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A Unifying View on Timing Problems and Algorithms[†]

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Abstract. Timing problems involve the choice of task execution dates within a predetermined processing sequence, and under various additional constraints or objectives such as time windows, time-dependent costs and so on. Their efficient solving is critical in branch and bound and neighborhood search methods for vehicle routing, scheduling with idle times, as well as in various applications in network optimization and statistical inference. Timing related problems have been studied for years, yet research on this subject suffers from a lack of consensus, and most knowledge is scattered among operations research and applied mathematics domains. This article introduces a classification for timing problems and features, as well as a unifying multidisciplinary analysis of timing algorithms. In relation to frequent application cases within branching schemes or neighborhood searches, the efficient resolution of series of similar timing subproblems is also analyzed. A dedicated formalism of re-optimization "by concatenation" is introduced to that extent. The knowledge developed through this analysis is valuable for modeling work and algorithmic design, for a wide range of combinatorial optimization problems with time characteristics, including rich vehicle routing settings and emerging non-regular scheduling applications, among others.

Keywords: Timing, scheduling, routing, statistical inference, branch and bound, neighborhood search.

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1 Introduction

Time-related constraints and objectives appear in a variety of flavors within scheduling, project management, data transmission, routing, network optimization and numerous other fields. Several problem settings in these domains involve the allocation and arrangement of elementary *activities* under time requirements, such as tasks, visits to customers, production of objects, and so on. They may thus be seen as a combination of three issues (Desrosiers et al. 1995): the repartition of activities among resources, called *resource allocation*, the choice of an execution order for activities on each resource, called *sequencing*, and finally the adjustment of activity execution dates with respect to this order, called *scheduling* or *timing*. We will favor the name *timing* in this paper, as the word *scheduling* is already employed for various other settings in the literature.

In most situations, the combinatorial aspect of resource allocation and sequencing issues leads to NP-hard problem settings. Most heuristic and exact solution approaches perform a search through a large number of sequence and resource allocation alternatives, using repeatedly a timing algorithm to produce adequate execution dates, filter feasible solutions and evaluate costs. In all these cases, the timing algorithm is called extensively, and thus its complexity impacts dramatically the performance of the solution method, making the difference between successful algorithmic approaches and failure.

Timing problems therefore are the cornerstone of many algorithms. Yet, the literature dedicated to this subject remains scarce and scattered. Most developments on timing are made in relation to particular fields such as project planning, shortest path, routing, scheduling, and statistical inference, which bring into play, quite unexpectedly, the same formulations. Few relationships between domains have been actually exploited and, thus, close concepts and solution methods are independently developed within different formalisms, being rarely assessed in a more general context. The large number of problem variants arising from real-life settings bring forth additional challenges related to the variety of timing problems that must be dealt with. Among typical features we find target execution dates, (possibly multiple) time windows on activities, penalized lateness and earliness, speed decisions, time-dependent activity durations and costs, congestion, learning issues, and so on and so forth. Although efficient timing algorithms have been designed for some of these characteristics taken separately, problems involving combinations of characteristics become much more complex, and there is generally no systematic way to evolve concepts developed for the separate problems into a methodology for the new ones.

This paper contributes to the *timing field*, by means of a multidisciplinary review and analysis of timing features, problems, and algorithmic approaches. A large assortment of problems, often treated independently in the literature under various names, are identified and classified in relation to their structure. Successful solution methods and solving concepts are inventoried and analyzed. In the most noticeable cases, this analysis led to identify more than 26 algorithms from different research fields as similar implementations of three main general approaches (Sections 5.3-5.4). Not only does this review gather the keys for a stand-alone resolution of a large variety of timing problems, but it also analyzes the efficient resolution of timing problems within the context of global search methods, e.g. neighborhood-based heuristics and branch-and-bound-based approaches. For these applications, managing global information through the successive resolution of similar timing instances can lead to dramatic reductions of the overall computational effort. To this extent, a *re-optimization framework* is introduced in the second

part of this paper. The body of knowledge developed in this paper is critical for both modeling work and algorithmic design, enabling to point out relationships between problems and their respective complexities. A portfolio of state-of-the-art timing algorithms is identified, which will prove useful to build more generalist solvers for many variants of difficult combinatorial optimization problems. To our knowledge, no such unifying review and methodological analysis of this rich body of issues has been performed in the past.

The remainder of this paper is organized as follows: Section 2 formally defines timing problems, while Section 3 presents examples of applications. Section 4 provides a detailed classification of the main timing features encountered in the literature as well as notations. Our methodological analysis of timing problems and their *independent resolution* is then organized in Sections 5, 6, and 7 relatively to the previous classification. Section 8 finally introduces a *re-optimization framework* that encompasses state-of-the-art approaches for solving *series* of related timing instances. Section 9 highlights a number of challenging avenues of research in the timing field, and concludes.

2 Problem statement

In this paper, the term *activities* is used to represent, independently of the field of application, elementary operations that must be managed. The term *date* always stands for a point in time, whereas the words *duration* or *time* are employed for relative values (e.g., processing time). Without loss of generality, objective minimization is considered.

Definition 1 (General timing problem). *Let $A = (a_1, \dots, a_n)$ be a sequence of n activities with processing times p_1, \dots, p_n . The execution dates of these activities $\mathbf{t} = (t_1, \dots, t_n)$ constitute the decision variables of the timing problem, and are required to follow a total order with respect to the subscripts, such that $t_i \leq t_{i+1}$ for $i \in \{1, \dots, n-1\}$. Additional so-called features complete the description of the problem, providing the means to address particular settings. Let m be the number of such features. A feature F^x for $x \in \{1, \dots, m\}$ corresponds to the specification of a set of m_x characteristic functions $f_y^x(\mathbf{t})$ for each $y \in \{1, \dots, m_x\}$. A role is associated to each feature F^x , either as “objective” ($F^x \in \mathcal{F}^{\text{OBJ}}$) or as “constraint” ($F^x \in \mathcal{F}^{\text{CONS}}$). The general timing problem aims to find a feasible timing solution $\mathbf{t} = (t_1, \dots, t_n)$, respecting order and features constraints (Equations 2-3), and minimizing the weighted sum of contributions from all feature objectives (Equation 1).*

$$\min_{\mathbf{t}=(t_1, \dots, t_n) \in \mathbb{R}^{n+}} \sum_{F^x \in \mathcal{F}^{\text{OBJ}}} \alpha_x \sum_{1 \leq y \leq m_x} f_y^x(\mathbf{t}) \quad (1)$$

$$s.t. \quad t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \quad (2)$$

$$f_y^x(\mathbf{t}) \leq 0 \quad F^x \in \mathcal{F}^{\text{CONS}}, \quad 1 \leq y \leq m_x \quad (3)$$

Two illustrative examples of features follow. The feature *deadlines* D , for example, completes the model with $m_D = n$ additional parameters, representing latest execution dates d_i for activities, and involve the characteristic functions $f_i^D(\mathbf{t}) = (t_i - d_i)^+$ ($i = 1, \dots, n$), where a^+ denotes $\max(a, 0)$. When D takes the role of a *constraint*, $f_i^D(\mathbf{t}) = (t_i - d_i)^+ \leq 0 \Leftrightarrow t_i \leq d_i$ yields the standard formulation of deadlines, while a role as objective leads to standard tardiness optimization criteria. The feature *time-lags* TL involves up to $m_{TL} = n^2$ additional parameters δ_{ij} on minimum elapsed time between the execution dates of activity pairs. The characteristic functions are $f_{ij}^{TL}(\mathbf{t}) = (t_j - \delta_{ij} - t_i)^+$.

Timing problems can be viewed as shifting activity execution dates on a single resource, without ever changing the processing order. This issue is equivalent to idle-time insertion when processing times are fixed. Most basic versions of timing are straightforward to solve, while various features arising from application cases can lead to dramatic increases in problem difficulty. Traditionally, in the scheduling domain, some constraints and objectives, such as due dates, are based on activity completion dates $C_i = t_i + p_i$. Without loss of generality, these problems are reformulated to involve only execution dates. It must also be noted that features have been defined independently from their role as constraint or objective for two main reasons. First, many algorithms are concerned with the effective calculation of some quantities, like total duration for example, that enable to tackle related constraints or objectives in the same way. Secondly, since constraints can be transformed into objectives by Lagrangian relaxation, it is sometimes artificial to discriminate problems involving a given feature either as constraint or objective. Our study will thus be targeted on features, independently of their role, the latter being specified only when relevant to the method.

To emphasize the relations with practical problem settings, Section 3 now details some major problems in the fields of operations research and applied mathematics leading to underlying timing structures.

