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Georges Dionne
Jingyuan Li

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Bureaux de Montréal :

Université de Montréal
C.P. 6128, succ. Centre-ville
Montréal (Québec)
Canada H3C 3J7
Téléphone : 514 343-7575
Télécopie : 514 343-7121

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Université Laval
2325, de la Terrasse, bureau 2642
Québec (Québec)
Canada G1V 0A6
Téléphone : 418 656-2073
Télécopie : 418 656-2624

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First-Order (Conditional) Risk Aversion, Backround Risk and Risk Diversification

Georges Dionne[1,*], Jingyuan Li[2]

[1] Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation (CIRRELT) and Canada Research Chair in Risk Management, HEC Montréal, 3000, Côte-Sainte-Catherine, Montréal, Canada H3T 2A7

[2] School of Management, Huazhong University of Science and Technology, Wuhan 430074, China

Abstract. In the literature, utility functions in the expected utility class are generically limited to second-order (conditional) risk aversion, while non-expected utility functions can exhibit either. First-order or second-order (conditional) risk aversion. This paper extends the concepts of orders of conditional risk aversion to orders of conditional dependent risk aversion. We show that first-order conditional dependent risk aversion is consistent with the framework of the expected utility hypothesis. We relate our results to risk diversification and provide additional insights into its application in different economic and finance examples.

Keywords. Expected utility theory, first-order conditional dependent risk aversion, background risk, risk diversification.

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* Corresponding author: Georges.Dionne@cirrelt.ca

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1 Introduction

The concepts of second-order and first-order risk aversion were coined by Segal and Spivak (1990). For an actuarially fair random variable $\tilde{\varepsilon}$, second-order risk aversion means that the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k^2 as $k \to 0$. Under first-order risk aversion, the risk premium is proportional to k. Loomes and Segal (1994) extend this notion to preferences about uninsured events, such as independent additive background risks. They introduce the concept of orders of conditional risk aversion. We define $\tilde{y}$ as an independent additive risk. The conditional risk premium is defined as the amount of money the decision maker is willing to pay to avoid $\tilde{\varepsilon}$ in the presence of $\tilde{y}$. The preference relation satisfies first-order conditional risk aversion if the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k as $k \to 0$. It satisfies second-order conditional risk aversion if the risk premium is proportional to k^2.

To the best of our knowledge, utility functions in the von Neumann-Morgenstern expected utility class can generically exhibit only second-order conditional risk aversion, while non-expected utility functions can exhibit either first-order or second-order (conditional) risk aversion[1]. First-order (conditional) risk aversion implies that small risks matter. Since expected utility theory is limited to second-order (conditional) risk aversion, it cannot take into account many real world results. For example, Epstein and Zin (1990) find that first-order risk aversion can help to resolve the equity premium puzzle. Schlesinger (1997) uses first-order risk aversion to explain why full insurance coverage may be optimal even when there is a positive premium loading. Further applications of first-order risk aversion appear in Schmidt (1999), Barberis et al. (2001), Barberis et al. (2006), and Chapman and Polkovnichenko (2009), among others.

In this paper, we extend the concept of order conditional risk aversion to order conditional dependent risk aversion, for which $\tilde{\varepsilon}$ and $\tilde{y}$ are dependent and $\tilde{y}$ may enter the agent's utility function in a rather arbitrary manner. We investigate whether first-order conditional dependent risk aversion appears in the framework of the expected utility hypothesis. The general answer to the above question is positive with some restrictions.

We propose conditions on the stochastic structure between $\tilde{\varepsilon}$ and $\tilde{y}$ that guarantee first-order conditional dependent risk aversion for expected utility agents with a certain type of risk preference, i.e., correlation aversion. Eeckhoudt et al. (2007) provide an economic interpretation

[1]See Eeckhoudt et al. (2005), Chapter 13, for more discussion.

of correlation aversion: a higher level of the background variable mitigates the detrimental effect of a reduction in wealth. It turns out that the concept of expectation dependence, proposed by Wright (1987), is the key element to such stochastic structure. Further, the more information that we possess about the sign of higher cross derivatives of the utility function,[2] the weaker dependence conditions on distribution we need. These weaker dependence conditions, which demonstrate the applicability of a weak version of expectation dependence (called N^{th}-order expectation dependence (Li, 2011)), induce weaker dependence conditions between $\tilde{\varepsilon}$ and $\tilde{y}$, to guarantee first-order conditional dependent risk aversion.

