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Thirty Years of Inventory-Routing

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Abstract. The Inventory-Routing problem dates back 30 years. It can be described as the combination of vehicle routing and inventory management problems, in which a supplier has to deliver products to a number of geographically dispersed customers, subject to side constraints. It provides integrated logistics solutions by simultaneously optimizing inventory management, vehicle routing and delivery scheduling. Some exact algorithms and several powerful metaheuristic and matheuristic approaches have been developed for this class of problems, especially in recent years. The purpose of this article is to provide a comprehensive review of this literature, based on a new classification of the problem. We categorize IRPs with respect to their structural variants and with respect to the availability of information on customer demand.

Keywords. Inventory-routing, survey, literature review, history.

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1 Introduction

The Inventory-Routing Problem (IRP) integrates inventory management, vehicle routing and delivery scheduling decisions. Its study is rooted in the seminal paper of Bell et al. [25] published 30 years ago. The IRP arises in the context of Vendor-Managed Inventory (VMI), a business practice aimed at reducing logistics costs and adding business value. In VMI, a supplier makes the replenishment decisions for products delivered to customers, based on specific inventory and supply chain policies [13, 89, 114]. This practice is often described as a win-win situation: vendors save on distribution and production costs since they can coordinate shipments made to different customers, and buyers also benefit by not allocating efforts to inventory control. The supplier has to make three simultaneous decisions: (1) when to serve a given customer, (2) how much to deliver to this customer when it is served, and (3) how to combine customers into vehicle routes.

1.1 Origins of the Inventory-Routing Problem

The first studies published on the IRP were mostly variations of models designed for the Vehicle Routing Problem (VRP) and heuristics developed to take inventory costs into consideration. The seminal paper of Bell et al. [25] dealt with the case where only transportation costs are included, demand is stochastic and customer inventory levels must be met. This was followed by a number of variants of the problem defined by the same authors. Some other early papers on the IRP are worthy of mention. Federgruen and Zipkin [65] have modified the VRP heuristic of Fisher and Jaikumar [67] to accommodate inventory and shortage costs in a random demand environment; Blumenfeld et al. [35] have considered distribution, inventory and production set-up costs; Burns et al. [38] have analyzed trade-offs between transportation and inventory costs, using an approximation of travel costs; Dror et al. [63] have studied short term solutions. The latter study was extended to stochastic demand by Dror and Ball [61]. The paper of Dror and Levy [62] adapts earlier VRP heuristics to the solution of a weekly IRP, while Anily and Federgruen [14] have proposed the first clustering algorithm for the IRP. Most of these papers assume that the consumption rate at the customer locations is known and deterministic. Despite the large number of contributions on distribution and on inventory problems before this period, the integration of these two features proved difficult to handle, not only because of limited computing power, but also because the available algorithms could not easily handle large and complex combinatorial problems, such as those combining routing and inventory management decisions.

1.2 Typologies of the Problem

We classify IRPs according to two schemes. The first one refers to the structural variants present in IRPs whereas the second is related to the availability of information on the demand.

Many variants of the IRP have been described over the past 30 years. There does not really exist a standard version of the problem. We will therefore refer to "basic versions" of the IRP, on which most of the research effort has concentrated, and to "extensions of the basic versions", which are more elaborate. The basic versions are presented in Table 1. They can be classified according to seven criteria, namely time horizon, structure, routing, inventory policy, inventory decisions, fleet composition and fleet size.

In Table 1, time refers to the horizon taken into account by the IRP model. It can either be finite or infinite. The number of suppliers and customers may vary, and therefore the structure can be one-to-one when there is only one supplier serving one customer, one-to-many in the most common case with one supplier and several customers, or less frequently, many-to-many with several suppliers and several customers. Routing can be direct when there is only one customer per route, multiple when there are several customers in the same route, or continuous when there is no central depot, like in several maritime applications. Inventory policies define pre-established rules to replenish customers. The two most common are the maximum level (ML) policy and the order-up-to level (OU) policy. Under an ML inventory policy, the replenishment level is flexible but bounded by the capacity available at each customer. Under an OU policy, whenever a customer is visited, the quantity delivered is that to fill its inventory capacity. Inventory decisions determine how

Table 1: Structural variants of the IRP

Criteria	Possible options		
Time horizon	Finite	Infinite	
Structure	One-to-one	One-to-many	Many-to-many
Routing	Direct	Multiple	Continuous
Inventory policy	Maximum level (ML)	Order-up-to level (OU)	
Inventory decisions	Lost sales	Back-order	Non-negative
Fleet composition	Homogeneous	Heterogeneous	
Fleet size	Single	Multiple	Unconstrained

Source: Adapted from Andersson et al. [12]

inventory management is modeled. If the inventory is allowed to become negative, then back-ordering occurs and the corresponding demand will be served at a later stage; if there are no back-orders, then the extra demand is considered as lost sales. In both cases there may exist a penalty for the stockout. In deterministic contexts, one can also restrict the inventory to be non-negative. Finally, the last two criteria refer to fleet composition and size. The fleet can either be homogeneous or heterogeneous, and the number of vehicles available may be fixed at one, fixed at many, or be unconstrained.

The second classification refers to the time at which information on demand becomes known. If it is fully available to the decision maker at the beginning of the planning horizon, the problem is then deterministic; if its probability distribution is known, then it is stochastic, which yields the Stochastic Inventory-Routing Problem (SIRP). Dynamic IRPs arise when demand is not fully known in advance, but is gradually revealed over time, as opposed to what happens in a static context. In this case, one can still exploit its statistical distribution in the solution process, yielding a Dynamic and Stochastic Inventory-Routing Problem (DSIRP).

1.3 Applications

Several applications of the IRP have been documented. Most arise in maritime logistics, namely in ship routing and inventory management. Literature reviews are provided in Ronen [110] and Christiansen et al. [50, 51]. The problems described in these surveys involve a many-to-many structure with continuous routes [45, 47, 48], direct deliveries [124], several products [22, 30, 101, 111], and stochastic demand [104]. More complex configurations involve the presence of time windows and the typical cost structure of the maritime environment (i.e., demurage and overage rates) [120], and soft time windows [46, 49]. Problems in which storage capacities, production and consumption rates are variable have been studied by Engineer et al. [64], Grønhaug et al. [79] and Uggen et al. [126]. Problems arising in the chemical components industry [96, 60] and in the oil and gas industries [10, 20, 52, 65, 79, 101, 120, 125] are also a frequent source of applications in a maritime environment.

Non-maritime applications of the IRP arise in a large variety of industries, including the distribution of gas using tanker trucks [25, 39, 77], road-based distribution of automobile components [11, 35, 36, 123] and of perishable items [65, 66]. Other applications include the transportation of groceries [59, 72, 94], cement [52], fuel [102], blood [81], and waste organic oil [9].

1.4 Aim and Organization of the Paper

The aim of this paper is to present a comprehensive literature review of the IRP, including its main variants, models and algorithms. It complements the survey of Andersson et al. [12], which puts more emphasis on industrial applications. In contrast, our contribution focuses on the methodological aspects of the problem. The remainder of the paper is organized as follows. In Section 2 we describe the basic versions of the IRP as well as its models and solutions procedures. A number of meaningful extensions of the problem are then presented in Section 3. This is followed by the description of the stochastic version of the problem in Section 4, and by the dynamic and stochastic IRP in Section 5. Benchmark instances are described in Section 6 and our conclusions follow in Section 7.

2 Basic Versions of the Inventory-Routing Problem

The basic IRP is defined on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V} = \{0, ..., n\}$ is the vertex set and $\mathcal{A} = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}$ is the arc set. Vertex 0 represents the supplier, and the vertices of $\mathcal{V}' = \mathcal{V} \setminus \{0\}$ represent customers. Both the supplier and customers incur unit inventory holding costs h_i per period ($i \in \mathcal{V}$), and each customer has an inventory holding capacity C_i. The length of the planning horizon is p and, at each time period $t \in \mathcal{T} = \{1, ..., p\}$, the quantity of product made available at the supplier is r^t. We assume the supplier has sufficient inventory to meet all the demand during the planning horizon and that inventories are not allowed to be negative. The variables I_0^t and I_i^t are defined as the inventory levels at the end of period t, respectively at the supplier and at customer i. At the beginning of the planning horizon the decision maker knows the current inventory level of the supplier and of all customers (I_0^0 and I_i^0 for $i \in \mathcal{V}'$), and has full knowledge of the demand d_i^t of each customer i for each time period t. A set $\mathcal{K} = \{1, ..., K\}$ of vehicles with capacity Q_k are available. Each vehicle is able to perform one route per time period to deliver products from the supplier to a subset of customers. A routing cost c_{ij} is associated with arc $(i, j) \in \mathcal{A}$.