3 Timing issues and major application fields

Production and project scheduling. The development of just-in-time policies leads to challenging non-regular scheduling settings for which earliness or idle times are a major concern. Time-dependent processing times and costs (Alidaee and Womer 1999, Cheng 2004, Gawiejnowicz 2008), especially, represent very active areas of research (Kanet and Sridharan 2000). In the earliness and tardiness (E/T) scheduling problem, for example, a sequence of activities (a_1, \dots, a_n) is given with target execution dates d_i and processing times p_i , as well as penalty factors for earliness ϵ_i and tardiness τ_i . The goal is to determine the sequence of activities and their execution dates on a single machine, such that linear penalties incurred for early or late processing are minimized. This scheduling problem presents a non-regular objective, and is NP-hard in the strong sense (Garey et al. 1988). Most recent approaches for (E/T) scheduling consider branch and bound, neighborhood search or other metaheuristic frameworks working on the activity sequence (Baker and Scudder 1990). For every sequence explored during the search, a timing algorithm is applied to compute the activity execution dates and thus the sequence cost. This timing problem is formulated in Equations (4-5).

$$\text{(E/T) timing: } \min_{(t_1, \dots, t_n) \in \mathcal{R}^{n+}} \sum_{i=1}^n \{\epsilon_i (d_i - t_i)^+ + \tau_i (t_i - d_i)^+\} \quad (4)$$

$$s.t. \quad t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \quad (5)$$

This problem involves two main features: release dates and deadlines. The characteristic functions of these features, involved in the objective, are $f_i(\mathbf{t}) = (r_i - t_i)^+$ and $f_i(\mathbf{t}) = (t_i - d_i)^+$, respectively. This timing problem is known to be solvable in $O(n \log n)$ (Sections 5.3 and 5.4). Yet, as the timing resolution is the main bottleneck for most (E/T) scheduling approaches, extensive research has been conducted to solve series of timing instances more efficiently within neighborhood searches. The use of global information through the search leads to timing “re-optimization” procedures working in amortized $O(\log n)$ complexity, and even $O(1)$ for some

particular cases, as described in Section 8.

Network optimization. Timing subproblems are also frequently encountered in network optimization settings, e.g., resource-constrained shortest paths, delivery-man and minimum latency, vehicle routing and scheduling, among others (Solomon and Desrosiers 1988, Desrochers et al. 1990, Desrosiers et al. 1995, Vidal et al. 2012). Thus, for example, the vehicle routing problem with time windows (VRPTW) consists in designing vehicle itineraries to service a set of geographically scattered customers within allowed time intervals. The VRPTW makes for one of the most intensively studied combinatorial optimization problem, as underlined by dozens of literature reviews on the subject (see Kallehauge et al. 2005, Bräysy and Gendreau 2005b,a, Gendreau and Tarantilis 2010, for the most recent). Timing subproblems arise when checking the feasibility, or estimating the minimal amount of violation with respect to time-windows, for the sequences (routes) produced in the course of the search. Time windows are a combination of both deadline and release date features, and the resulting timing formulations are closely related to those present in (E/T) scheduling.

One also observes a recent important focus on “richer” VRPs (Hartl et al. 2006, Vidal et al. 2012), which explicitly take into account various combined constraints and objectives issued from application cases. These complex combinatorial optimization problems frequently involve temporal considerations, time-dependent travel speed, crew costs, customer requirements in terms of visit times, employee breaks and duty times, learning or fatigue effects, fair repartition of working time among employees, and so on. Such characteristics must be directly managed within route evaluations in heuristics and exact methods, thus leading to a large variety of timing subproblems. Having at hand efficient and generic timing solvers is thus an important step towards designing robust solvers for rich VRP settings.

Energy optimization. Norstad et al. (2010) introduce a ship routing problem with convex speed optimization, which presents two interlaced issues: the design of a ship itinerary, and the optimization of arrival dates and speed (PATSO) to reduce fuel consumption. For a fixed sequence of visits, the latter subproblem is formulated in Equations (6-9).

$$\text{(PATSO): } \min_{\mathbf{t}, \mathbf{v}} \sum_{i=1}^{n-1} d_{i,i+1} c(v_{i,i+1}) \quad (6)$$

$$s.t. \quad t_i + p_i + d_{i,i+1}/v_{i,i+1} \leq t_{i+1} \quad 1 \leq i \leq n-1 \quad (7)$$

$$r_i \leq t_i \leq d_i \quad 1 \leq i \leq n \quad (8)$$

$$v_{min} \leq v_{i,i+1} \leq v_{max} \quad 1 \leq i \leq n-1 \quad (9)$$

In the previous formulation, the decision variables are the travel speeds $v_{i,i+1}$ for $i \in \{1, \dots, n-1\}$ for each port-to-port leg, and the arrival dates at ports t_i for $i \in \{1, \dots, n\}$. The objective is to minimize the fuel consumption on all trips, $c(v)$ representing the energy consumption per mile as a convex and increasing function of speed. Let v_{opt} be the minimum of this function. $d_{i,i+1}$ represents the leg distances, and p_i stands for processing times at ports. Equations (7-9) ensure that port arrival and departure dates are consistent with leg speeds, that time windows at port arrivals are respected, and finally that speeds are within a feasible range.

This problem can be reformulated to rely exclusively on arrival dates by defining an extended cost/speed function $\hat{c}(v)$, which accounts for the fact that waiting times can be used in case of

sub-optimal low speed values (Equations 10-13).

$$\text{(PATSO-2): } \min_{\mathbf{t}} \sum_{i=1}^n d_{i,i+1} \hat{c} \left(\frac{d_{i,i+1}}{t_{i+1} - t_i} \right) \quad (10)$$

$$\text{s.t. } t_i + p_i + d_{i,i+1}/v_{max} \leq t_{i+1} \quad 1 \leq i \leq n-1 \quad (11)$$

$$r_i \leq t_i \leq d_i \quad 1 \leq i \leq n \quad (12)$$

$$\text{with } \hat{c}(v) = \begin{cases} c(v_{opt}) & \text{if } v \leq v_{opt} \\ c(v) & \text{otherwise} \end{cases} \quad (13)$$

The latter model falls into the category of timing problems. It involves time-window features characterized by functions $f_i(\mathbf{t}) = (t_i - d_i)^+ + (r_i - t_i)^+$ with a role as constraints, as well as flexible processing times characterized by convex functions $f_i(\mathbf{t}) = c_i(t_{i+1} - t_i)$ in the objective, such that $c_i(\Delta t_i) = d_{i,i+1} \hat{c}(d_{i,i+1}/\Delta t_i)$. Norstad et al. (2010) introduced a Recursive Smoothing Algorithm (RSA) to solve the previous timing problem with a worst case complexity of $O(n^2)$. A similar algorithm was then used in the context of a vehicle routing problem with CO_2 minimization (Demir et al. 2012). Several other timing algorithms and re-optimization procedures are known for these settings (Sections 6.3 and 8.5.6).

Statistical Inference. The isotonic regression problem under a total order (IRC) constitutes an intensively studied particular case of our models. Given a vector $\mathbf{N} = (N_1, \dots, N_n)$ of n real numbers, IRC seeks a vector of non-decreasing values $\mathbf{t} = (t_1, \dots, t_n)$ as close as possible to \mathbf{N} according to a distance metric $\| \cdot \|$ (generally the Euclidean distance), as in Equations (14-15):

$$\text{(IRC): } \min_{\mathbf{t}=(t_1, \dots, t_n)} \| \mathbf{t} - \mathbf{N} \| \quad (14)$$

$$\text{s.t. } t_i \leq t_{i+1} \quad 1 \leq i < n \quad (15)$$

As underlined by the seminal books of Barlow et al. (1972) and Robertson et al. (1988), IRC is the key to performing many restricted maximum likelihood estimates in statistics, and is linked with various applications such as image processing and data analysis. It appears here as a timing problem with separable convex costs, similar to those encountered when solving vehicle routing problems with time windows or (E/T) scheduling settings.

Other applications. Timing formulations also appear under more general mathematical formalisms, under the name of *projection onto order simplexes* in Grotzinger and Witzgall (1984), or as particular cases of several convex optimization problems with underlying network structures (Hochbaum 2002, Ahuja et al. 2003). We now develop a new classification, as well as notations, for the main classes of timing features and problems.

4 Features : classification and reductions

This section introduces a classification of the main timing features in the literature, and levers notations for the related problems. Reduction relationships between features are then investigated.

4.1 Classification and notations

The features are here classified relatively to the structure of their characteristic functions. We rely to that extent on a *feature dimension* measure ξ , which illustrates the links that a feature creates between decision variables.

Definition 2 (Feature dimension). *The dimension $\xi(F^x)$ of a feature F^x is defined as the maximum number of variables involved together in any characteristic function $f_y^x(\mathbf{t})$ for $y \in \{1, \dots, m_x\}$.*

Table 1 presents the most common features in the literature relatively to their dimension ξ . The first column provides an abbreviation for each feature. The next columns describe the parameters, characteristic functions and dimensions of these features. Finally, we report the most frequent roles of each feature in the literature.

