0.75	0.64	0.55	0.00
			7	45.6	0.89	0.65	0.55	0.49	0.00
	3	20	7	41.7	1.11	0.80	0.59	0.39	0.00
		25	7	44.3	0.78	0.49	0.38	0.30	0.00
		30	7	46.1	0.97	0.63	0.50	0.43	0.00
2	1	5	5	74.2	2.57	2.57	2.50	2.44	0.00
			7	76.0	1.07	1.05	0.97	0.91	0.00
		10	5	75.4	1.12	1.06	0.92	0.78	0.00
			7	77.0	1.31	1.18	1.06	0.98	0.00
		15	5	74.9	0.67	0.62	0.55	0.48	0.00
			7	74.6	0.79	0.66	0.55	0.47	0.00
	2	5	5	95.3	6.17	6.17	6.15	6.08	0.00
			7	80.7	6.65	6.64	6.62	6.60	0.00
		10	5	89.4	7.68	7.67	7.57	7.34	0.00
			7	90.4	6.38	6.38	6.36	6.27	0.00
		15	5	92.7	13.00	13.00	12.82	11.66	0.00
			7	91.2	4.61	4.58	4.53	4.41	0.00
	3	20	7	83.4	4.93	4.81	4.66	4.07	0.00
		25	7	88.6	2.98	2.92	2.75^{1}	2.53	0.00
		30	7	92.2	7.46^{2}	7.47^{4}	6.99^{1}	6.22^{2}	0.00^{1}
Average					**2.80**	**2.69**	**2.56**	**2.39**	**0.00**

*Superscript numbers in some of the entries indicate the number of instances that could not be solved to optimality within the two-hour time limit.

new strong formulation within the branch-and-cut algorithm for the nominal case of the problem (i.e. a deterministic IRP with backlogging), which is able to optimally solve instances six times larger than the only previously available MIP formulation (Abdelmaguid and Dessouky 2006, Abdelmaguid et al. 2009).

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