The objective of the problem is to minimize the total inventory-distribution cost while meeting the demand of each customer. The replenishment plan is subject to the following constraints:

- the inventory level at each customer can never exceed its maximum capacity;
- inventory levels are not allowed to be negative;
- the supplier's vehicles can perform at most one route per time period, each starting and ending at the supplier;
- vehicle capacities cannot be exceeded.

The solution to the problem determines which customers to serve in each time period, which of the supplier's vehicles to use, how much to deliver to each visited customer, as well as the delivery routes. Obviously, the IRP just defined is deterministic and static because consumption rates are fixed and known beforehand.

The basic IRP is NP-hard since it subsumes the classical VRP. As a result, most papers propose heuristics for its solution, but a number of exact algorithms are also available. We present in Table 2 the papers mentioned in this section on the deterministic IRP. These will be further described when we present exact algorithms in Section 2.1 and heuristics in Section 2.2.

2.1 Exact Algorithms

All formulations presented in this section were developed assuming the cost matrix is symmetric. In such cases, it is natural to define the problem on an undirected graph $G = (V, E)$, where $E = \{(i, j) : i, j \in \mathcal{V}, i < j\}$ and to use routing variables associated with the edges, which is computationally more efficient. It is straightforward to extend edge based formulations to the directed case. Archetti et al. [15] have developed the first branch-and-cut algorithm for a single-vehicle IRP. Their model works with binary variables x_{ij}^t equal to the number of times edge (i, j) is traversed on the route of the supplier's vehicle in period t. Let the quantity of product delivered from the supplier to each customer i in each time period t be q_i^t, and let y_0^t be a binary variable equal to one if and only if there exists a route to perform in that period. Finally, let y_i^t be a binary

Table 2: Classification of the papers on the basic versions of the IRP

Reference	**Time horizon**		**Structure**			**Routing**			**Inventory policy**		**Inventory decisions**			**Fleet composition**		**Fleet size**		
	Finite	Infinite	One-to-one	One-to-many	Many-to-many	Direct	Multiple	Continuous	Maximum level (ML)	Order-up-to level (OU)	Lost sales	Backlogging	Non-negative	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Bell et al. [25]	✓			✓			✓		✓		✓				✓		✓	
Burns et al. [38]	✓			✓		✓	✓		✓		✓			✓				✓
Dror et al. [63]	✓			✓			✓			✓	✓			✓			✓	
Dror and Levy [62]	✓			✓			✓		✓				✓	✓			✓	
Dror and Ball [61]	✓		✓	✓			✓		✓		✓			✓			✓	
Chien et al. [44]	✓			✓			✓		✓			✓			✓		✓	
Anily and Federgruen [14]		✓		✓			✓		✓		✓			✓				✓
Gallego and Simchi-Levi [70]		✓		✓		✓			✓		✓			✓				✓
Campbell et al. [40]	✓			✓			✓		✓					✓				✓
Christiansen [45]	✓				✓			✓	✓		✓				✓		✓	
Bertazzi et al. [32]	✓			✓			✓			✓			✓	✓		✓		
Ribeiro and Lourenço [109]	✓			✓			✓		✓		✓			✓				✓
Abdelmaguid [1]	✓			✓			✓		✓			✓		✓			✓	
Campbell and Savelsbergh [39]	✓			✓			✓		✓		✓			✓			✓	
Abdelmaguid and Dessouky [2]	✓			✓			✓		✓			✓		✓			✓	
Aghezzaf et al. [8]		✓		✓			✓		✓		✓			✓			✓	
Archetti et al. [15]	✓			✓			✓		✓	✓			✓	✓		✓		
Raa and Aghezzaf [105]		✓		✓			✓		✓				✓	✓			✓	
Savelsbergh and Song [113]	✓				✓			✓	✓		✓			✓			✓	
Zhao et al. [129]		✓		✓			✓		✓				✓	✓				✓
Abdelmaguid et al. [3]	✓			✓			✓		✓			✓			✓		✓	
Boudia and Prins [37]	✓			✓			✓		✓				✓	✓			✓	
Raa and Aghezzaf [106]		✓		✓			✓		✓		✓			✓			✓	
Geiger and Sevaux [74]	✓			✓			✓		✓		✓			✓				✓
Solyalı and Süral [118]	✓			✓			✓			✓			✓	✓		✓		
Adulyasak et al. [6]	✓			✓			✓		✓	✓			✓	✓	✓		✓	
Archetti et al. [17]	✓			✓			✓		✓	✓			✓	✓		✓		
Coelho and Laporte [55]	✓			✓			✓		✓	✓			✓	✓	✓		✓	
Coelho et al. [56]	✓			✓			✓		✓	✓			✓	✓		✓		
Coelho et al. [57]	✓			✓			✓		✓	✓			✓	✓			✓	
Michel and Vanderbeck [95]	✓			✓			✓			✓			✓	✓			✓	

variable equal to one if and only if customer i is served in period t. The problem formulated by Archetti et al. [15] consists of minimizing the following objective function:

$$\text{minimize} \sum_{i \in \mathcal{V}} \sum_{t \in \mathcal{T}} h_i I_i^t + \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}, i<j} \sum_{t \in \mathcal{T}} c_{ij} x_{ij}^t \tag{1}$$

subject to the following constraints:

$$I_0^t = I_0^{t-1} + r^t - \sum_{i \in \mathcal{V}'} q_i^t \qquad t \in \mathcal{T} \tag{2}$$

$$I_0^t \geq 0 \qquad t \in \mathcal{T} \tag{3}$$

$$I_i^t = I_i^{t-1} + q_i^t - d_i^t \qquad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{4}$$

$$I_i^t \geq 0 \qquad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{5}$$

$$q_i^t \geq C_i y_i^t - I_i^{t-1} \qquad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{6}$$

$$q_i^t \leq C_i - I_i^{t-1} \qquad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{7}$$

$$q_i^t \leq C_i y_i^t \qquad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{8}$$

$$\sum_{i \in \mathcal{V}'} q_i^t \leq Q \qquad t \in \mathcal{T} \tag{9}$$

$$\sum_{i \in \mathcal{V}} q_i^t \leq Q y_0^t \qquad t \in \mathcal{T} \tag{10}$$

$$\sum_{j \in \mathcal{V}', i<j} x_{ij}^t + \sum_{j \in \mathcal{V}', i>j} x_{ji}^t = 2y_i^t \qquad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{11}$$

$$\sum_{i \in \mathscr{S} u} \sum_{j \in \mathscr{S}, i<j} x_{ij}^t \leq \sum_{i \in \mathscr{S}} y_i^t - y_m^t \qquad \mathscr{S} \subseteq \mathcal{V} \quad t \in \mathcal{T} \quad m \in \mathscr{S} \tag{12}$$

$$q_i^t \geq 0 \qquad i \in \mathcal{V} \quad t \in \mathcal{T} \tag{13}$$

$$x_{ij}^t \in \{0,1\} \qquad i,j \in \mathcal{V}, i \neq j \quad t \in \mathcal{T} \tag{14}$$

$$x_{i0}^t \in \{0,1,2\} \qquad i \in \mathcal{V} \quad t \in \mathcal{T} \tag{15}$$

$$y_i^t \in \{0,1\} \qquad i \in \mathcal{V} \quad t \in \mathcal{T}. \tag{16}$$

Constraints (2) define the inventory level at the supplier at the end of period t by its inventory level at the end of period $t-1$, plus the quantity r^t made available in period t, minus the total quantity shipped to the customers using the supplier's vehicle in period t. Constraints (3) avoid stockouts at the supplier by imposing that the supplier's inventory cannot be negative. Constraints (4) and (5) are similar and apply to customers. Constraints (6)−(8) define the quantities delivered. These sets of constraints enforce the OU policy. More specifically, they ensure that the quantity delivered by the supplier's vehicle to each customer $i \in \mathcal{V}'$ in each period $t \in \mathcal{T}$ will fill the customer's inventory capacity if the customer is served, and will be zero otherwise. If customer i is not visited in period t, then constraints (8) mean that the quantity delivered to it will be zero (while constraints (6) and (7) are still respected). Otherwise, if customer i is visited in period t, then constraints (8) limit the quantity delivered to the customer's inventory holding capacity, and this bound is tightened by constraints (7), making it impossible to deliver more than what would fill this capacity. Constraints (6) model the OU replenishment policy, ensuring that the quantity delivered will be exactly the bound provided by constraints (7). Constraints (9) state that the vehicle capacity is not exceeded. Constraints (10)−(12) guarantee that a feasible route is determined to visit all customers served in period t. Finally, constraints (13)−(16) enforce integrality and non-negativity conditions on the variables.