Table 1: Classification of timing features and notations

Symbol	Parameters	Char. functions	ξ	Most frequent roles
C	Deadline t_{max} on last activity	$f(\mathbf{t}) = (t_n - t_{max})^+$	1	Deadline on last activity, lateness of last activity, makespan
W	Weights w_i	$f_i(\mathbf{t}) = w_i t_i$	1	Weighted execution dates
D	Deadlines d_i	$f_i(\mathbf{t}) = (t_i - d_i)^+$	1	Deadline constraints, tardiness
R	Release dates r_i	$f_i(\mathbf{t}) = (r_i - t_i)^+$	1	Release-date constraints, earliness.
TW	Time windows $TW_i = [r_i, d_i]$	$f_i(\mathbf{t}) = (t_i - d_i)^+ + (r_i - t_i)^+$	1	Time-window constraints, soft time windows.
MTW	Multiple TW $MTW_i = \cup[r_{ik}, d_{ik}]$	$f_i(\mathbf{t}) = \min_k [(t_i - d_{ik})^+ + (r_{ik} - t_i)^+]$	1	Multiple time-window constraints
$\Sigma c_i^{cvx}(t_i)$	Convex $c_i^{cvx}(t_i)$	$f_i(\mathbf{t}) = c_i^{cvx}(t_i)$	1	Separable convex objectives
$\Sigma c_i(t_i)$	General $c_i(t)$	$f_i(\mathbf{t}) = c_i(t_i)$	1	Separable objectives, time-dependent activity costs
DUR	Total dur. δ_{max}	$f(\mathbf{t}) = (t_n - \delta_{max} - t_1)^+$	2	Duration or overall idle time
NWT	No wait	$f_i(\mathbf{t}) = (t_{i+1} - p_i - t_i)^+$	2	No wait constraints, min idle time
IDL	Idle time ι_i	$f_i(\mathbf{t}) = (t_{i+1} - p_i - \iota_i - t_i)^+$	2	Limited idle time by activity, min idle time excess
$P(t)$	Time-dependent proc. times $p_i(t_i)$	$f_i(\mathbf{t}) = (t_i + p_i(t_i) - t_{i+1})^+$	2	Processing-time constraints, min activities overlap
TL	Time-lags δ_{ij}	$f_i(\mathbf{t}) = (t_j - \delta_{ij} - t_i)^+$	2	Min excess with respect to time-lags
$\Sigma c_i(\Delta t_i)$	General $c_i(t)$	$f_i(\mathbf{t}) = c_i(t_{i+1} - t_i)$	2	Separable functions of durations between successive activities, flex. processing times
$\Sigma c_i(t_i, t_{i+1})$	General $c_i(t, t')$	$f_i(\mathbf{t}) = c_i(t_i, t_{i+1})$	2	Separable objectives or constraints by successive pairs of variables
$\Sigma c_{ij}(t_i, t_j)$	General $c_{ij}(t, t')$	$f_{ij}(\mathbf{t}) = c_{ij}(t_i, t_j)$	2	Separable objectives or constraints by any pairs of variables
$c(\mathbf{t})$	General $c(t)$	$f(\mathbf{t}) = c(\mathbf{t})$	–	Any feature

Most of the features presented in Table 1 are well-known in the scheduling or vehicle routing literature. Features D , C , and W , can be qualified as *regular*, as they involve non-decreasing characteristic functions $f_y^x(\mathbf{t})$. The set of *active schedules* “such that no operation can be made to

start sooner by permissible left shifting” (Giffler and Thompson 1960) is dominating for regular features. Solving the timing problem is then straightforward by means of a minimum idle time policy (Section 5.1). However, these regular features present notably different behaviors with regards to re-optimization, thus motivating a detailed study. Other features from Table 1 are *non-regular*. They lead to more complex timing problems for which the insertion of idle time can improve the objective or the satisfaction of constraints.

Single- and two-dimensional features are directly linked to physical quantities, execution dates and durations, respectively. As seen in Table 1, such features are frequently encountered in timing formulations. Higher-dimension features are more unusual in the literature, and can, by definition, lead to a wide range of difficult problems. Indeed, any mathematical program can be viewed as a combination of unary and binary mathematical operators of the form $x = f(y, z)$, and thus can be reformulated with constraints and objectives separable in groups of three variables. Hence, any problem on continuous totally ordered variables can be viewed as a timing problem with three-dimensional features. A reasonable limit to define “*what a timing problem is*” is then to consider, as in the present paper, only applications and problems presenting explicitly the aspect of an activity sequence.

We introduce a notation specifying for each problem the features considered, as well as information regarding their role. Each problem is tagged as a two-component string $\{O|C\}$, where O is a list of features involving the objective and C lists features that participate to constraints. Separating features in the field O with a comma indicates a weighted sum of objectives. The sign \cup is used for multi-objective problems and the sign $>$ indicates an order of priority. Particular parameter characteristics are reported in parentheses after the feature symbol. For example, problems with common deadlines can be marked with $(d_i = d)$, null processing times as $(p_i = 0)$, and so on.

To illustrate, consider the problem of speed optimization of Section 3. This problem presents a separable and convex objective as a function of durations between successive activities, along with time-window constraints. It can thus be categorized as $\{\Sigma c^{cvx}(\Delta t)|TW\}$. The (E/T) timing problem presents linear penalties around a target execution date. These penalties can be assimilated to relaxed simultaneous release dates and deadlines, leading to the notation $\{R, D(r_i = d_i)|\emptyset\}$. Finally, the vehicle routing literature includes problem settings with a hierarchical objective aiming first to minimize the amount of time-window violations, then duration excess, and finally time-lag violations (Cordeau and Laporte 2003, Berbeglia et al. 2011). Such a problem setting can be characterized as $\{TW > D > TL|\emptyset\}$.

4.2 Feature reductions

We use reduction relationships to illustrate the level of generality and complexity of timing features. A rich body of polynomial reductions has been developed in the scheduling literature. Most timing problems, however, are polynomially solvable, and the use of polynomial reduction relationships leads to consider most problems in the same class of equivalence. We thus seek stronger reduction properties to distinguish them. We also aim to build relationships between features instead of complete problems, leading to the following definition of feature reductions:

Definition 3 (Reducibility among timing features). *A feature F is said to be reducible to feature F' if any timing problem \mathcal{T} involving F and other features $\{F^1, \dots, F^k\}$ admits a linear many-one reduction to a timing problem \mathcal{T}' involving $F' \cup \{F^1, \dots, F^k\}$.*

An overview of feature reductions is given in Figure 1, where an arrow from feature F^i

to F^j indicates that feature F^i can be reduced to F^j . Four different categories of features are identified by different shades of gray. On the left, we present features involving at most one decision variable (the first part of Table 1) and separable costs. Progressing to the right, the next gray shade represents two-dimensional features that involve only pairs of consecutive activities, then features involving any pair of activities, and finally other features. We also demarcate the area of “NP-hard” features, which alone are sufficient to lead to NP-hard timing problems.

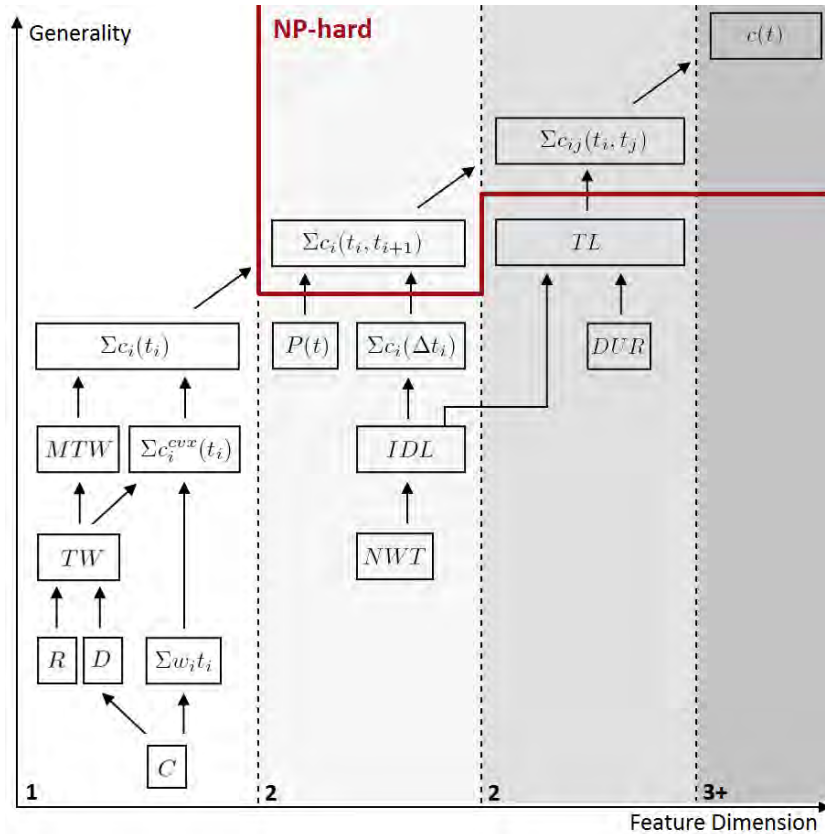


Figure 1: Hierarchy of timing features

The hierarchy of reductions presented in Figure 1 gives an indication on the level of generality of the features. Some features, such as $\Sigma c_i^{cvx}(t_i)$, $\Sigma c_i(t_i)$, $\Sigma c_i(\Delta t_i)$, and TL , generalize many other features while remaining polynomially solvable. An algorithm addressing such general features can tackle many problems, while specialized algorithms for simpler combinations of features may be more efficient. Both specialized and general algorithms are critical for practical applications and deserve a detailed study. We thus provide a methodological review ordered by increasing generality of the main features, timing problems and solution methods in the literature. We start with the most simple cases of single-dimensional features in Section 5, and follow with two-dimensional features in Section 6, to conclude our analysis of stand-alone methods for timing in Section 7.