Risk premium is an important concept in economics and finance. Intuition suggests that the risk premium for a diversified risk should relate to the number of trials n. We investigate a correlation averse risk premium for a naive diversified risk in the presence of a dependent background risk. The naive diversified risk is defined as one in which a fraction $\frac{1}{n}$ of wealth is allocated to each of the n risks. In the absence of a dependent background risk, the population mean value of the naive diversified risk approximates the expected value. The "Law of Large Numbers" states that the risk premium converges to zero when n is large. This is often called the benefit of diversification. Given that, in real life, an agent can diversify wealth only on a limited number of risks, a natural question is how small is the risk premium in the presence of a dependent background risk? In other words, what is the convergence rate or approximation error? Our results show that the convergence rate is at the order of $\frac{1}{n^2}$ in the presence of an independent background risk compared with $\frac{1}{n}$ in the presence of a dependent background risk. This difference is a quantitative statement on the benefice of diversification which provides information on how background risk affects the risk premium of a a naive diversified risk. This result also provides additional insights regarding previous results on insurance supply, public investment decisions, naive diversified portfolio pricing, bank lending and lottery business in the presence of a dependent background risk.

The paper proceeds as follows. Section 2 sets up the model. Section 3 discusses the concept of orders of conditional risk aversion. Section 4 investigates the orders of conditional dependent risk aversion. Section 5 discusses some weaker dependence conditions. Section 6 applies the results to different economic and financial examples. Section 7 concludes this paper.

[2]Eeckhoudt et al. (2007) provide a context-free interpretation for the sign of higher cross derivatives of the utility function.

2 The model

We consider an agent whose preference for a random wealth, $\tilde{w}$, and a random outcome, $\tilde{y}$, can be represented by a bivariate expected utility function. Let $u(w, y)$ be the utility function, and let $u_1(w, y)$ denote $\frac{\partial u}{\partial w}$ and $u_2(w, y)$ denote $\frac{\partial u}{\partial y}$, and follow the same subscript convention for higher derivatives $u_{11}(w, y)$ and $u_{12}(w, y)$ and so on. We assume that all partial derivatives required for any definition exist. We make the standard assumption that $u_1 > 0$.

Let us assume that $\tilde{z} = k\tilde{\varepsilon}$. Parameter k can be interpreted as the size of the risk. One way to measure an agent's degree of risk aversion for $\tilde{z}$ is to ask her how much she is ready to pay to get rid of $\tilde{z}$. The answer to this question will be referred to as the risk premium $\pi(k)$ associated with that risk. For an agent with utility function u and non random initial wealth w, the risk premium, $\pi(k)$, must satisfy the following condition:

$$u(w + Ek\tilde{\varepsilon} - \pi(k), E\tilde{y}) = Eu(w + k\tilde{\varepsilon}, E\tilde{y}). \tag{1}$$

Segal and Spivak (1990) give the following definitions of first and second-order risk aversion:

Definition 2.1 *(Segal and Spivak 1990) The agent's attitude towards risk at w is of first order if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'(0) \neq 0$.*

Definition 2.2 *(Segal and Spivak 1990) The agent's attitude towards risk at w is of second order if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'(0) = 0$ but $\pi''(0) \neq 0$.*

They provide the following results linking properties of a utility function to its order of risk aversion given level of wealth w_0:

(a) If a risk averse von Neumann-Morgenstern utility function u is not differentiable at w_0 but has well-defined and distinct left and right derivatives at w_0, then the agent exhibits first-order risk aversion at w_0.

(b) If a risk averse von Neumann-Morgenstern utility function u is twice differentiable at w_0 with $u_{11} \neq 0$, then the agent exhibits second-order risk aversion at w_0.

Segal and Spivak (1997) point out that, if the von Neumann-Morgenstern utility function is increasing, then it must be differentiable almost everywhere, and one may therefore convincingly argue that non-differentiability is not often observed in the expected utility model. Therefore first-order risk aversion cannot be taken into account in this model.

3 Order of conditional risk aversion

Loomes and Segal (1994) introduced the order of conditional risk aversion by examining the characteristic of $\pi(k)$ in the presence of independent uninsured risks. For an agent with utility function u and initial wealth w, the conditional risk premium, $\pi_c(k)$, must satisfy the following condition:

$$Eu(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}). \tag{2}$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are independent.

Definition 3.1 *(Loomes and Segal 1994) The agent's attitude towards risk at w is first order conditional risk aversion if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi_c'(0) \neq 0$.*

Definition 3.2 *(Loomes and Segal 1994) The agent's attitude towards risk at w is second order conditional risk aversion if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi_c'(0) = 0$ but $\pi_c''(0) \neq 0$.*

It is obvious that the definitions of first and second order conditional risk aversion are more general than the definitions of first and second order risk aversion

We can extend the above definitions to the case $E\tilde{\varepsilon} \neq 0$. Since u is differentiable, fully differentiating (2) with respect to k yields

$$E\{[E\tilde{\varepsilon} - \pi_c'(k)]u_1(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y})\} = E[\tilde{\varepsilon}u_1(w + k\tilde{\varepsilon}, \tilde{y})]. \tag{3}$$

Since $\tilde{\varepsilon}$ and $\tilde{y}$ are independent, then

$$\pi_c'(0) = \frac{E\tilde{\varepsilon}Eu_1(w, \tilde{y}) - E[\tilde{\varepsilon}u_1(w, \tilde{y})]}{Eu_1(w, \tilde{y})} = 0. \tag{4}$$

Therefore, not only does $\pi_c(k)$ approach zero as k approaches zero, but also $\pi_c'(0) = 0$. This implies that, at the margin, accepting a small zero-mean risk has no effect on the welfare of risk-averse agents. This is an important property of expected-utility theory: "in the small", the expected-utility maximizers are risk neutral.