To solve the IRP under the ML policy, it suffices to drop constraints (6) and (8). In the branch-and-cut scheme of Archetti et al. [15], constraints (12) are relaxed and added as cuts in the search tree whenever an incumbent solution violates them. These authors have also derived some valid inequalities to strengthen the model and were able to solve instances with up to 50 customers in a three-period horizon, and 30 customers in a six-period horizon within two hours of computing time.

Despite considering only one vehicle, this model is somewhat more general than others because it incorporates not only inventory holding costs at the customers, but also at the supplier. It was later improved by Solyalı and Süral [118] who used a stronger formulation with shortest path networks representing customer replenishments, as well as a heuristic to provide an initial upper bound to the branch-and-cut algorithm. These authors considered only the OU policy and solved larger instances with up to 15 customers and 12 periods, 25 customers and nine periods, and 60 customers in a three-period horizon.

Recently, algorithms capable of solving multi-vehicle versions of the IRP exactly have been introduced. Coelho and Laporte [55] and Adulyasak et al. [6] have proposed an extension of the above formulation under the OU and ML policies to account for multiple vehicles, and have solved it in a branch-and-cut fashion. Assuming again that the transportation cost matrix is symmetric, their proposed model is undirected in order to reduce the number of variables. Thus, their model uses variables x_{ij}^{kt} equal to the number of times edge (i, j) is used on the route of vehicle k in period t. It also uses variables y_i^{kt} equal to one if and only if vertex i (the supplier or a customer) is visited by vehicle k in period t. Let I_i^t denote the inventory level at vertex $i \in \mathcal{V}$ at the end of period $t \in \mathcal{T}$, and q_i^{kt} the quantity of product delivered from the supplier to customer i using vehicle k in time period t. Assuming again that the OU inventory policy applies, the problem can then be formulated as

$$\text{minimize} \sum_{i\in\mathcal{V}}\sum_{t\in\mathcal{T}} h_i I_i^t + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V},i<j}\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} c_{ij}x_{ij}^{kt} \tag{17}$$

subject to

$$I_0^t = I_0^{t-1} + r^t - \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{V}'} q_i^{kt} \quad t\in\mathcal{T} \tag{18}$$

$$I_0^t \geq 0 \quad t\in\mathcal{T} \tag{19}$$

$$I_i^t = I_i^{t-1} + \sum_{k\in\mathcal{K}} q_i^{kt} - d_i^t \quad i\in\mathcal{V}' \quad t\in\mathcal{T} \tag{20}$$

$$I_i^t \geq 0 \quad i\in\mathcal{V} \quad t\in\mathcal{T} \tag{21}$$

$$I_i^t \leq C_i \quad i\in\mathcal{V} \quad t\in\mathcal{T} \tag{22}$$

$$\sum_{k\in\mathcal{K}} q_i^{kt} \leq C_i - I_i^{t-1} \quad i\in\mathcal{V}' \quad t\in\mathcal{T} \tag{23}$$

$$q_i^{kt} \geq C_i y_i^{kt} - I_i^{t-1} \quad i\in\mathcal{V}' \quad k\in\mathcal{K} \quad t\in\mathcal{T} \tag{24}$$

$$q_i^{kt} \leq C_i y_i^{kt} \quad i\in\mathcal{V}' \quad k\in\mathcal{K} \quad t\in\mathcal{T} \tag{25}$$

$$\sum_{i\in\mathcal{V}'} q_i^{kt} \leq Q_k y_0^{kt} \quad k\in\mathcal{K} \quad t\in\mathcal{T} \tag{26}$$

$$\sum_{j\in\mathcal{V},i<j} x_{ij}^{kt} + \sum_{j\in\mathcal{V},i>j} x_{ji}^{kt} = 2y_i^{kt} \quad i\in\mathcal{V} \quad k\in\mathcal{K} \quad t\in\mathcal{T} \tag{27}$$

$$\sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S},i<j} x_{ij}^{kt} \leq \sum_{i\in\mathcal{S}} y_i^{kt} - y_m^{kt} \quad \mathcal{S}\subseteq\mathcal{V}' \quad k\in\mathcal{K} \quad t\in\mathcal{T} \quad m\in\mathcal{S} \tag{28}$$

$$q_i^{kt} \geq 0 \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{29}$$

$$x_{i0}^{kt} \in \{0, 1, 2\} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{30}$$

$$x_{ij}^{kt} \in \{0, 1\} \quad i, j \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{31}$$

$$y_i^{kt} \in \{0, 1\} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T}. \tag{32}$$

Constraints (18) define the inventory at the supplier while constraints (19) prevent stockouts at the supplier; constraints (20) and (21) are similar and apply to the customers. Constraints (22) impose maximal inventory level at the customers. Note that these constraints assume that the inventory at the end of the period cannot exceed the maximum available holding capacity, which means that during the period, before all demand has happened the inventory capacity could be temporarily exceeded. This is a usual assumption in IRP models. Constraints (23)$-$(25) link the quantities delivered to the routing variables. In particular, they only allow a vehicle to deliver products to a customer if the customer is visited by this vehicle and enforce the OU policy. Constraints (26) ensure that vehicle capacities are respected, while constraints (27) and (28) are degree constraints and subtour elimination constraints, respectively. The latter are relaxed and added as cuts whenever they are violated in the search tree. Constraints (29)$-$(32) enforce integrality and non-negativity conditions on the variables.

This formulation can be solved by branch-and-cut by making use of the capabilities of modern MIP solvers. Instances with up to 45 customers, three periods and three vehicles have been solved to optimality with CPLEX. Adulyasak et al. [6] have compared this model with a two-index formulation which yielded better lower bounds on larger instances that could not be solved exactly with the three-index formulation.

2.2 Heuristic Algorithms

Most of the early papers on the IRP have applied simple heuristics to simplified versions of the problem. These explore the solution space through the use of simple neighborhood structures such as interchanges and typically decompose the IRP into hierarchical subproblems, where the solution to one subproblem is used in the next step. Examples include an assignment heuristic [63], an interchange algorithm [62], trade-offs based on approximate routing costs [38] and a clustering heuristic [14].

Current heuristic algorithms are rather involved and are able to obtain high quality solutions to difficult optimization problems. They rely on the concept of *metaheuristics* which apply local search procedures and a strategy to avoid local optima, and perform a thorough evaluation of the search space [75]. New developments in this area include the hybridization of different metaheuristic concepts to create more powerful algorithms [107] and also the hybridization of a heuristic and of a mathematical programming algorithm, yielding so-called *matheuristic* algorithms [93]. Recent IRP papers using some of these techniques include iterated local search [109], variable neighborhood search [129], greedy randomized adaptive search [39], memetic algorithms [37], tabu search [17], and adaptive large neighborhood search [56].