5 Single-dimensional features

Problems with single-dimensional features are analyzed according to their difficulty and generality, starting with simple regular features, following with time-window TW features, separable convex costs $\Sigma c_i^{\text{CVX}}(t_i)$ and, finally, general separable costs $\Sigma c_i(t_i)$. The latter feature encompasses multiple time windows MTW and generalizes all problems in this category.

5.1 Makespan, deadlines and weighted execution dates

Maximum execution dates C , deadlines D , and weighted execution dates W features lead to several well-documented objectives in the scheduling literature, aiming to minimize makespan, tardiness, lateness or total weighted starting or completion times among others (Graham et al. 1979, Pinedo 2008). W as an objective also arises in various routing settings, such as the delivery-man problem (Fischetti et al. 1993), the minimum latency problem (Blum et al. 1994), and the cumulative TSP or VRP (Bianco et al. 1993, Ngueveu et al. 2010), where the goal is to service a set of customer as early as possible. These features are *regular*, as any backward shift of execution date is beneficial for both feasibility and objective value. A very simple algorithm follows, which will be referred to as the “minimum idle time policy”: *For each activity a_i of A in the sequence order, schedule a_i at its earliest possible execution date. If a_i cannot be scheduled, declare problem infeasibility and stop. If all activities have been successfully scheduled, declare problem feasibility.* An optimal solution is thus retrieved in n searches of the earliest feasible execution date, leading to a $O(n)$ complexity.

5.2 Release dates and time windows

Release dates and time-window features appear frequently in vehicle routing and scheduling applications. Time-window features generalize release dates R and deadlines D , as any release date r_i or deadline d_i can be transformed into a time window with an infinite value on the right $[r_i, +\infty]$ or the left $[-\infty, d_i]$. Two main issues are often considered regarding these features. The first involves stating on the feasibility of a sequence of activities under time-window constraints, whereas the second problem involves the minimization of infeasibility with respect to the time windows, and thus involves characteristic functions $f_i(\mathbf{t}) = (t_i - d_i)^+ + (r_i - t_i)^+$ in the objective.

Feasibility problem. Solving the feasibility problem $\{\emptyset|TW\}$ is straightforward, as the minimum idle time policy, presented in Section 5.1, is dominating in this respect. For a sequence of n activities (a_1, \dots, a_n) , the algorithm starts with $t_1 = r_1$, then chooses each subsequent activity execution date to minimize idle time: $t_{i+1} = \max(t_i + p_i, r_{i+1})$. Hence, feasibility can be checked in $O(n)$ *from scratch*. Yet, more efficient feasibility checking procedures are available to solve *series* of timing instances in local-search context. These procedures are presented in Section 8.

Infeasibility minimization. Many real-case applications allow lateness or earliness, the so-called *soft* time-window settings, as a way to gain flexibility. Relaxations are also very frequently used in heuristics, as managing intermediate infeasible solutions contributes to improve the capabilities of exploration. Different conventions for soft time windows have been reported in the literature, relatively to earliness allowance and the way infeasibility is penalized. Several contributions, such as Taillard et al. (1997) and Cordeau et al. (2001), focus on the problem

$\{D|R\}$, where late activities are allowed with penalties, but not early activities. This case falls within the scope of regular features (Section 5.1), and choosing for each activity the earliest execution date is optimal. The problem can thus be solved with linear complexity $O(n)$.

However, when early activities are allowed, as in $\{TW|\emptyset\}$ (Garey et al. 1988, Baker and Scudder 1990, Koskosidis et al. 1992, Balakrishnan 1993, Ibaraki et al. 2005), the objective function is no longer non-decreasing. Supposing that activity a_i is finished earlier than the beginning of the time-window of a_{i+1} , a choice must be made whether to insert idle time to reach a_{i+1} , or pay a penalty to better satisfy the time windows of remaining activities. The resulting timing setting can thus become more complex. As an example, we show in Appendix A that the problem of minimizing the number of time-window infeasibilities, $\{TW(unit)|\emptyset\}$, generalizes the Longest Increasing Subsequence Problem (LISP). LISP has been the subject of extensive research, and admits a computational lower bound of $\Omega(n \log n)$ in the comparison tree model (Fredman 1975). It should be noted that $\{TW|\emptyset\}$ is also special case of separable piecewise linear convex cost functions. Sections 5.3 and 5.4 provide general efficient algorithms to address this wide range of problems, leading to an $O(n \log n)$ algorithm for soft time-window relaxations.

5.3 Separable convex costs

Separable convex costs Σc_i^{CVX} include a wide range of problem settings as particular cases. The feature TW , and thus R , D , and C , can be reduced to $\Sigma c_i^{\text{CVX}}(t_i)$, as any time-window constraint can be formulated as a piecewise convex cost by associating arbitrary large costs to both sides of the feasibility interval. This feature also encompasses various other settings from the literature such as earliness-tardiness scheduling (Baker and Scudder 1990), isotonic regression problems with respect to a total order (Barlow et al. 1972, Robertson et al. 1988), extensions of team orienteering problems (Feillet et al. 2005) in which the profit value can decrease with time (Erkut and Zhang 1996), various convex penalty functions for time-window infeasibility (Sexton and Bodin 1985a,b, Ioachim et al. 1998, Ibaraki et al. 2005) and time-dependent convex processing costs (Tagmouti et al. 2007), among others. The timing problem $\{\Sigma c_i^{\text{CVX}}(t_i)|\emptyset\}$ is formulated in Equations (16-17). Functions $c_i^{\text{CVX}}(t_i)$ are supposed to take infinite value for $t_i < 0$.

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^n} \sum_{i=1}^n c_i^{\text{CVX}}(t_i) \quad (16)$$

$$\text{s.t. } t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \quad (17)$$

Many methods have been proposed for this setting. We study, in this subsection, approaches specially designed for separable convex cost functions. Dynamic programming based algorithms, relying on fundamentally different concepts, are grouped in Section 5.4.

We base our analysis of algorithms for $\{\Sigma c_i^{\text{CVX}}(t_i)|\emptyset\}$ on a set of optimality conditions using the active set formalism (Chakravarti 1989, Best and Chakravarti 1990, Best et al. 2000). The necessary and sufficient conditions we propose are more general than those developed previously in the literature, being applicable to any set of proper convex cost functions, including non-smooth cases.

Definition 4 (Activity blocks and prefix blocks). *A block B is defined as a sequence of activities $(a_{B(1)}, \dots, a_{B(|B|)})$ processed consecutively such that $t_i + p_i = t_{i+1}$ for all $i \in \{B(1), \dots, B(|B|) - 1\}$. For $k \in \{B(1), \dots, B(|B|) - 1\}$, we also define the prefix block $B^k = (a_{B(1)}, \dots, a_k)$. Let p_{ij}*

for $1 \leq i \leq j \leq n$ be the cumulative processing duration of activities (a_i, \dots, a_j) . The execution cost C_B of a block B as a function of its first activity execution date $t_{B(1)}$ is given in Equation (18).

$$C_B(t_{B(1)}) = c_{B(1)}(t_{B(1)}) + \sum_{i=B(1)+1}^{B(|B|)} c_i(t_{B(1)} + p_{B(1)i-1}) \quad (18)$$

When the costs are proper convex functions (such that $\exists x | f(x) < +\infty$ and $\forall x, f(x) > -\infty$), the set of execution dates for the first activity minimizing this block execution cost is an interval $[T_B^{-*}, T_B^{+*}]$. These assumptions enable to state the following necessary and sufficient optimality conditions:

Theorem 1. *Let costs $c_i(t_i)$ for $i \in \{1, \dots, n\}$ be proper convex, possibly non-smooth, functions. A solution $\mathbf{t}^* = (t_1^*, \dots, t_n^*)$ of $\{\Sigma c_i^{\text{cvx}}(t_i) | \emptyset\}$ with activity blocks (B_1, \dots, B_m) , is optimal if and only if the three following conditions are satisfied:*

1. *Blocks are optimally placed, $t_{B_i(1)}^* \in [T_{B_i}^{-*}, T_{B_i}^{+*}]$ for each block B_i ;*
2. *Blocks are strictly spaced, $t_{B_i(1)}^* + p_{B_i(1)B_i(|B_i|)} < t_{B_{i+1}(1)}^*$ for each pair of blocks (B_i, B_{i+1}) ;*
3. *Blocks are consistent, $T_{B_i^k}^{+*} \geq t_{B_i(1)}^*$ for each block B_i and prefix block B_i^k .*

Condition 2 and 3 are direct consequences of the primal and the dual feasibility, respectively. Proof is given in Appendix B. Surveying the literature, we distinguish two main categories of methods: those who maintain primal feasibility, and those who maintain dual feasibility. These algorithms are issued from various domains. In the case of isotonic regression in particular, only precedence constraints among decision variables are considered (and thus $p_i = 0$ for all i), yet these methods can generally be extended to solve problems with processing times with only minor modifications. Hence, we illustrate all algorithms on a simple problem, for which the cost functions and the processing times are given in Figure 2.