Using a Taylor expansion of π_c around $k = 0$, we obtain that

$$\pi_c(k) = \pi_c(0) + \pi_c'(0)k + O(k^2) = O(k^2). \tag{5}$$

This result is the Arrow-Pratt approximation, which states that the conditional risk premium is approximately proportional to the square of the size of the risk.

Within the von Neumann-Morgenstern expected-utility model, if the random outcome and the background risk are independent, then second-order conditional risk aversion relies on the assumption that the utility function is differentiable. Hence, with an independent background risk, utility functions in the von Neumann-Morgenstern expected utility class can generically exhibit only second-order conditional risk aversion and cannot explain the rejection of a small, independent, and actuarially favorable gamble.

4 Order of conditional dependent risk aversion

We now introduce the concept of order of conditional dependent risk aversion. For an agent with utility function u and initial wealth w, the conditional dependent risk premium, $\pi_{cd}(k)$, must satisfy the following condition:

$$Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}). \tag{6}$$

where $\tilde{\varepsilon}$ and $\tilde{y}$ are not necessarily independent[3].

Definition 4.1 *The agent's attitude towards risk at w is first order conditional dependent risk aversion if for every $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k)$.*

Definition 4.2 *The agent's attitude towards risk at w is second order conditional dependent risk aversion if for every $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k^2)$.*

$\pi_{cd}(k) - \pi_c(k)$ measures how dependence between risks affects risk premium. Second order conditional dependent risk aversion implies that, in the presence of a dependent background risk, small risk has no effect on risk premium, while first order conditional dependent risk aversion implies that, in the presence of a dependent background risk, small risk affects risk premium.

We denote $F(\varepsilon, y)$ and $f(\varepsilon, y)$ the joint distribution and density functions of $(\tilde{\varepsilon}, \tilde{y})$, respectively. $F_\varepsilon(\varepsilon)$ and $F_y(y)$ are the marginal distributions.

Wright (1987) introduces the following idea in the economic literature.

[3] In the statistical literature, the sequence b_k is at most of order k^λ, denoted as $b_k = O(k^\lambda)$, if for some finite real number $\Delta > 0$, there exists a finite integer K such that for all $k > K$, $|k^\lambda b_k| < \Delta$ (see, White 2000, p16).

Definition 4.3 *(Wright 1987) If*

$$ED(y) = [E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \leq y)] \geq 0 \ for \ all \ y, \tag{7}$$

$$and \ there \ is \ at \ least \ some \ y_0 \ for \ which \ a \ strong \ inequality \ holds,$$

then $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$. Similarly, $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ if (7) holds with the inequality sign reversed.

Wright (1987, p115) interprets negative first-degree expectation dependence as follows: "when we discover $\tilde{y}$ is small, in the precise sense that we are given the truncation $\tilde{y} \leq y$, our expectation of $\tilde{\varepsilon}$ is revised upward". This definition of dependence is useful for deriving an explicit value of $\pi_{cd}(k)$.

Lemma 4.4

$$\pi_{cd}(k) = -k\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w,y)F_y(y)dy}{Eu_1(w,\tilde{y})} + O(k^2). \tag{8}$$

Proof From the definition of $\pi_{cd}(k)$, we know that

$$Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}). \tag{9}$$

Differentiating with respect to k yields

$$\pi'_{cd}(k) = \frac{E\tilde{\varepsilon}Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) - E[\tilde{\varepsilon}u_1(w + k\tilde{\varepsilon}, \tilde{y})]}{Eu_1(w - \pi_{cd}(k), \tilde{y})}. \tag{10}$$

Since $\pi_{cd}(0) = 0$, we have

$$\pi'_{cd}(0) = \frac{E\tilde{\varepsilon}Eu_1(w,\tilde{y}) - E[\tilde{\varepsilon}u_1(w,\tilde{y})]}{Eu_1(w,\tilde{y})}. \tag{11}$$