Bell et al. [25] have analyzed the case where only transportation costs are included, but inventory levels must be met at the customers. A short term solution is presented in Dror and Ball [61] and in Dror et al. [63], based on the assignment of customers to optimal replenishment periods, and on the computation of the expected increase in cost when the customer is visited in another period. Dror et al. [63] offered the first algorithmic comparison for the IRP with two major simplifications: (1) an OU policy applies and (2) customers are only visited once during the planning period. Dror and Ball [61] also applied the OU policy, which has been widely used by many researchers.

Building on the idea of adapting previous VRP algorithms and heuristics, Dror and Levy [62] have proposed a vertex interchange algorithm for a weekly IRP. They have generated an initial solution to a VRP by keeping track of vehicle capacities and customer inventories, thus improving the initial solution scheme presented in Dror et al. [63]. Burns et al. [38] have developed formulas based on the trade-offs between transportation and inventory costs, using an approximation of traveling costs. They showed that under direct shipping the optimal delivery size is the economic order quantity.

Clustering heuristics were proposed by Anily and Federgruen [14] and by Campbell and Savelsbergh [39]. Direct deliveries were studied by Gallego and Simchi-Levi [70] who evaluated their long-term effectiveness: direct shipping proved to be 94% effective whenever the vehicle capacity is at least 71% used. Aghezzaf et al. [8] have allowed vehicles to perform more than one route per period and have modified the approach employed by Anily and Federgruen [14] by using heuristic column generation. Their work was later extended by Raa and Aghezzaf [106] who have added driving time constraints. Construction and improvement heuristics were proposed by Chien et al. [44] for a version of the problem with a heterogeneous fleet. Considering backlogging, a construction heuristic was put forward by Abdelmaguid [1] and was later outperformed by the genetic algorithm of Abdelmaguid and Dessouky [2]. Heuristics for the IRP with backlogging were later reviewed by Abdelmaguid et al. [3].

Savelsbergh and Song [113] have solved a problem in which a single producer cannot usually meet the demand of its customers because they are too far away. This leads to the formulation of a problem with several suppliers and trips lasting longer than one period. This problem is called the IRP with continuous moves and is solved through a local search algorithm applied on an initial solution generated by a randomized greedy heuristic.

Considering a cyclic planning approach where a long-term distribution pattern can be derived, Raa and Aghezzaf [105] have developed an algorithm allowing vehicles to perform multiple tours. Initially, customers are partitioned over vehicles using a column generation algorithm. Then, for each vehicle, the set of customers assigned to it is partitioned over different tours for which frequencies are then determined. For each partition of customers over tours and each combination of tour frequencies, a delivery schedule is then made to check feasibility.

With the aim of identifying Pareto-optimal solutions, Geiger and Sevaux [74] have compared different solutions with respect to the two opposing terms in the objective function. When a customer is visited very frequently, its inventory cost is low but routing becomes expensive, and vice versa. This is important when considering changes in some of the parameters, for example when fuel prices increase or when focusing on the computation of "green" solutions.

A heuristic column generation algorithm is used to solve a tactical IRP in Michel and Vanderbeck [95], in which customer demands are deterministic and are clustered to be served by different vehicles. Routing costs are approximated. This heuristic yields solutions that deviate by approximately 6% from the optimum and improve upon industrial practice by 10% with respect to travel distances and the number of vehicles used.

A two-phase heuristic based on a linear programming model was proposed by Campbell et al. [40]. In the first phase, the exact visiting period and quantity to be delivered to each customer are calculated. Then, in the second phase, customers are sequenced into vehicle routes. The model definition and formulation are as follows. Let d_i denote the constant usage rate of customer i, $L_i^t = \max\{0, td_i - I_i^0\}$ denote a lower bound on the total volume to be delivered to customer i by period t, and $U_i^t = td_i + C_i - I_i^0$ be an upper bound on the total volume that can be delivered to customer i by period t. If q_i^t represents the delivery quantity to customer i in period t, then in order to prevent stockouts or exceeded inventory capacity one must ensure that

$$L_i^t \leq \sum_{1 \leq s \leq t} q_i^s \leq U_i^t \qquad i \in \mathcal{V}' \quad t \in \mathcal{T}. \tag{33}$$

The total volume that can be delivered in a single period is constrained by a combination of capacity and time windows. Since vehicles are allowed to make more than one trip per period, the authors model the problem based on the resource constraints as follows. Let $\mathcal{R}$ be the set of all possible delivery routes r, T_r the duration of route r (as a fraction of a period), and c_r the cost of executing route r. For simplicity, we write $i \in r$ if i belongs to route r. Let x_r^t be a binary variable indicating whether route r is used in period t or not, and let q_{ir}^t be a continuous variable representing the delivery volume to customer i on route r in period t. Also let Q denote the vehicle capacity and m the time available for a vehicle to perform its routes in a single period. The problem can then be formulated as

$$\text{minimize} \sum_{t\in\mathcal{T}} \sum_{r\in\mathcal{R}} c_r x_r^t \tag{34}$$

subject to

$$L_i^t \leq \sum_{1\leq s\leq t} \sum_{r:i\in r} q_{ir}^t \leq U_i^t \qquad i\in\mathcal{V} \quad t\in\mathcal{T} \tag{35}$$

$$\sum_{i:i\in r} q_{ir}^t \leq Q x_r^t \qquad r\in\mathcal{R} \quad t\in\mathcal{T} \tag{36}$$

$$\sum_{r\in\mathcal{R}} T_r x_r^t \leq m \qquad t\in\mathcal{T}. \tag{37}$$

Constraints (36) ensure that vehicle capacities are not exceeded, while constraints (37) mean that the time available to perform the routes is sufficient. This model is difficult to solve due to the high number of possible routes, and also because of the length of the planning horizon. Considering a small set of routes and aggregating periods towards the end of the horizon makes the model more tractable. The output of this first phase specifies how much to deliver to each customer in each period of the planning horizon. This information then becomes the input of a standard algorithm for the VRP with Time Windows which is solved for each period in the second phase. Since decisions are taken separately in the two phases, the second phase can only be optimal with respect to the solution obtained from the first phase. Besides, this model considers time constraints explicitly but does not include any consideration for the inventory holding costs.

Bertazzi et al. [32] have proposed a fast local search algorithm for the single-vehicle case in which an OU inventory policy is applied. This policy decreases the flexibility of the decision maker by restricting the set of possible solutions to the problem. The simplified problem is solved heuristically. A first step creates a feasible solution, and a second one is applied as long as a given minimum improvement is made to the total cost function. This is achieved by removing all possible customer pairs and computing a series of shortest paths to determine the periods in which the customers should be reinserted. Specifically, shortest paths are computed on acyclic networks $\mathcal{N}_i$, one for each customer i. Each node of $\mathcal{N}_i$ corresponds to a discrete time instant between 0 and $p+1$, and an arc (t,t') is defined if no stockout occurs at customer i whenever it is not visited in the interval $[t,t']$; the quantity delivered to i at each time period will be that to fill the customer capacity, and each arc cost is the sum of the inventory and routing costs associated with visiting customer i in the interval $[t,t']$. The authors consider both inventory and transportation costs, and it is relevant to note that the supplier also incurs inventory costs, which was not generally the case in previous papers. Computational experiments have shown that this heuristic works extremely fast but the optimality gap is sometimes larger than 5%.

Archetti et al. [17] have proposed a more involved heuristic combining tabu search with the exact solution of mixed integer linear programs (MILPs) used to approximate routing decisions. It operates with a combination of a tabu search heuristic embedded within four neighborhood search operators and two MILPs to further refine the solutions. Starting from a feasible solution, the algorithm explores the neighborhood of the current solution and performs occasional jumps to new regions of the search space. Infeasible solutions are temporarily accepted, namely due to a stockout at the supplier or exceeded vehicle capacity. Results show that the heuristic performs remarkably well on benchmark instances, with an optimality gap usually below 0.1%.

Coelho et al. [56] have developed an adaptive large neighborhood search (ALNS) matheuristic which can solve the IRP as a special case of a broader problem including transshipments. This algorithm works in two phases, first creating vehicle routes by means of the ALNS operators and then determining delivery quantities through the use of an exact min-cost network flow algorithm. When no transshipments are considered, this matheuristic performs slightly worse than the algorithm of Archetti et al. [17]. Finally, Coelho et al. [57] have proposed an extension of the previous algorithm to the multi-vehicle version of the IRP. In this problem, the

ALNS creates vehicle routes, and delivery quantities are again optimized by means of a min-cost network flow algorithm. Better solutions are obtained by approximating the costs of inserting or removing customers from existing solutions through the exact solution of a MILP, as in Archetti et al. [17].