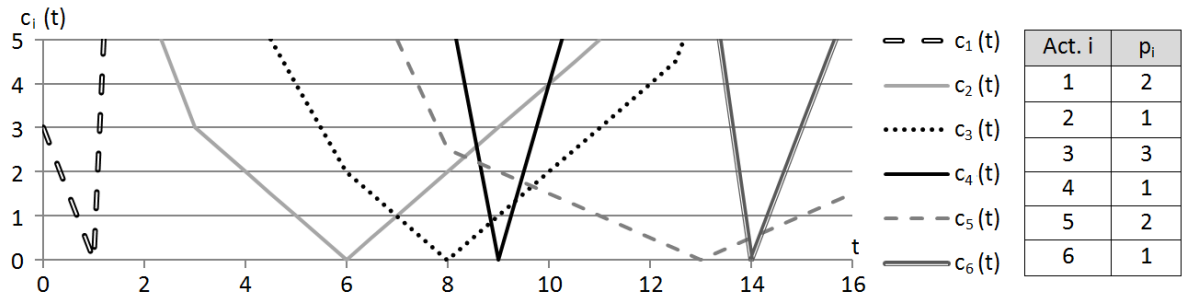


Figure 2: Illustrative example with six activities: cost functions and durations

Primal methods. A first category of methods is based on respecting the primal feasibility conditions and iteratively restoring the dual conditions. The first method of this kind, called *Minimum Lower Set (MLS)* algorithm, has been proposed by Brunk (1955) for isotonic regression problems. The MLS algorithm starts with a single big block, then iteratively finds for each block B the biggest prefix block B^k violating dual conditions. If no such violation is found, this

block is optimal, otherwise the current block is split in two and the procedure is recursively called on each sub-block until no dual conditions violation may be found. The algorithm can be implemented in $O(n^2)$ unimodal function minimizations.

Later on, Best and Chakravarti (1990) introduced a primal feasible algorithm for IRC in $O(n)$ unimodal function minimizations. Again, activities are sequentially examined in each block to find the first violation of dual conditions (and not the most important violation). If such a violation exists, the block under consideration is split at this place. The leftmost block has an earlier optimal starting date, and thus can possibly be merged with one or several previously scheduled blocks to reach an optimal execution date. In presence of quadratic costs, a closed form exists for the function minimums, and the complexity of this algorithm becomes $O(n)$ elementary operations.

The method of Garey et al. (1988), originally designed for (E/T) scheduling, iterates on activities in the sequence order and yields at any step i an optimal solution to the subproblem containing only the first i activities. Each new activity is inserted at the end of the schedule, to be left-shifted and possibly merged with previous activity blocks until no improvement may be achieved. The algorithm runs in $O(n \log n)$ when relying on heap data structures.

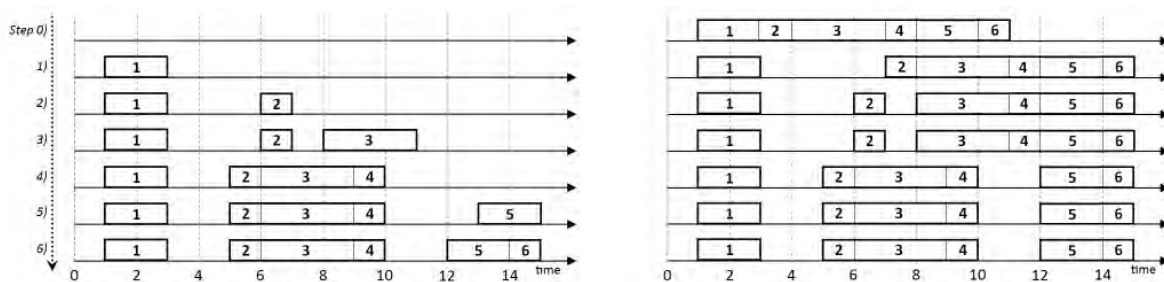


Figure 3: Comparison between Garey et al. (1988) algorithm (left part of the figure), and Best and Chakravarti (1990) algorithm (right part of the figure)

Figure 3 illustrates the algorithms of Best and Chakravarti (1990) and Garey et al. (1988) on the example of Figure 2. The problem is solved in six steps, illustrated from top to bottom along with the incumbent solutions, representing activities as rectangles with a length proportional to the processing time. The activity blocks in presence are very similar. Indeed, these two algorithms can be viewed as two variations of the same underlying primal feasible method, with the exception that Garey et al. (1988) considers non-inserted activities as non-existing in the current solution, whereas Best and Chakravarti (1990) maintains these non-scheduled activities in one final block which does not respect Condition 3.

The method of Garey et al. (1988) was extended by Lee and Choi (1995) and Pan and Shi (2005) to address (E/T) scheduling problems with distinct penalty weights for earliness and tardiness, that is $\{D, R(d_i = r_i)\emptyset\}$, in $O(n \log n)$ elementary operations. Szwarc and Mukhopadhyay (1995) and Feng and Lau (2007) also proposed to identify the tasks that are necessarily processed without idle time (in the same block) before solving. Chrétienne and Sourd (2003) applied the algorithm to project scheduling with piecewise convex cost functions, and Hendel and Sourd (2007) to timing problems with convex piecewise linear or quadratic costs. These algorithms work in $O(n)$ unimodal function minimizations, but differ in terms of the data structures used to represent the functions and thus on the complexity of the function minimizations. When the cost functions are Piecewise Linear (PiL), the method of Hendel and

Sourd (2007) attains a complexity of $O(\varphi_c \log n)$, where φ_c is the total number of pieces in the activity cost functions of the sequence. Finally, Davis and Kanet (1993) proposed another primal method for (E/T) scheduling similar to Garey et al. (1988), and generalized to PiL convex costs by Wan and Yen (2002). Activities are iteratively added to the solution in reverse sequence order. Each activity is scheduled at date 0, and then shifted onwards (while possibly merging blocks), until no improvement can be achieved.

Dual feasible methods. Simultaneously with the MLS algorithm, another seminal method for IRC was proposed by Ayer et al. (1955) under the name of *Pool Adjacent Violators (PAV)*. Starting with an initial solution consisting of n separate blocks, one for each activity, successive pairs of blocks (B_i, B_{i+1}) not satisfying primal conditions are iteratively identified. Such blocks are merged, and the next iteration is started. The order in which these block couples are identified does not affect the final result of the algorithm. An illustration of the method on the previous example is given in Figure 4. The algorithm iteratively merges the first pair of blocks that does not verify primal conditions. We notice that the optimal solution is reached after three merges (at Step 3).

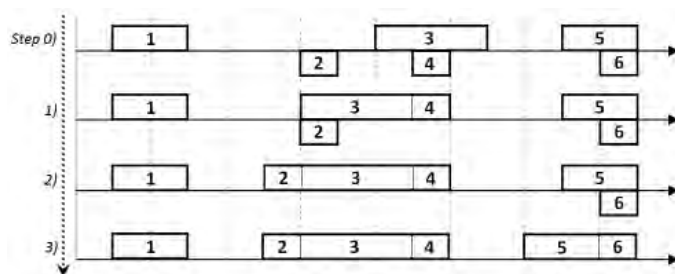


Figure 4: The PAV algorithm illustrated on timing problems

Chakravarti (1989) proved that PAV is a dual feasible method for the linear formulation of the problem when the distance considered is $\| \cdot \|_1$ ($c(\mathbf{t}) = \sum |t_i - N_i|$), while Grotzinger and Witzgall (1984) and Best and Chakravarti (1990) showed that PAV is a dual algorithm for IRC when quadratic costs are considered (Euclidean distance).

The PAV algorithm was also generalized to convex functions by Best et al. (2000) and Ahuja and Orlin (2001), achieving a complexity of $O(n)$ unimodal minimizations. It is noteworthy that, under a totally different formalism, an equivalent algorithm was discovered by Dumas et al. (1990) for a general vehicle routing setting with convex time-dependent service costs. For IRC with the $\| \cdot \|_1$ distance, the PAV algorithm can be implemented in $O(n \log^2 n)$ elementary operations using balanced search trees (Pardalos 1995), or $O(n \log n)$ complexity using scaling techniques (Ahuja and Orlin 2001). Finally, $O(n)$ algorithms are known in the case of quadratic costs (Grotzinger and Witzgall 1984, Dumas et al. 1990, Pardalos and Xue 1999).

5.4 Separable costs and multiple time windows

Without the previous convexity assumption, the timing problems $\{\Sigma c_i(t_i) | \emptyset\}$ become more complex, and many authors have focused on separable PiL costs. The feature MTW especially (Christiansen and Fagerholt 2002, Tricoire et al. 2010) can also be linearly reduced to a $\{\Sigma c_i(t_i) | \emptyset\}$ problem when a closed form representation of the multiple time windows, such as a list of feasible intervals, is available.