Note that

$$E[\tilde{\varepsilon}u_1(w,\tilde{y})] = E\tilde{\varepsilon}Eu_1(w,\tilde{y}) + Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) \tag{12}$$

and the covariance can always be written as (see, Cuadras (2002), Theorem 1)

$$Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F(\varepsilon,y) - F_\varepsilon(\varepsilon)F_Y(y)]d\varepsilon du_1(w,y). \tag{13}$$

Since we can always write (see, e.g., Tesfatsion (1976), Lemma 1)

$$\int_{-\infty}^{\infty}[F_\varepsilon(\varepsilon|\tilde{y} \leq y) - F_\varepsilon(\varepsilon)]d\varepsilon = E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \leq y), \tag{14}$$

hence, by straightforward manipulations we find

$$\begin{aligned} Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F(\varepsilon,y)-F_\varepsilon(\varepsilon)F_y(y)]u_{12}(w_0,y)d\varepsilon dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F_\varepsilon(\varepsilon|\tilde{y}\leq y)-F_\varepsilon(\varepsilon)]d\varepsilon F_y(y)u_{12}(w,y)dy \\ &= \int_{-\infty}^{\infty}[E\tilde{\varepsilon}-E(\tilde{\varepsilon}|\tilde{y}\leq y)]F_y(y)u_{12}(w,y)dy \qquad (by\ (14)) \\ &= \int_{-\infty}^{\infty}ED(y)u_{12}(w,y)F_y(y)dy. \end{aligned} \tag{15}$$

Finally, we get

$$\pi'_{cd}(0) = -\frac{\int_{-\infty}^{\infty}ED(y)u_{12}(w,y)F_y(y)dy}{Eu_1(w,\tilde{y})}. \tag{16}$$

Using a Taylor expansion of π around $k=0$, we obtain that

$$\pi_{cd}(k) = \pi_{cd}(0) + \pi'_{cd}(0)k + O(k^2) = -k\frac{\int_{-\infty}^{\infty}ED(y)u_{12}(w,y)F_y(y)dy}{Eu_1(w,\tilde{y})} + O(k^2). \tag{17}$$

Q.E.D.

Lemma 4.4 shows the general condition for first order risk aversion. The condition involves two important concepts u_{12} and $ED(y)$. The sign of u_{12} indicates how this first element acts on utility u. Eeckhoudt et al. (2007) provide a context-free interpretation of the sign of u_{12}. They show that $u_{12} \leq 0$ is necessary and sufficient for “correlation aversion”, meaning that a higher level of the background variable mitigates the detrimental effect of a reduction in wealth. This condition involves the expectation dependence between two risks and the cross derivative of the utility function. It captures the welfare interaction between the two risks. The sign of the first-degree expectation dependence indicates whether the movements on background risk tend to reinforce the movements on wealth (positive first-degree expectation dependence) or to counteract them (negative first-degree expectation dependence). Lemma (4.4) allows a quantitative treatment of the direction and size of first-degree expectation dependence effect on first order risk aversion. To clarify this, consider the following cases: (1) assume the agent is correlation neutral ($u_{12} = 0$) or the background risk is independent ($ED(y) = 0$), then the agent's attitude towards risk is second order conditional dependent risk aversion; (2) Assume $u_{12} < 0$ and $ED(y) > 0$ ($ED(y) < 0$), then the agent's attitude towards risk is first order conditional dependent risk aversion and her marginal risk premium for a small risk is positive (negative) (i.e., $\lim_{k\to 0^+}\pi'_{cd}(k) > (<)0$).

From Lemma (4.4) and Equation (5), we obtain

Proposition 4.5 *(i) If $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$ and $u_{12} < 0$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$;*

(ii) If $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ and $u_{12} > 0$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$;

(iii) If $\tilde{\varepsilon}$ is positive expectation dependent on $\tilde{y}$ and $u_{12} > 0$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$;

(iv) If $\tilde{\varepsilon}$ is negative expectation dependent on $\tilde{y}$ and $u_{12} < 0$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$.

We consider two examples to illustrate Proposition 4.5.

Example 1. Consider the additive background risk case $u(x, y) = U(x + y)$. Here x may be the random wealth of an agent and y may be a random income risk which cannot be insured. Since $u_{12} < 0 \Leftrightarrow U'' < 0$, part (i) and (iv) of Proposition 4.5 implies that, if the agent is risk averse and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) > (<)\pi_c(k)$.

Example 2. Consider the multiplicative background risk case $u(x, y) = U(xy)$. Here x may be the random wealth of an agent and y may be a random interest rate risk which cannot be hedged. Since $u_{12} < 0 \Leftrightarrow -xy\frac{U''(xy)}{U'(xy)} > 1$ (relative risk aversion greater than 1), Proposition 4.5 implies that, (i) if $-xy\frac{U''(xy)}{U'(xy)} > 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) > (<)\pi_c(k)$; (ii) if $-xy\frac{U''(xy)}{U'(xy)} < 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) < (>)\pi_c(k)$.