3 Extensions of the Basic Versions

Almost every combination of the criteria presented in Table 1 has been studied at some point over the past 30 years. Specific versions of the IRP include the IRP with a single customer [30, 61, 117, 122], the IRP with multiple customers [15, 25, 44, 56, 55], the IRP with direct deliveries [29, 70, 71, 80, 87, 98], the multi-item IRP [22, 104, 115, 121], the IRP with several suppliers and customers [26] and the IRP with heterogeneous fleet [44, 45, 55, 101], among others. However, some common criteria are more relevant and have received more attention. Table 3 presents the papers cited in this section, which covers deterministic extensions of the IRP.

Table 3: Classification of the papers on extensions of the basic versions of the IRP

Reference	Time horizon		Products			Structure			Routing		Inventory policy		Inventory decisions		Fleet composition		Fleet size		
	Finite	Infinite	Single	Two	Many	One-to-one	One-to-many	Many-to-many	Direct	Multiple	Maximum level (ML)	Order-up-to level (OU)	Lost sales	Backlogging	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Blumenfeld et al. [35]		✓			✓		✓	✓	✓		✓		✓		✓				✓
Roundy [112]		✓	✓				✓			✓	✓		✓		✓				✓
Gallego and Simchi-Levi [70]		✓	✓				✓		✓		✓		✓		✓				✓
Hall [80]		✓	✓				✓		✓		✓		✓		✓				✓
Gallego and Simchi-Levi [71]		✓	✓				✓		✓		✓		✓		✓				✓
Speranza and Ukovich [121]		✓			✓		✓		✓		✓		✓		✓				✓
Carter et al. [41]	✓				✓		✓			✓	✓			✓	✓			✓	
Speranza and Ukovich [122]		✓			✓		✓			✓	✓		✓			✓		✓	
Bertazzi et al. [31]		✓			✓		✓			✓	✓		✓		✓				✓
Herer and Roundy [82]		✓	✓				✓			✓	✓		✓		✓			✓	
Viswanathan and Mathur [127]		✓			✓		✓			✓	✓		✓		✓	✓		✓	
Bausch et al. [22]	✓				✓			✓		✓	✓			✓		✓			
Bertazzi and Speranza [30]	✓				✓	✓			✓		✓		✓		✓			✓	
Persson and Göthe-Lundgren [101]	✓				✓			✓		✓	✓		✓			✓		✓	
Sindhuchao et al. [115]	✓				✓		✓			✓	✓		✓		✓			✓	
Stacey et al. [123]		✓			✓		✓			✓	✓		✓		✓			✓	
Zhao et al. [128]		✓	✓				✓		✓	✓	✓		✓		✓			✓	
Bertazzi [29]		✓			✓		✓		✓		✓		✓		✓				✓
Solyalı and Süral [117]	✓		✓			✓			✓			✓	✓		✓		✓		
Li et al. [91]		✓	✓				✓		✓		✓		✓		✓				✓
Benoist et al. [26]	✓		✓					✓		✓	✓		✓			✓		✓	
Moin et al. [99]	✓				✓		✓			✓	✓		✓		✓			✓	
Coelho and Laporte [55]	✓		✓				✓			✓	✓	✓	✓		✓	✓		✓	
Coelho et al. [56]	✓		✓				✓			✓	✓	✓	✓		✓		✓		
Coelho et al. [57]	✓		✓				✓			✓	✓	✓	✓		✓			✓	
Popović et al. [102]	✓				✓		✓			✓	✓		✓		✓			✓	
Ramkumar et al. [108]	✓				✓			✓		✓	✓		✓		✓			✓	

3.1 The Production-Routing Problem

Because VMI provides advantages both to the supplier and to the customers, it is natural to think that integrating one more element of the supply chain may lead to an even better performance. This extra element may be external (the supplier of the supplier) or may include other activities of the supplier, such as production planning. This leads to the production-inventory-routing problem, also called production-routing problem.

Chandra [42] and then Chandra and Fisher [43] were among the first to integrate production decisions within the IRP. They were followed by Chandra and Fisher [43], Herer and Roundy [82], Fumero and Vercellis [69], Bertazzi et al. [33], Bard and Nananukul [18, 19]. More recent works in this direction include those of Archetti et al. [16] and of Adulyasak et al. [5].

In the same vein, other levels of integration have been proposed. For instance, Blumenfeld et al. [35] considered distribution, inventory and production set-up costs. Javid and Azad [86] have proposed a broader mechanism which simultaneously optimizes location, allocation, capacity, inventory and routing decisions in supply chain design under stochastic demand.

3.2 The IRP with Multiple Products

In some versions of the IRP, several products are handled at once. Speranza and Ukovich [121, 122] studied the case with predetermined frequencies for a multi-product flow for a single customer. Bertazzi et al. [31] later extended these studies to handle multiple customers. Carter et al. [41] have also proposed a two-phase heuristic to solve the multi-product version of the IRP. A particular case of the multi-item IRP was analyzed by Popović et al. [102] in which different types of fuel are delivered to a set of customers by vehicles with compartments. The problem was solved by means of a variable neighborhood search heuristic since the proposed MILP could only handle the smallest instance from a practical application. A variation of the multi-product version which also considers multiple suppliers but only one customer (many-to-one structure) was analyzed by Moin et al. [99]. The authors derive lower and upper bounds after solving a linear mathematical formulation with a commercial solver and then compute better upper bounds by means of a genetic algorithm. Building up on the previous structure, Ramkumar et al. [108] studied the many-to-many case and proposed a MILP formulation for a multi-item multi-depot IRP. However, their computational results show the limitations of the method since several small instances with only two vehicles, two products, two suppliers, three customers and three periods could not be solved to optimality in eight hours of computing time.

3.3 The IRP with Direct Deliveries and Transshipment

Another extension of the IRP deals with direct deliveries, as the one studied by Kleywegt et al. [87] and by Bertazzi [29]. Making exclusive use of direct deliveries simplifies the problem since it removes the routing dimension from it. Direct deliveries are shown to be effective when economic order quantities for the customers are close to the vehicle capacities [70, 71]. Li et al. [91] developed an analytic method for performance evaluation of this delivery strategy, whose effectiveness can be represented as a function of system parameters.

A number of replenishment policies have been proposed in this context. Power-of-two policies were analyzed by Herer and Roundy [82], a fixed partition policy combined with a tabu search heuristic was studied by Zhao et al. [128], and a stationary nested joint replenishment policy was developed by Viswanathan and Mathur [127] for a multi-product case. Most of the IRP literature considers continuous decision variables for the delivery times. Under this assumption, the optimal replenishment time may be non-integer, which can constitute an inconvenience for some suppliers. Roundy [112] studies the case with multiple customers receiving direct deliveries at discrete times, and defines frequency based policies proven to be within 2% of the optimum in the worst case. In this model, inventory holding costs are linear, but there are fixed ordering and delivery costs.

Direct deliveries from the supplier and lateral transshipments between customers have also been used in conjunction with multi-customer routes in order to increase the flexibility of the system. Transshipments were formally introduced within the IRP framework by Coelho et al. [56]. These authors have included planned transshipment decisions within a deterministic framework as a way of reducing distribution costs. Coelho et al. [58] have later used transshipments within a DSIRP framework as a means of mitigating stockouts when demand exceeds the available inventory. Emergency transshipments were shown to be a valuable option for decreasing average stockouts while significantly reducing distribution costs.