In presence of non-negative and Lower Semi-Continuous (LSC) costs ($c_i(t_i) \leq \lim_{\epsilon \rightarrow 0} \min\{c_i(t_i + \epsilon), c_i(t_i - \epsilon)\}$ at every discontinuity point), the timing problem $\{\Sigma c_i(t_i) | \emptyset\}$ can be efficiently solved by dynamic programming. A large range of backward and forward approaches has thus been proposed in the routing and scheduling literature (Yano and Kim 1991, Sourd 2005, Ibaraki et al. 2005, Hendel and Sourd 2006, Ibaraki et al. 2008).

Solving $\{\Sigma c_i(t_i) | \emptyset\}$ by forward dynamic programming involves the *forward minimum cost* function $F_i(t)$, which evaluates the minimum cost to execute the sequence of activities (a_1, \dots, a_i) while starting the last activity before t ($t_i \leq t$). $F_i(t)$ functions can be computed by means of Equation (19), starting from the case $i = 1$ with a single activity where $F_1(t) = \min_{x \leq t} c_1(x)$. The optimal solution value of the timing problem is $z^* = F_n(+\infty)$, and the optimal activity execution dates can be retrieved from the $F_i(t)$ functions.

$$F_i(t) = \min_{0 \leq x \leq t} \{c_i(x) + F_{i-1}(x - p_{i-1})\} \quad 1 < i \leq n \quad (19)$$

The symmetric way to solve this problem by backward programming involves the *backward minimum cost* function $B_i(t)$, which evaluates the minimum cost to execute the sequence of activities (a_i, \dots, a_n) , while executing the first activity a_i after t ($t_i \geq t$). $B_i(t)$ functions are computed by backward recursion, starting with $B_n(t) = \min_{x \geq t} c_n(x)$ and using Equation (20). The optimal solution value of the timing problem is $z^* = B_1(-\infty)$.

$$B_i(t) = \min_{x \geq t} \{c_i(x) + B_{i+1}(x + p_i)\} \quad 1 \leq i < n \quad (20)$$

These methods can be implemented in $O(n\varphi_c)$, where φ_c stands for the total number of pieces in the activity cost functions c_i . When the costs are also convex, the use of efficient tree data structures leads to a complexity of $O(\varphi_c \log \varphi_c)$ (Ibaraki et al. 2008), matching the best available approaches in $O(n \log n)$ for the particular cases related to IRC, (E/T) scheduling, and soft time windows. Finally, besides their good complexity for solving independent timing problems, these dynamic programming open the way to efficient re-optimization procedures for solving series of similar timing instances. These latter methods are surveyed in Section 8.

5.5 State-of-the-art: single-dimensional features

We introduced and analyzed in this section the main single-dimensional features from the literature, independently of the field of application. Single-dimensional features are related to many prominent problems such as LISP and IRC, which have been the subject of extensive research. Various algorithms were examined and state-of-the-art methods for each particular feature and problem were identified. In the particular case of $\{\Sigma c_i^{\text{cvx}}(t_i) | \emptyset\}$, 26 methods from various fields, such as routing, scheduling and isotonic regression, were classified into three main families: primal, dual, and dynamic programming methods.

Efficient linear methods are known for the most general problem of this category, $\{\Sigma c_i^{\text{cvx}}(t_i) | \emptyset\}$, in presence of either convex or LSC and piecewise linear cost functions. Linear complexity is the best possible complexity when addressing the problem *from scratch*. As illustrated in Section 8, the resolution of series of similar timing instances in a local search context can be performed more efficiently by means of re-optimization procedures, and research challenges remain open in this area, even for single-dimensional features.

6 Two-dimensional features

We now focus on the problems with two-dimensional features. These features are also often considered in presence of time-window TW constraints. The presentation is structured in relation to the level of problem complexity and generality. Starting with the duration feature DUR , which involves exclusively the first and last activities together, we then examine two-dimensional features involving successive activities: no-wait NWT , idle time IDL , and flexible $c_i(\Delta t_i)$ or time-dependent $P(t)$ processing times. Finally, features involving any pair of activities, such as time-lags TL , and cost functions separable by pairs of variables $\Sigma c_{ij}(t_i, t_j)$ are analyzed.

6.1 Total duration and total idle time

Accounting for total duration or idle-time is meaningful when one has the possibility to delay the beginning of operations. Otherwise, considering the maximum execution date feature C is sufficient. Whereas delaying the start of production is generally not an option in scheduling problems, it becomes particularly relevant in routing, as real-life objectives and driver's wages are frequently based on duration. We mention Savelsbergh (1992) for duration minimization under time-window and duration constraints in VRPs, Cordeau et al. (2004) that generalizes the previous approach for soft time-windows and duration constraints, Desaulniers and Villeneuve (2000) for shortest path settings with linear idle-time costs and time windows, and Desaulniers et al. (1998) and Irnich (2008b) for a general framework that enables to tackle duration and idle time features, among others, in various time-constrained routing and crew scheduling problems. It should be noted that computing the *total duration* or the *total idle time* is equivalent in the presence of fixed processing times. Therefore, w.l.o.g. we will focus on duration in this section.

To manage duration features in $\{DUR|TW\}$ and $\{\emptyset|DUR, TW\}$, Savelsbergh (1992) proposed to first rely on a minimum idle time policy, and then shift activity execution dates forward to reduce the total duration. The related amount of shift was introduced many years ago in the project scheduling literature (Malcolm et al. 1959), as the latest processing date for an activity that does not cause a delay in the calendar. It is also known in the VRP literature under the name of *forward time slack* (Savelsbergh 1985, 1992). The following quantities are computed for each activity a_i : the earliest feasible execution date T_i , the cumulative idle time W_i on the subsequence (a_1, \dots, a_i) according to these execution dates, and the partial forward time slack F_i on the subsequence (a_1, \dots, a_i) . These values are computed recursively by means of Equations (21-23), starting with the case of a single activity where $T_1 = r_1$, $W_1 = 0$ and $F_1 = d_1 - r_1$.

$$T_i = \max(T_{i-1} + p_{i-1}, r_i) \quad 1 < i \leq n \quad (21)$$

$$W_i = W_{i-1} + T_i - T_{i-1} - p_{i-1} \quad 1 < i \leq n \quad (22)$$

$$F_i = \min(F_{i-1}, d_i - T_i + W_i) \quad 1 < i \leq n \quad (23)$$

The problem admits a feasible solution if and only if $T_i \leq d_i$ for all i . The execution date of the first activity in an optimal solution is given by $t_1^* = r_1 + \min\{F_n, W_n\}$. The other dates are computed using the minimum idle time policy. Both feasibility checking and duration minimization problems are thus solved in $O(n)$. Kindervater and Savelsbergh (1997), Desaulniers and Villeneuve (2000) and Irnich (2008b) proposed different calculations of this optimal schedule. As pointed out by Parragh et al. (2012), all these approaches are equivalent.

Tricoire et al. (2010) recently considered a more complex timing problem aiming to minimize duration under MTW constraints $\{DUR|MTW\}$. Each activity a_i is associated with a set of k_i time windows, $MTW_i = \{[r_{i1}, d_{i1}], \dots, [r_{ik_i}, d_{ik_i}]\}$. The authors proposed a procedure that first removes some unnecessary time-window segments, not suitable for any feasible solution, while detecting infeasible timing problems. In a second step, the procedure examines a subset of *dominant* schedules, such that “no better solution exists with the same last activity execution date”. For a given execution date t_n of the last activity, a *dominant* schedule with minimum duration can easily be found using the backward recursion of Equation (24).

$$t_{i-1} = \max\{t \mid t \leq t_i - p_i \wedge t \in MTW_i\} \quad (24)$$

Starting from the dominant schedule \bar{t} with earliest completion time, the method iteratively identifies the last activity a_i followed by idle time: $\bar{t}_i + p_i < \bar{t}_{i+1}$. If activity a_i does not admit a later time window, the algorithm terminates. Otherwise, the execution date of activity a_i is set to the beginning of the next time window, and the execution dates of activities situated afterwards in the sequence are re-computed with a minimum idle time policy. This leads to a *dominant* schedule which becomes \bar{t} in the next iteration. Tricoire et al. (2010) proved that at least one dominant schedule explored in the course of the algorithm is optimal. If each customer is associated to at least one time window, the overall method can be implemented in $O(n\varphi_{MTW})$, φ_{MTW} representing the number of time windows in the problem.