5 First-order conditional dependent risk aversion and N^{th}-order expectation dependent background risk

Li (2011) considers the following weaker dependence: suppose $\tilde{y} \in [c, d]$, where c and d are finite. Rewriting $1^{th}ED(\tilde{x}|y) = FED(\tilde{x}|y)$, $2^{th}ED(\tilde{x}|y) = SED(\tilde{x}|y) = \int_c^y FED(\tilde{x}|t)F_y(t)dt$, and repeated integrals defined by

$$N^{th}ED(\tilde{x}|y) = \int_c^y (N-1)^{th}ED(\tilde{x}|t)dt, \ for \ N \geq 3. \tag{18}$$

Definition 5.1 *(Li 2011) If $m^{th}ED(\tilde{x}|d) \geq 0$, for $m = 2, ..., N-1$ and*

$$N^{th}ED(\tilde{x}|y) \geq 0 \ for \ all \ y \in [c, d], \tag{19}$$

then $\tilde{x}$ is positive N^{th}-order expectation dependent (NED) on $\tilde{y}$. The family of all distributions F satisfying (19) will be denoted by $\mathcal{H}_N$. Similarly, $\tilde{x}$ is negative N^{th}-order expectation dependent on $\tilde{y}$ if (19) holds with the inequality sign reversed, and the family of all negative N^{th}-order expectation dependent distributions will be denoted by $\mathcal{I}_N$.

From this definition, we know that $\mathcal{H}_N \supset \mathcal{H}_{N-1}$ and $\mathcal{I}_N \supset \mathcal{I}_{N-1}$. In the following lemma, we obtain the risk premium in the presence of an N^{th}-order expectation dependent background risk.

Lemma 5.2

$$\begin{aligned} & \pi_{cd}(k) \\ = & -k\frac{\sum_{m=2}^{N}(-1)^m u_{12^{(m-1)}}(w,d)m^{th}ED(\tilde{x}|d) + \int_c^d (-1)^{N+1}u_{12^{(N)}}(w,y)N^{th}ED(\tilde{x}|y)dy}{Eu_1(w,\tilde{y})} \\ & +O(k^2). \end{aligned} \tag{20}$$

Proof From (12) and (14), we know that

$$E[\tilde{\varepsilon}u_1(w,\tilde{y})] = E\tilde{\varepsilon}Eu_1(w,\tilde{y}) + Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) = E\tilde{\varepsilon}Eu_1(w,\tilde{y}) + \int_{-\infty}^{\infty} ED(y)u_{12}(w,y)F_y(y)dy. \tag{21}$$

We simply integrate the last term of (21) by parts again and again until we obtain:

$$\begin{aligned} Cov(\tilde{\varepsilon}, u_1(w,\tilde{y})) &= \sum_{m=2}^{N}(-1)^m u_{12^{(m-1)}}(w,d)m^{th}ED(\tilde{x}|d) \\ &+ \int_c^d (-1)^{N+1}u_{12^{(N)}}(w,y)N^{th}ED(\tilde{x}|y)dy, \ for N \geq 2. \end{aligned} \tag{22}$$

From (11), we have

$$\begin{aligned} & \pi'_{cd}(0) \\ = & \frac{E\tilde{\varepsilon}Eu_1(w,\tilde{y}) - E[\tilde{\varepsilon}u_1(w,\tilde{y})]}{Eu_1(w,\tilde{y})} \\ & -k\frac{\sum_{m=2}^{N}(-1)^m u_{12^{(m-1)}}(w,d)m^{th}ED(\tilde{x}|d) + \int_c^d (-1)^{N+1}u_{12^{(N)}}(w,y)N^{th}ED(\tilde{x}|y)dy}{Eu_1(w,\tilde{y})}. \end{aligned} \tag{23}$$

Using a Taylor expansion of π around $k=0$, we obtain that

$$\begin{aligned} & \pi_{cd}(k) && (24) \\ = \; & \pi_{cd}(0)+\pi'_{cd}(0)k+O(k^2) \\ = \; & -k\frac{\sum_{m=2}^{N}(-1)^m u_{12^{(m-1)}}(w,d)m^{th}ED(\tilde{x}|d)+\int_c^d(-1)^{N+1}u_{12^{(N)}}(w,y)N^{th}ED(\tilde{x}|y)dy}{Eu_1(w,\tilde{y})} \\ & +O(k^2). \end{aligned}$$

Q.E.D.