3.4 The Consistent IRP

Some authors have noted that a cost-optimal solution may sometimes result in inconveniences both to the supplier and to the customers. This is the case, for example, when very small deliveries take place on consecutive days, followed by a very large delivery, after which the customer is not visited for a long period. Another example, this time undesirable for the supplier, is that it could be optimal to dispatch a mix of almost full and almost empty vehicles, which does not yield a proper load balancing and may irritate some drivers. It is possible to alleviate some of these problems by introducing some consistency features into the basic IRP, as has already been done in the context of the VRP. Thus some authors have included workforce management within the periodic VRP for assigning territories to drivers as in Christofides and Beasley [53], Beasley [24], Barlett and Ghoshal [21] or Zhong et al. [130]. This is an indirect way of enforcing driver consistency, which was formally put forward by Groër et al. [78]. Smilowitz et al. [116] have analyzed potential trade-offs between workforce management and travel distance goals in a multi-objective PVRP. Another example of consistency is the spacing of deliveries to customers, which ensure smoother operations (see, e.g., Ohlmann et al. [100]). This type of requirement is often modeled as constraints in the context of the periodic VRP [53, 68]. Finally, the quantities delivered to customers were also controlled in order to avoid large variations over time, which are negatively perceived by customers [23].

Quality of service features were incorporated in IRP solutions by Coelho et al. [57]. This was achieved by ensuring consistent solutions from three different aspects: quantities delivered, frequency of the deliveries and workforce management. These authors have shown through extensive computational experiments on benchmark instances that ensuring consistent solutions over time increases the cost of the solution between 1% and 8% on average.

4 Stochastic Inventory-Routing

In the SIRP, the supplier knows customer demand only in a probabilistic sense. Demand stochasticity means that shortages may occur. In order to discourage them, a penalty is imposed whenever a customer runs out of stock, and this penalty is usually modeled as a proportion of the unsatisfied demand. Unsatisfied demand is typically considered to be lost, that is, there is no backlogging. The objective of the SIRP remains the same as in the deterministic case, but is written so as to accommodate the stochastic and unknown future parameters: the supplier must determine a distribution policy that minimizes its expected discounted value (revenue minus costs) over the planning horizon, which can be finite or infinite. Typical problems dealing with SIRP applications arise in the oil and gas industry [20, 65, 125] and in maritime transportation [22, 50, 110, 111]. Table 4 lists the papers cited in this section.

4.1 Heuristic Algorithms

Several heuristic algorithms exist for the SIRP. Federgruen and Zipkin [65] have modified the VRP heuristic of Fisher and Jaikumar [67] to accommodate inventory and shortage costs in a random demand environment. Federgruen et al. [66] have extended the work of Federgruen and Zipkin [65] to allow multiple products, in their case, perishable items. Qu et al. [104] develop a periodic policy for a multi-item IRP and Huang and Lin [83] solve it by means of an ant colony optimization algorithm. Using a different approach, Golden et al. [77] determine which customers to visit based on their degree of urgency, before solving the routing problem heuristically by means of the Clarke and Wright [54] algorithm.

Table 4: Classification of the papers on the Stochastic IRP

Reference	Time horizon		Structure			Routing			Inventory policy		Inventory decisions		Fleet composition		Fleet size		
	Finite	Infinite	One-to-one	One-to-many	Many-to-many	Direct	Multiple	Continuous	Maximum level (ML)	Order-up-to level (OU)	Lost sales	Backlogging	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Federgruen and Zipkin [65]	✓			✓			✓		✓		✓			✓		✓	
Golden et al. [77]	✓			✓			✓		✓		✓		✓			✓	
Federgruen et al. [66]	✓			✓			✓		✓		✓			✓		✓	
Trudeau and Dror [125]	✓		✓				✓		✓		✓		✓			✓	
Minkoff [97]		✓		✓			✓		✓		✓		✓				✓
Bard et al. [20]	✓				✓		✓		✓		✓		✓			✓	
Campbell et al. [40]		✓		✓			✓		✓		✓		✓				✓
Qu et al. [104]		✓		✓			✓		✓		✓		✓		✓		
Berman and Larson [28]		✓		✓			✓		✓		✓		✓		✓		
Kleywegt et al. [87]		✓		✓		✓			✓		✓		✓				✓
Ronen [111]	✓				✓			✓	✓		✓		✓				✓
Adelman [4]		✓		✓			✓		✓		✓		✓			✓	
Kleywegt et al. [88]		✓		✓			✓		✓		✓		✓			✓	
Aghezzaf [7]		✓		✓			✓		✓		✓		✓			✓	
Hvattum and Løkketangen [84]		✓		✓			✓		✓		✓		✓			✓	
Hvattum et al. [85]		✓		✓			✓		✓		✓		✓			✓	
Huang and Lin [83]		✓		✓		✓				✓	✓		✓			✓	
Geiger and Sevaux [73]	✓			✓			✓		✓	✓	✓		✓				✓
Liu and Lee [92]	✓			✓			✓		✓		✓		✓			✓	
Solyalı et al. [119]	✓			✓			✓		✓			✓	✓		✓		

Given the size and the complexity of the SIRP, Minkoff [97] proposes a heuristic approach based on a Markov decision model to a problem somewhat similar to the IRP, called the Dispatch Delivery Problem. He simplifies the objective function, making it a sum of smaller and simpler objective functions, one for each customer, and solves the problem heuristically. This model is one of the few to work with an unconstrained fleet. Berman and Larson [28] also use dynamic programming to solve the case where the demand probability distributions are known, adjusting the amount of goods delivered to each customer, in order to minimize the expected sum of penalties associated with early or late deliveries.

Hvattum and Løkketangen [84] and Hvattum et al. [85] solve the problem heuristically, capturing the stochastic information over a short horizon. In Hvattum and Løkketangen [84] the problem is solved using a GRASP which successively increases the volume delivered to customers. Hvattum et al. [85] state that it is sufficient to capture the stochastics of the SIRP over a finite horizon which is achieved through truncated scenario trees, both breadthwise and depthwise.

Geiger and Sevaux [73] have studied a problem with unknown demand varying within 10% of a mean value. They proposed several policies based on delivery frequencies for each customer. They provide the Pareto front approximation of such policies when moving from a total routing-optimized solution to an inventory-optimized one. In order to solve the problem for many periods, they apply the record-to-record travel heuristic of Li et al. [90].

The classical road-based IRP with time windows was solved by Liu and Lee [92]. Their algorithm uses a combination of variable neighborhood search and tabu search. However, the effectiveness of the algorithm cannot be completely assessed because the computational comparison is made against three algorithms designed for the VRP with Time Windows.

4.2 Dynamic Programming

Campbell et al. [40] introduced a dynamic programming model for the SIRP in which only transportation and stockout costs are taken into account. To simplify the model, no inventory holding costs are incurred. At the beginning of each period the supplier knows the inventory level at each of the customers and decides which customers to visit, how much to deliver to each, how to combine them into routes and which routes to assign to each of the available vehicles. The components of their Markov decision process are the following:

- The state x is the current inventory at each customer and the state space $\mathcal{X}$ is $[0, C_1] \times [0, C_2] \times \ldots \times [0, C_n]$. Let $X_t \in \mathcal{X}$ denote the state at time t.
- The action space $\mathcal{A}(x)$ for each state x is the set of all itineraries satisfying constraints such as vehicle capacities and customer inventory capacities. Let $\mathcal{A} \equiv \bigcup_{x \in \mathcal{X}} \mathcal{A}(x)$ denote the set of all possible itineraries and $A_t \in \mathcal{A}(X_t)$ denote the itinerary chosen at time t.
- The Markov transition function R obtained from the known demand probability distribution. For any state $x \in \mathcal{X}$, any itinerary $a \in \mathcal{A}(x)$, and any (measurable) subset $B \subseteq \mathcal{X}$, the transition follows

$$P[X_{t+1} \in B \mid X_t = x, A_t = a] = \int_B R[dy \mid x, a]. \tag{38}$$

- The only costs taken into account are transportation costs, which depend on the vehicle tours, and a stockout penalty cost. Let $c(x, a)$ denote the expected daily cost if the process is in state x and itinerary $a \in \mathcal{A}(x)$ is chosen.
- Let $\alpha \in [0, 1)$ denote the discount factor. The objective is to minimize the expected total discounted cost over an infinite horizon. Let $V^*(x)$ denote the optimal expected cost given that the initial state is x, i.e.,

$$V^*(x) \equiv \inf_{\{A_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \alpha^t c(X_t, A_t) \mid X_0 = x\right]. \tag{39}$$