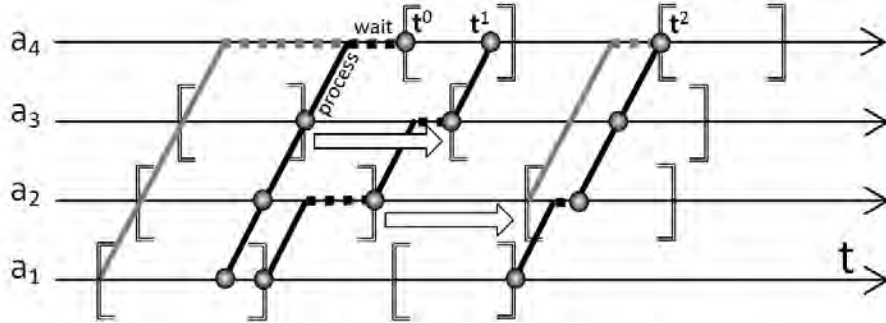


Figure 5: Duration minimization under multiple time-window constraints

This algorithm is illustrated in Figure 5 on a small example with four activities. Activities are represented from bottom to top with their time windows. The earliest completion date is computed with a minimum idle time policy, illustrated in gray lines. The initial dominant schedule \bar{t}^0 , in black, is then determined by backward recursion using Equation (24). This schedule presents waiting time after activity a_3 , and thus the execution date of this activity is delayed to the next time window, leading to a dominant schedule \bar{t}^1 . Now the latest activity followed by waiting time is a_2 . Its execution date is delayed, and leads to the dominant schedule \bar{t}^2 . The latest activity followed by waiting time is a_1 . As there is no later time window for this activity, the algorithm terminates. Among the three dominant schedules explored by the method, the best solution with minimum duration has been reached by \bar{t}^2 , and is optimal.

6.2 No wait and idle time

No-wait *NWT* and idle-time *IDL* features appear in various settings involving, among others, deterioration of products, maximum waiting times in passenger transportation, fermentation

processes in the food industry, and cooling in metal-casting processes. *IDL* reduces to *NWT* when $\iota_i = 0$. No-wait constraints $t_i = t_{i+1}$ can also be addressed by problem reformulation, merging unnecessary variables. When no waiting time is allowed on the entire activity sequence, the timing problem becomes a minimization problem of a sum of single-variable functions. Two main categories of problems have been considered in the literature for *NWT* and *IDL*: the feasibility problem under idle-time and time-window constraints, and the optimization problem when some of these features appear in the objective function, treated in Section 6.3 in a more general context.

Feasibility checking under maximum idle time and time-window constraints has been frequently studied in the routing literature. Hunsaker and Savelsbergh (2002) designed an algorithm to check the feasibility of itineraries in dial-a-ride settings. This algorithm contains a $O(n)$ checking method for the special case of $\{\emptyset|IDL, TW\}$. The solution to $\{\emptyset|IDL, TW\}$ is found in two scans of the activity sequence. The first pass considers the relaxed subproblem $\{\emptyset|TW\}$, determining for each activity i the earliest feasible execution dates T_i complying with time-windows and processing times. This calculation is performed by means of Equation (21). In a second pass, starting with $T'_n = T_n$, the algorithm proceeds backwards in the sequence to determine the earliest feasible execution dates $T'_i = \max(T'_{i+1} - p_i - \iota_i, T_i)$ and check idle-time constraints. The problem is declared infeasible if for any $i \in \{1, \dots, n\}$, $T_i > d_i$ or $T'_i > d_i$.

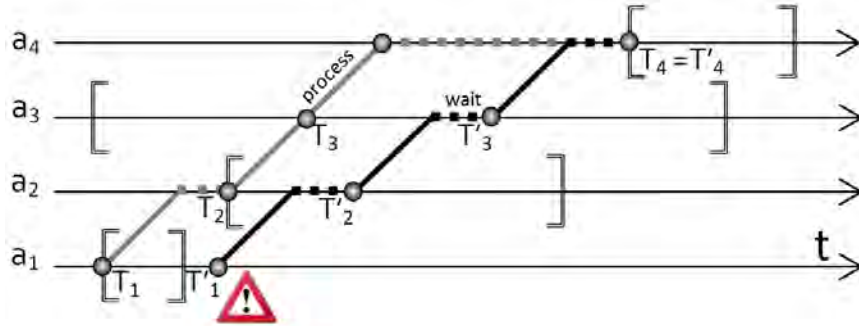


Figure 6: Feasibility checking under time-windows and idle-time constraints

The two passes are illustrated in Figure 6 for a small problem with four activities, represented from bottom to top with their time windows. The first pass (T_i values) has been represented in gray, while the backward scan, in black, provides the earliest feasible execution date for each activity T'_i . As shown in the figure, this value exceeds the time window of activity a_1 ($T'_1 > d_1$), and thus the timing problem illustrated is infeasible.

6.3 Flexible processing times

Resource allocation, time-resource trade-offs, and project crashing problems have been thoroughly studied in the operations research literature, since they are critical in many applications, including energy optimization (see Section 3), project scheduling, portfolio selection, production economics, optimization of search effort, advertising, and so on and so forth (Kelley and Walker 1959, Talbot 1982, Ibaraki and Katoh 1988, Patriksson 2008). The processing time or resource allocation in all these problems can be increased or reduced at a cost. We refer to this feature as “flexible processing times” $\Sigma c_i(\Delta t_i)$. The characteristic function of this feature is a general separable function of successive execution time differences $t_{i+1} - t_i$. As such, it

generalizes both *NWT* and *IDL*.

When only *NWT*, *IDL*, and $\Sigma c_i(\Delta t_i)$ features are encountered, the timing problem $\{\Sigma c_i(\Delta t_i)|\emptyset\}$ (Equations 25-27) can be decomposed along $t_{i+1} - t_i$ values, and the independent minimization of every $c_i(\Delta t_i)$ leads to the optimal solution where $t_{i+1} - t_i = \arg \min_{\Delta t_i} \Sigma c_i(\Delta t_i)$.

When the c_i functions are convex and in presence of an additional constraint on the total duration, the problem $\{\Sigma c_i(\Delta t_i)|DUR\}$ can be reformulated (Equations 28-30) as a Simple Resource Allocation Problem (SRA) by setting $t_0 = 0$ w.l.o.g. and using the change of variables $x_i = t_{i+1} - t_i$.

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^{n+1}} \sum_{i=1}^{n-1} c_i(t_{i+1} - t_i) \quad (25)$$

$$s.t. \quad t_n - t_1 \leq d_{max} \quad (26)$$

$$t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \quad (27)$$

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^n} \sum_{i=1}^{n-1} c_i(x_i) \quad (28)$$

$$s.t. \quad \sum_{i=1}^{n-1} x_i \leq d_{max} \quad (29)$$

$$x_i \in [p_i, +\infty] \quad 1 \leq i < n \quad (30)$$

Non-linear resource allocation problems have been the subject of hundreds of papers. Two main families of methods for this problem are generally discerned. *Lagrangian methods* (Charnes and Cooper 1958, Everett 1963, Srikantan 1963) seek to iteratively progress towards the optimal value of the single dual variable associated to Equation (29) while respecting primal feasibility. *Pegging* algorithms (Sanathanan 1971, Luss and Gupta 1975, Kodialam and Luss 1998) relax the constraints of Equations (29-30), iteratively solving the relaxed problems and fixing the values of variables that do not satisfy primal constraints. A third approach originates from concepts originally developed for the discrete problem version. Hochbaum (1994) proposes a greedy algorithm with arbitrary increments coupled with scaling techniques, yielding the best known computational complexity for this problem, $O(n \log n \log(d_{max}/\epsilon n))$, where ϵ is the target precision. The method increments progressively the x_i variable yielding the least reduced cost, leading to the optimum of this convex problem. Finally, a dynamic programming approach has been introduced in Karush (1962), which iteratively characterizes the best cost as a function of the resource consumption (total time in our case).

In particular, relationships can be established between these approaches and the three classes of methods surveyed in Section 5.3 for timing problems with separable convex costs. *Pegging* algorithms work similarly to the dual feasible methods of Section 5.3, such as PAV or the algorithm of Dumas et al. (1990), by iteratively solving relaxed problems and re-introducing the constraints. These constraints are here re-introduced by setting variables to constraint bounds, rather than joining pairs of activities (merging blocks) as in the PAV algorithm. The equivalent to the primal methods of Section 5.3 is less immediate. The greedy method of Hochbaum (1994) is one of this kind, but requires advanced scaling techniques to solve the continuous case. Also, even though most authors opted to optimize the single Lagrangian variable μ associated to Equation (29) rather than the primal variables, a consequence of the

KKT conditions is that optimizing μ is equivalent to choosing iteratively the slope for each $c_i(x_i)$ and, in fact, iteratively fixing primal variables. Thus, Lagrangian approaches for this problem were eventually qualified by Patriksson (2008) as *explicitly dual* but also *implicitly primal*.

Adding time-window constraints to the model of Equations (25-27) leads to other timing settings, which can no longer be assimilated to resource allocation problems. In the special case of Norstad et al. (2010), a timing problem $\{\Sigma c_i^{\text{CVX}}(\Delta t_i) | TW\}$ is addressed with the objective $z(t) = \sum_{i=1}^n d_{i,i+1} \bar{c} \left(\frac{t_{i+1} - t_i}{d_{i,i+1}} \right)$ and non-increasing and convex $\bar{c}(\Delta t)$ functions. In presence of such functions, removing time-window constraints leads to optimal solutions presenting a constant ratio $\frac{t_{i+1} - t_i}{d_{i,i+1}}$ for all i , and thus constant speed on all legs. The *recursive smoothing algorithm* (RSA) exploits this property by maintaining this ratio constant on subsequences and progressively re-introducing violated time-window constraints. The overall method works in $O(n^2)$ elementary operations once the minimum of each function $\bar{c}(\Delta t)$ is given. It is noteworthy that this method is another example of a *dual feasible* algorithm based on the relaxation and re-introduction of constraints, conceptually similar to PAV and *pegging* methods.