From Lemma (5.2) and Equation (5), we obtain

Proposition 5.3 *(i) If* $(\tilde{\varepsilon},\tilde{y})\in\mathcal{H}_N$ *and* $(-1)^m u_{12^{(m-1)}}\le 0$ *for* $m=1,2,...,N+1$*, then the agent's attitude towards risk is first order conditional dependent risk aversion and* $\pi_{cd}(k)-\pi_c(k)=|O(k)|$*;*

(ii) If $(\tilde{\varepsilon},\tilde{y})\in\mathcal{I}_N$ *and* $(-1)^m u_{12^{(m-1)}}\ge 0$ *for* $m=1,2,...,N+1$*, then the agent's attitude towards risk is first order conditional dependent risk aversion and* $\pi_{cd}(k)-\pi_c(k)=|O(k)|$*;*

(iii) If $(\tilde{\varepsilon},\tilde{y})\in\mathcal{H}_N$ *and* $(-1)^m u_{12^{(m-1)}}\ge 0$ *for* $m=1,2,...,N+1$*, then the agent's attitude towards risk is first order conditional dependent risk aversion and* $\pi_{cd}(k)-\pi_c(k)=-|O(k)|$*;*

(iv) If $(\tilde{\varepsilon},\tilde{y})\in\mathcal{I}_N$ *and* $(-1)^m u_{12^{(m-1)}}\le 0$ *for* $m=1,2,...,N+1$*, then the agent's attitude towards risk is first order conditional dependent risk aversion and* $\pi_{cd}(k)-\pi_c(k)=-|O(k)|$*.*

Eeckhoudt et al. (2007, p120) also provide an intuitive interpretation for the meaning of the sign of the higher order cross derivatives of utility function, $u_{12^{(k)}}$. For example, $u_{122}>0$ is a necessary and sufficient condition for "cross-prudence in wealth", meaning that higher wealth reduces the detrimental effect of the background risk. We consider two examples to illustrate Proposition 5.3.

Example 3. Consider the additive background risk case $u(x,y)=U(x+y)$. Since $(-1)^m u_{12^{(m-1)}}\le 0 \Leftrightarrow (-1)^m U^{(m)}\le 0$, parts (i) and (iv) of Proposition 4.5 imply that, if the agent is kth degree risk averse (See Ekern, 1980 and Eeckhoudt and Schlesinger, 2006 for more discussions of kth degree of risk aversion.) for $m=1,2,...,N+1$ and $\tilde{\varepsilon}$ is positive (negative) Nth expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k)>(<)\pi_c(k)$.

Example 4. Consider the multiplicative background risk case $u(x,y)=U(xy)$. Since

$$(-1)^m u_{12^{(m-1)}}\le 0 \Leftrightarrow (-1)^m xy\frac{U^{(m+1)}(xy)}{U^{(m)}(xy)}\ge m,\ for\ m=1,2,...,N+1 \qquad (25)$$

$$(multiplicative\ risk\ apportionment\ of\ order\ m\ for\ m = 1, 2, ..., N + 1)$$

(See Eeckhoudt et al., 2009, Wang and Li, 2010 and Chiu et al., 2010 for more discussions of multiplicative risk apportionment of order m.) Proposition 4.5 implies that, (i) if $(-1)^m xy \frac{U^{(m+1)}(xy)}{U^{(m)}(xy)} \geq m$ for $m = 1, 2, ..., N + 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) > (<)\pi_c(k)$; (ii) if $(-1)^m xy \frac{U^{(m+1)}(xy)}{U^{(m)}(xy)} \leq m$ for $m = 1, 2, ..., N + 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk $\tilde{y}$, then the agent's attitude towards risk is first order conditional dependent risk aversion and $\pi_{cd}(k) < (>)\pi_c(k)$.

6 Applications: the importance of background risk in risk diversification

In this section we illustrate the applicability of our results. In particular, we demonstrate how our results can be used to gain additional insight into risk diversification in the presence of a dependent background risk. We also show how our framework extends the understanding of insurance supply, public investment decisions, naive diversified portfolio pricing, bank lending and lottery business in the presence of a dependent background risk.

6.1 Background risk and risk diversification

Common wisdom suggests that diversification is a good way to reduce risk. Consider a set of n lotteries whose net gains are characterized by $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$,...,$\tilde{\varepsilon}_n$ that are assumed to be independent and identically distributed. Define the sample mean $\tilde{\varepsilon} = \frac{1}{n}\sum_{i=1}^n \tilde{\varepsilon}_i$, then, when w is not random,

$$Eu(w + E\tilde{\varepsilon} - \pi_c(\frac{1}{n}), \tilde{y}) = Eu(w + \tilde{\varepsilon}, \tilde{y}), \quad where\ \tilde{\varepsilon}\ and\ \tilde{y}\ are\ independent, \tag{26}$$

and

$$Eu(w + E\tilde{\varepsilon} - \pi_{cd}(\frac{1}{n}), \tilde{y}) = Eu(w + \tilde{\varepsilon}, \tilde{y}), \quad where\ \tilde{\varepsilon}\ and\ \tilde{y}\ are\ not\ necessary\ independent. \tag{27}$$

From (5), we know that $\pi_c(\frac{1}{n}) = O(\frac{1}{n^2})$. When $n \to \infty$, $\pi_c(\frac{1}{n}) \to 0$ because diversification is an efficient way to reduce risk. With an independent background risk, diversification can eliminate idiosyncratic risk at the rate of $\frac{1}{n^2}$ and the agent is second order risk aversion. This is the well known benefit of diversification. However, with a dependent background risk, it is not clear that the benefit of diversification holds for a correlation averse agent.