The actions are restricted in the sense that A_t depends only on the history of the system; when one decides which itinerary to choose, one does not know what the future holds. Under certain usual conditions, equation (39) can be written as

$$V^*(x) \equiv \inf_{a \in \mathcal{A}(x)} \left\{ c(x,a) + \alpha \int_X V^*(y) R[dy \mid x, a] \right\}. \tag{40}$$

Equation (40) can only be solved using classical dynamic programming algorithms if the state space $\mathcal{X}$ is small, which is not the case for practical instances of the SIRP. Campbell et al. [40] state that it is possible to solve the problem by approximating the value function $V^*(x)$ with a function $\hat{V}(x, \beta)$ that depends on a vector of parameters β. This is the approach followed by Kleywegt et al. [87, 88] who, as in Campbell et al. [40], use a Markov decision process to formulate the SIRP. Here, a set of customers must be served from a warehouse by means of a fleet of homogeneous capacitated vehicles. Each customer has an inventory capacity, and the problem is modeled in discrete time. Inventory at each customer at any given time is known to the supplier. Customer demands are stochastic and independent from each other, and the supplier knows the joint probability distribution of their demands, which does not change over time. The supplier must decide which customers to visit, how much to deliver to them, how to combine customers into routes, and which routes to assign to each vehicle. The set of admissible decisions is constrained by vehicle and customer capacities, driver working hours, possible time windows at the customers, and by any other constraint imposed by the system or the application. Although demands are stochastic, the cost of each decision is known to the supplier. Thus, Kleywegt et al. [87, 88] define the following costs:

- traveling costs c_{ij} on the arcs (i, j) of the network;
- shortages, if they occur, are proportional to the amount of unsatisfied demand s_i at customer i and cost $s_i(p_i)$. In this model unsatisfied demand is lost;
- inventory holding costs are incurred on the existing inventory x_i at customer i, plus the amount q_i delivered to this customer, and are equal to $(x_i + q_i)h_i$;
- finally, if the supplier delivers q_i at customer i, he then earns a revenue $r_i(q_i)$.

The problem is formulated so as to maximize the expected discounted value over an infinite horizon as a discrete time Markov decision process as follows. Let X_{it} denote the inventory level at customer i at time t. Thus x is the current inventory at each customer and the state space $\mathcal{X}$ is $[0, C_1] \times [0, C_2] \times \ldots \times [0, C_n]$. Let $X_t = (X_{1t}, X_{2t}, \ldots, X_{nt})$ denote the state at time t. The action space $\mathcal{A}(x)$ for each state x is the set of feasible decisions, that is, those that satisfy the constraints of the problem such as vehicle and customer capacities and any other constraint needed. Let $A_t \in \mathcal{A}(X_t)$ denote the decision made at time t. Let $k_{ij}(a)$ denote the number of times that arc (i, j) is traversed while executing decision a, for any a and arc (i, j). Finally, for any customer i, let $q_i(a)$ denote the quantity delivered to customer i while executing decision a.

Let d_{it} denote the demand at customer i at time t. Since there is no backlogging, consumption cannot exceed the amount available. In the way Kleywegt et al. [87] formulate the problem, the customer's inventory, plus the amount delivered are available for use in the same period. Thus the amount of product used by customer i at any time t is given by $\min\{d_{it}, X_{it} + q_i(A_t)\}$ and the shortage at customer i at any time t is $S_{it} = \max\{0, d_{it} - (X_{it} + q_i(A_t))\}$.

Kleywegt et al. [87] studied the case with direct deliveries only, whereas Kleywegt et al. [88] limited the routing to at most three customers per route. In the paper of Adelman [4] there is no limit on the number of customers to be served in a route, except for the limits resulting from maximal route duration and vehicle capacity. The approach taken by this author is a little different and works as follows. Using a value function not made up of individual customer values, but of marginal transportation costs, he compares stockout costs with replenishment policies, choosing the one that maximizes the value. A linear program is derived from the value function, and its optimal dual prices are used to calculate the optimal policy of the semi-Markov decision process. In the direct deliveries study of Kleywegt et al. [87] optimal solutions were obtained on instances with up to 60 customers and up to 16 vehicles, whereas in Kleywegt et al. [88] instances with up to 15 customers and five vehicles were solved.

4.3 Robust Optimization

A different way to model and solve the SIRP is through the use of robust optimization. This solution framework is appropriate to deal with uncertainty where no information is available on the parameter probability distributions. This is achieved by optimizing the problem while ensuring feasibility for all possible realizations of the bounded uncertain parameters, also called a minimax solution. Usually studies on the SIRP assume one knows the probability distribution of demand, which is generally not the case in practice. Aghezzaf [7] considers the case of normally distributed customer demands and travel times with constant averages and bounded standard deviations. He uses robust optimization to determine the distribution plan through a non-linear mixed-integer programming formulation which is feasible for all possible realizations of the random variables. Monte Carlo simulation is used to improve the plan's critical parameters (replenishment cycle times and safety stock levels). Solyalı et al. [119] proposed such an exact approach based on robust optimization, which we will now describe.

In their model, a supplier distributes a single product to n customers, using a vehicle of capacity Q, over a finite discrete time horizon p. The dynamic uncertain demand at each customer $i \in \mathcal{V} = \{1, \ldots, n\}$ in period $t \in \mathcal{T} = \{1, \ldots, p\}$ is d_i^t. The probability distribution of the demand is unknown, but one knows that it can take any value in the interval $[\bar{d}_i^t - \hat{d}_i^t, \bar{d}_i^t + \hat{d}_i^t]$, where $\bar{d}_i^t$ is the nominal value (point estimate), and $\hat{d}_i^t$ is the maximum deviation for the demand of i in period t. An inventory holding cost equal to h_i^t per unit at customer i in period t is incurred at the customers. Backlogging is allowed and each unit backlogged in period t at customer i costs g_i^t, where $g_i^t > h_i^t$. There is a fixed vehicle dispatching cost f_t for using the vehicle in period t. If the vehicle leaves customer $i \in \mathcal{V}' = \mathcal{V} \cup \{0\}$ heading to customer j, it incurs a cost c_{ij}, and transportation costs are assumed to be symmetric.

The problem is formulated as follows. Let q_{itk} be the total inventory cost of replenishing customer i in period $t \in \mathcal{T}$ to satisfy its demand in period $k \in \mathcal{T}$; $q_{i,T+1,k}$ the total inventory cost of not meeting the demand of customer i in period $k \in \mathcal{T}$; w_{itk} the fraction of the demand of customer i in period $k \in \mathcal{T}$ delivered in period $t \in \mathcal{T}$; and $w_{i,T+1,k}$ the fraction of the unsatisfied demand of customer i in period $k \in \mathcal{T}$. Additionally let y_{it} be 1 if customer i is replenished in period $t \in \mathcal{T}$ and 0 otherwise; y_{0t} be 1 if the vehicle is used in period $t \in \mathcal{T}$ and 0 otherwise; and x_{ij}^t be the number of times the edge (i, j) is traversed in period $t \in \mathcal{T}$. The robust IRP is then formulated as

$$\text{minimize} \sum_{t \in \mathcal{T}} f_t y_{0t} + \sum_{i \in \mathcal{V}'} \sum_{j \in \mathcal{V}', i<j} \sum_{t \in \mathcal{T}} c_{ij} x_{ij}^t + \sum_{i \in \mathcal{V}} \sum_{t=1}^{p+1} \sum_{k=1}^{p} d_i^k q_{ikt} w_{itk} \tag{41}$$

subject to

$$\sum_{t=1}^{p+1} w_{itk} = 1 \qquad i \in \mathcal{V} \quad k \in \mathcal{T}; \tag{42}$$

$$w_{itk} \leq y_{it} \qquad i \in \mathcal{V} \quad t, k \in \mathcal{T} \quad d_i^k > 0; \tag{43}$$

$$\sum_{i \in \mathcal{V}} \sum_{k=1}^{p} d_i^k w_{itk} \leq Q y_{0t} \qquad t \in \mathcal{T}; \tag{44}$$