From Proposition 5.3 and equation (5), we obtain:

Proposition 6.1 *(i) If* $(\tilde{\varepsilon}, \tilde{y}) \in \mathcal{H}_N$ *and* $(-1)^m u_{12(m-1)} \leq 0$ *for* $m = 1, 2, ..., N+1$*, then* $\pi_{cd}(\frac{1}{n}) = |O(\frac{1}{n})|$*;*

(ii) If $(\tilde{\varepsilon}, \tilde{y}) \in \mathcal{I}_N$ *and* $(-1)^m u_{12(m-1)} \geq 0$ *for* $m = 1, 2, ..., N+1$*, then* $\pi_{cd}(\frac{1}{n}) = |O(\frac{1}{n})|$*;*

(iii) If $(\tilde{\varepsilon}, \tilde{y}) \in \mathcal{H}_N$ *and* $(-1)^m u_{12(m-1)} \geq 0$ *for* $m = 1, 2, ..., N+1$*, then* $\pi_{cd}(\frac{1}{n}) = -|O(\frac{1}{n})|$*;*

(iv) If $(\tilde{\varepsilon}, \tilde{y}) \in \mathcal{I}_N$ *and* $(-1)^m u_{12(m-1)} \leq 0$ *for* $m = 1, 2, ..., N+1$*, then* $\pi_{cd}(\frac{1}{n}) = -|O(\frac{1}{n})|$*.*

Proposition 6.1 signs the effect of dependent background risk on the benefits of diversification: if $\tilde{\varepsilon}$ and $\tilde{y}$ are positive (negative) expectation dependent and the agent is correlation aversion, then $\pi_{cd}(\frac{1}{n})$ will be greater (less) than zero. Proposition 6.1 also shows that, in the presence of an expectation dependent background risk, diversification can eliminate idiosyncratic risk ($\pi_{cd}(\frac{1}{n}) \to 0$, as $n \to \infty$). Therefore, for correlation averse agents, the benefit of diversification holds. However, the convergence rate is $\frac{1}{n}$ rather than $\frac{1}{n^2}$ which implies that if we use zero to approximate $\pi_{cd}(\frac{1}{n})$, then the error will be much larger in the presence of an expectation dependent background risk.

6.2 Insurance supply

It is well known that the "Law of Large Numbers" is the actuarial basis of insurance pricing: by pooling the risks of many policyholders, the insurer can take advantage of the "Law of Large Numbers". While Li (2011) and Soon et al. (2011) investigate how dependent background risk affects the demand for insurance, Proposition 6.1 shows how dependent background risk affects insurance supply. If $\frac{1}{n}\sum_{i=1}^{n} \tilde{\varepsilon}_i$ and $\tilde{y}$ are positive (negative) expectation dependent and the insurer is correlation averse, then the insurance premium will be higher (lower) than the actuarially fair premium. Suppose that $\tilde{\varepsilon}_i$ is the loss for insured i, and $\pi_{cd}(\frac{1}{n})$ and $\pi_c(\frac{1}{n})$ are the risk premiums of the insurance company for the individual loss $\tilde{\varepsilon}_i$. Proposition 6.1 implies that, in the presence of a dependent background risk, the insurer can not always take advantage of the benefit of diversification because the insurance risk will be eliminated only at the rate of $\frac{1}{n}$.

6.3 Public investment decisions

Arrow and Lind (1970) investigated the implications of uncertainty for public investment decisions. They considered the case where all individuals have the same preferences U, and their disposable incomes are identically distributed random variables represented by $\tilde{A}$. Suppose that the government undertakes an investment with returns represented by $\tilde{B}$, which are independent of $\tilde{A}$. Let $\bar{B} = E\tilde{B}$ and $\tilde{X} = \tilde{B} - \bar{B}$. Consider a specific taxpayer and denote his fraction of this investment by s with $0 \leq s \leq 1$. Suppose that each taxpayer has the same tax rate and that there are n taxpayers, then $s = \frac{1}{n}$. Arrow and Lind (1970) show that

$$EU(\tilde{A} + \frac{\bar{B}}{n} + r(n)) = EU(\tilde{A} + \frac{\bar{B} + \tilde{X}}{n}), \tag{28}$$

where $r(n)$ is the risk premium of the representative individual. They show that not only does $r(n)$ vanish, but so does the total of the risk premiums for all individuals: $nr(n)$ approaches zero as n rises.