$$\sum_{j\in\mathcal{V}',i<j} x_{ij}^t + \sum_{j\in\mathcal{V}',i>j} x_{ji}^t = 2y_{it} \qquad i\in\mathcal{V}' \quad t\in\mathcal{T}; \tag{45}$$

$$\sum_{i\in\mathscr{S}} \sum_{j\in\mathscr{S},i<j} x_{ij}^t \le \sum_{i\in\mathscr{S}} y_{it} - y_{mt} \qquad \mathscr{S}\subseteq\mathcal{V} \quad t\in\mathcal{T} \quad m\in\mathscr{S}; \tag{46}$$

$$y_{it} \le y_{0t} \qquad i\in\mathcal{V} \quad t\in\mathcal{T}; \tag{47}$$

$$x_{ij}^t \in \{0,1\} \qquad i,j\in\mathcal{V}, i<j \quad t\in\mathcal{T}; \tag{48}$$

$$x_{i0}^t \in \{0,1,2\} \qquad i\in\mathcal{V} \quad t\in\mathcal{T}; \tag{49}$$

$$y_{it} \in \{0,1\} \qquad i\in\mathcal{V}' \quad t\in\mathcal{T}; \tag{50}$$

$$w_{itk} \ge 0 \qquad i\in\mathcal{V} \quad k\in\mathcal{T} \quad 1\le t\le p+1, \tag{51}$$

where $q_{itk} = \sum_{l=t}^{k-1} h_i^l$ if $t \le k$ and $q_{itk} = \sum_{l=k}^{t-1} g_i^l$ if $t > k$.

The objective function (41) is the sum of the fixed vehicle dispatching, transportation, inventory holding and shortage costs. Constraints (42) specify that the demand of customer i in period k is either met from periods 1 through p, or lost. Constraints (43) allow the vehicle to serve customer i in period t only if a replenishment to customer i takes place in period t. Contraints (44) ensure that the vehicle capacity is not exceeded. Constraints (45) are degree constraints, guaranteeing that if i is visited in period t, then there are two edges incident to it. Constraints (46) are subtour elimination constraints. Constraints (47) ensure the vehicle starts its tour from the supplier and are used to strengthen the formulation. Constraints (48)−(50) and (51) are integrality constraints and non-negativity constraints, respectively.

If d_i^k is replaced by $\bar{d}_i^t$ for $i \in \mathcal{V}$, $t \in \mathcal{T}$, then it is called the nominal formulation, since it does not incorporate any robustness. The derivation of the robust formulation is rather involved and the reader is referred to Solyalı et al. [119] for details. Their final robust formulation ensuring feasibility for any $d_i^k \in [\bar{d}_i^t - \hat{d}_i^t, \bar{d}_i^t + \hat{d}_i^t]$ is

$$\text{minimize} \sum_{t\in\mathcal{T}} f_t y_{0t} + \sum_{i\in\mathcal{V}'} \sum_{j\in\mathcal{V}',i<j} \sum_{t\in\mathcal{T}} c_{ij} x_{ij}^t + \sum_{i\in\mathcal{V}} \sum_{t=1}^{p+1} \sum_{k=1}^{p} q_{itk} w'_{itk} \tag{52}$$

subject to (45)−(50) and to

$$\sum_{i\in\mathcal{V}} \sum_{k=1}^{p} w'_{itk} \le Q y_{0t} \qquad t\in\mathcal{T}; \tag{53}$$

$$w'_{itk} \ge 0 \qquad i\in\mathcal{V} \quad k\in\mathcal{T} \quad 1\le t\le p+1; \tag{54}$$

$$\sum_{t=1}^{p+1} w'_{itk} \ge \bar{d}_i^t + \hat{d}_i^t \qquad i\in\mathcal{V} \quad k\in\mathcal{T}; \tag{55}$$

$$w'_{itk} \le (\bar{d}_i^t + \hat{d}_i^t) y_{it} \qquad i\in\mathcal{V} \quad t\in\mathcal{T} \quad k\in\mathcal{T}, \tag{56}$$

where $w'_{itk} = d_{ik} w_{itk}$. Using this formulation the authors have solved instances with up to seven periods and 30 customers within a reasonable computing time.

5 Dynamic and Stochastic Inventory-Routing Problem

In the dynamic IRP, customer demand is gradually revealed over time, e.g., at the end of each period, and one must solve the problem repeatedly with the available information. We are aware of two studies on this problem, both making use of probabilistic knowledge of the demand. In Dynamic and Stochastic Inventory-Routing Problems (DSIRP), customer demand is known in a probabilistic sense and revealed over time, thus yielding a dynamic and stochastic problem.

Solving a dynamic problem consists of proposing a *solution policy* as opposed to computing a static output [27]. A possible policy is to optimize a static instance whenever new information becomes available. The drawback of such a method is that it is often very time consuming to solve a large number of instances. A more common policy is to apply the static algorithm only once and then reoptimize the problem through a heuristic whenever new information is made available. A third policy, which can be combined with either of the first two, is to take advantage of the probabilistic knowledge of future information and make use of forecasts. For more information on the solution of dynamic problems, see Psaraftis [103], Ghiani et al. [76] and Berbeglia et al. [27].

Recently, Bertazzi et al. [34] and Coelho et al. [58] have solved DSIRPs, whose goal is to minimize the total inventory, distribution and shortage costs. The paper of Coelho et al. is more general than that of Bertazzi et al. in that it develops and compares several policies instead of only one. In particular, Bertazzi et al. considered only the OU policy while Coelho et al. studied both the OU and ML policies and compared their costs. Moreover, while the first authors used probabilistic information as a proxy for future demands using only averages, the latter proposed a method that can make use of historical data in the form of forecasts in order to take future unknown demands into account, thus being able to efficiently solve instances in which the demand presents a trend or seasonalities. Both groups of researchers have implemented their algorithms in a rolling horizon framework. In addition, Coelho et al. have studied the impact of varying several system parameters as well as other variants of the problem.

Bertazzi et al. [34] tested their algorithm on instances with up to 35 customers and three periods, 15 customers and six periods, and 10 customers and nine periods. Coelho et al. [58] solved instances containing up to 200 customers and 20 periods.

6 Benchmark Instances

Benchmark instance sets are now available to the research community and allow for a better assessment and comparison of algorithms. We have aggregated these instances into a single website in order to make their access easier and to encourage other researchers to use them. They are all available at `http://www.leandro-coelho.com/instances`. The first set was proposed by Archetti et al. [15] and comprises 160 instances ranging from five to 50 customers, with three and six periods. These were used to evaluate the algorithms of Bertazzi et al. [32], Archetti et al. [15], Solyalı and Süral [118], Archetti et al. [17], Coelho et al. [56, 57] and Coelho and Laporte [55]. A newer, larger and more challenging data set proposed by Archetti et al. [17] contains 60 instances with six periods and up to 200 customers. This set has been used to evaluate the algorithms of Archetti et al. [17], Coelho et al. [57] and Coelho and Laporte [55]. Finally, Coelho et al. [58] have proposed a large test bed for the DSIRP, containing 450 instances.

7 Conclusions

The IRP was introduced 30 years ago by Bell et al. [25] and has since evolved into a rich research area. Several versions of the problem have been studied, and applications are encountered in many settings, primarily in maritime transportation. Our survey provides a classification of the IRP literature under two dimensions: the structure of the problem and the time at which information becomes available. Because IRPs are typically very hard to solve, most algorithms are heuristics. These have gradually evolved from simple interchange schemes to more sophisticated metaheuristics, sometimes combined with exact methods. In recent years, we have also witnessed the emergence of exact branch-and-cut algorithms which can be implemented within the framework of general purpose solvers. Over the years, part of the research effort has shifted toward the study of rich extensions of the basic IRP model. These include the production-routing problem, the IRP with multiple products, the IRP with direct deliveries and transshipment, and the consistent IRP. Finally, several authors have moved away from the deterministic and static version of the IRP and have proposed models and algorithms capable of handling its stochastic and dynamic versions. We believe this paper has helped unify the body of knowledge on the IRP and will stimulate other researchers to pursue the study of this fascinating field.

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