Proposition 6.1 allows us to investigate the cases where $\tilde{A}$ and $\tilde{B}$ are dependent. Since (28) can be rewritten as

$$EU(\tilde{A} + \frac{\bar{B}}{n} + r(n)) = EU(\tilde{A} + \frac{\tilde{B}}{n}), \tag{29}$$

from Proposition 6.1, we obtain:

Proposition 6.2 *(i) If $(\tilde{B}, \tilde{A}) \in \mathcal{H}_N$ and $(-1)^k u_{12^{(k-1)}} \leq 0$ for $k = 1, 2, ..., N+1$, then $r(n) = -|O(\frac{1}{n})|$;*

(ii) If $(\tilde{B}, \tilde{A}) \in \mathcal{I}_N$ and $(-1)^k u_{12^{(k-1)}} \geq 0$ for $k = 1, 2, ..., N+1$, then $r(n) = -|O(\frac{1}{n})|$;

(iii) If $(\tilde{B}, \tilde{A}) \in \mathcal{H}_N$ and $(-1)^k u_{12^{(k-1)}} \geq 0$ for $k = 1, 2, ..., N+1$, then $r(n) = |O(\frac{1}{n})|$;

(iv) If $(\tilde{B}, \tilde{A}) \in \mathcal{I}_N$ and $(-1)^k u_{12^{(k-1)}} \leq 0$ for $k = 1, 2, ..., N+1$, then $r(n) = |O(\frac{1}{n})|$.

Therefore, when $\tilde{A}$ and $\tilde{B}$ are expectation dependent, $r(n)$ can not vanish as n becomes large. Proposition 6.2 shows that if the return of the investment and the disposable incomes are positive (negative) expectation dependent and the society is risk averse, then the risk premium of the representative individual will remain less (greater) than zero for any large n.

6.4 Naive diversified portfolio pricing

The naive portfolio diversification rule is defined as one in which a fraction $\frac{1}{n}$ of wealth is allocated to each of the n assets available for investment at each rebalancing date. This rule is easy to implement because it does not rely either on estimation or optimization. Many investors continue to use this simple rule for allocating their wealth across assets (see, Benartzi and Thaler 2001; Huberman and Jiang 2006). DeMiguel et al. (2009) find that there is no single model that consistently delivers a Sharpe ratio or a certainty-equivalent return that is higher than that of the $\frac{1}{n}$ portfolio rule.

Suppose that $\tilde{\varepsilon}_i$ is the return of stock i, $\tilde{\varepsilon}$ is the return of a portfolio consisting of $\frac{1}{n}$ shares of each stock, and $\pi_{cd}(\frac{1}{n})$ and $\pi_c(\frac{1}{n})$ are minimum risk premiums the investor will demand for this portfolio. Proposition 6.1 shows that, in the presence of a dependent background risk, the investor can not always take advantage of the benefit of diversification and the portfolio risk will be eliminated only at the rate of $\frac{1}{n}$. If $\tilde{\varepsilon}$ and $\tilde{y}$ are positive (negative) expectation dependent and the investor is correlation averse, then the return of the naive diversified portfolio will be higher (lower) than that corresponding to the portfolio's expected return.

6.5 Other examples

We can also apply our result to other examples. Suppose that $\tilde{\varepsilon}_i$ is the default risk of borrower i, and $\pi_{cd}(\frac{1}{n})$ and $\pi_c(\frac{1}{n})$ are the yield spread charged by the banker. Proposition 6.1 shows that if $\tilde{\varepsilon}$ and $\tilde{y}$ are positive (negative) expectation dependent and the banker is correlation averse, then the yield spread will be higher (lower) than that corresponding to the expected loss of default risk.

It is believed that the lottery business is rather safe, because the "Law of Large Numbers" entails that the average of the results from a large number of independent bets is quasi constant (with a very small variance). Suppose that $\tilde{\varepsilon}_i$ is the payment to a winner i, $\pi_{cd}(\frac{1}{n})$ and $\pi_c(\frac{1}{n})$ are the average risk premiums for a lottery ticket. Proposition 6.1 shows that if $\tilde{\varepsilon}$ and $\tilde{y}$ are positive (negative) expectation dependent and the lottery business is correlation averse, then the price for a lottery ticket must be higher (lower) than the expected payment of the lottery game.

7 Conclusion

In this study, we have generated the concepts of orders of conditional risk aversion to orders of conditional dependent risk aversion. We have shown that first-order conditional dependent risk aversion can appear in the framework of the expected utility function hypothesis. Our contribution provides insight into the difficulty of obtaining risk diversification in the presence of a dependent background risk.

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