In more general settings, when flexible processing times are combined with single-dimensional features such as time windows or time-dependent activity costs, the problem can become more difficult. Sourd (2005) and Hashimoto et al. (2006) have independently studied $\{\Sigma c_i(\Delta t_i), \Sigma c_i(t_i) | \emptyset\}$ (Equation 31) with piecewise linear functions in the context of (E/T) scheduling and vehicle routing, and report its NP-hardness.

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^{n+}} \sum_{i=1}^n c_i(t_i) + \sum_{i=1}^{n-1} c'_i(t_{i+1} - t_i) \quad (31)$$

When the functions c_i and c'_i are PiL with integer breakpoints, a dynamic programming algorithm is proposed to solve the problem in $O(T^2)$, where T represents an upper bound on the schedule durations. Note that this dynamic programming algorithm can be viewed as an extension of the resource allocation algorithm of Karush (1962). The method can be implemented with a forward dynamic programming function $F_i(t)$ (Equations 32-33), which evaluates the minimum cost to process the subsequence of activities (a_1, \dots, a_i) , starting the last activity exactly at time t ($t_i = t$). The resulting optimal cost is given by $z^* = \min_t F_n(t)$.

$$F_1(t) = c_1(t) \quad (32)$$

$$F_i(t) = c_i(t) + \min_{0 \leq x \leq t} \{F_{i-1}(x) + c'_{i-1}(t - x)\} \quad 1 < i \leq n \quad (33)$$

A polynomial dynamic programming algorithm working in $O(n(\varphi_c + \widehat{\varphi}_c \times \varphi'_c))$ exists for the case where the functions $c'_i(\Delta t)$ are PiL and convex (Sourd 2005, Hashimoto et al. 2006). φ_c and φ'_c represent the total number of pieces in the cost functions c_i and c'_i , and $\widehat{\varphi}_c$ stands for the number of convex pieces in the cost functions c_i . Efficient re-optimization procedures have also been proposed (Section 8). Finally, *DUR* involved in the objective can be seen as a special case of $\Sigma c_i(\Delta t_i)$ where $c_i^{\text{DUR}}(\Delta t_i) = \Delta t_i$. Since *MTW* is also reducible to $\Sigma c_i(t_i)$, the previous algorithm thus provides an alternative way to solve $\{DUR | MTW\}$ or $\{\emptyset | DUR, MTW\}$ in $O(n + n\varphi_{\text{MTW}})$, where φ_{MTW} represents the total number of time windows.

6.4 Time-dependent processing times

In several application settings, activity processing times may vary as a function of the execution dates. In machine and project scheduling for example, learning, deterioration effects and other

time-dependencies can have a large impact (see Alidaee and Womer 1999, Cheng 2004). Network congestion is a major concern for vehicle routing and data transmission (Van Woensel et al. 2008, Kok et al. 2009) and, thus, the time-dependent processing-time feature $P(t)$ appears in various network optimization problems: shortest path (Cooke and Halsey 1966, Dreyfus 1969, Halpern 1977), traveling salesman (Malandraki and Dial 1996), vehicle routing (Beasley 1981, Malandraki and Daskin 1992, Ichoua et al. 2003, Donati et al. 2008, Hashimoto et al. 2008), and so on.

The literature on the subject can generally be separated between discrete and continuous settings. Discrete optimization models generally involve time-space networks which are less likely to present the timing issues studied in this article, whereas several continuous models have led to explicit timing problems with $P(t)$ features, as in Ichoua et al. (2003), Fleischmann et al. (2004), Donati et al. (2008), and Hashimoto et al. (2008). These models involve constraints of the type $t_i + p_i(t_i) \leq t_{i+1}$ within a timing formulation with other additional features. A *FIFO* assumption on functions p_i can be made:

$$\text{FIFO assumption: } \forall i \ x \geq y \implies x + p_i(x) \geq y + p_i(y) \quad (34)$$

FIFO implies that any delay in an activity execution date results in a delay in its completion date. The assumption is meaningful in several settings, e.g. vehicle routing, as two vehicles that behave similarly on the same route are supposed to remain in the same arrival order, whatever congestion happens (Ichoua et al. 2003).

Time-dependent processing-time features are generally assumed to result in more complex timing problems. However, one should clearly identify the source of the difficulty, which is frequently imputable to the calculation and access to $p_{ij}(t)$ throughout the search, and not necessarily to the timing problem solving method. Assuming that $p_{ij}(t)$ can be evaluated in constant time and under FIFO, $\{D|R, P(t)\}$ is still solvable in $O(n)$ by means of a *minimum idle time policy* (Fleischmann et al. 2004). In the same spirit, Donati et al. (2008) apply the time-slack approach of Savelsbergh (1992) to $\{\emptyset|TW, P(t)\}$. Still, dedicated methodologies are necessary for other settings such as $\{DUR|TW, P(t)\}$ and for re-optimization procedures (Section 8).

A general time-dependent timing problem $\{\sum c_i(t_i)|P(t)\}$, Equations (35-36), is addressed in Hashimoto et al. (2008). All functions considered are PiL, non-negative and lower semicontinuous.

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^{n+}} \sum_{i=1}^n c_i(t_i) \quad (35)$$

$$\text{s.t. } t_i + p_i(t_i) \leq t_{i+1} \quad 1 \leq i < n \quad (36)$$

The authors propose a dynamic programming approach, which extends the method of Section 5.4. It involves the functions $F_i(t)$, which represent the minimum cost to process the subsequence of activities (a_1, \dots, a_i) while starting the last activity before t ($t_i \leq t$). These functions can be computed by means of Equations (37-38), and the optimal solution cost is given by $F_n(+\infty)$.

$$F_1(t) = \min_{0 \leq x \leq t} c_1(x) \quad (37)$$

$$F_i(t) = \min_{0 \leq x \leq t} \{c_i(x) + \min_{x' + p_i(x') \leq x} F_{i-1}(x')\} \quad 1 < i \leq n \quad (38)$$

Under the assumption of Equation (39), which is slightly weaker than FIFO, the method can be implemented in $O(n(\varphi_c + \varphi_p))$, where φ_c and φ_p are the total number of pieces in cost and processing-time functions. This method for $\{\Sigma c_i(t_i)|P(t)\}$ thus presents the same quadratic complexity as in the case without time dependency (Section 5.4). When the previous assumption does not hold, the dynamic programming method of Hashimoto et al. (2008) is not polynomial, and the question remains open whether $\{\Sigma c_i(t_i)|P(t)\}$ is polynomially solvable.

$$\text{(HYI) assumption: } \forall i \ x + p_i(x) = y + p_i(y) \implies x + p_i(x) = z + p_i(z) \ \forall z \in [x, y] \quad (39)$$

6.5 Time lags

The two-dimensional features surveyed in the previous sections involved linking constraints and objectives between the first and last tasks, in the case of *DUR*, or between pairs of successive variables in the case of *NWT* and *IDL*. We now review the time-lag *TL* feature, which brings into play a time difference $t_j - t_i$ between any two activity execution dates t_i and t_j . This feature is thus a generalization of *NWT*, *IDL* and *DUR*.

To the best of our knowledge, early research on time lags has been conducted by Mitten (1959) for flowshop scheduling problems. This feature has been used since to model many problem characteristics in various domains, such as the deterioration of food or chemical products, glue drying, customer requirements in some dial-a-ride problems, elevator dispatching, quarantine durations, and so on. Time-lag scheduling problems on a single machine have also been shown by Brucker et al. (1999) to generalize all shop, multi-purpose machines, and multi-processor scheduling problems. Hence, timing problems with *TL* are likely to be difficult.

The most basic problem with *TL* feature relates to feasibility checking under time-lag constraints of the form $t_i + \delta_{ij} \leq t_j$. When $\delta_{ij} \geq 0$, the constraint is called positive time lag, and corresponds to a minimum delay between activities a_i and a_j , whereas $\delta_{ij} \leq 0$ corresponds to a negative time lag, and involves a maximum delay of $-\delta_{ij}$ between the activities a_j and a_i . Equality constraints $t_i + \delta_{ij} = t_j$ involve both positive and negative time lags. The resulting timing problem $\{\emptyset|TL\}$ can be seen as a special case of project scheduling on a chain of activities, and the METRA potential method (MPM) of Roy (1959, 1962) can be applied. In MPM, the time-lag constraints are represented on a graph $G = (V, A)$, where each activity a_i is associated with a node $v_i \in V$, and each arc (v_i, v_j) , associated with a weight w_{ij} , represents a temporal constraint of the form $t_j - t_i \geq w_{ij}$. The feasibility of $\{\emptyset|TL\}$ is equivalent to the non-existence of a positive length cycle in this graph (Bartusch et al. 1988, Dechter et al. 1991). The algorithm of Floyd-Warshall can be employed to solve this problem in $O(n^3)$, but the longest-path procedure of Hurink and Keuchel (2001), also in $O(n^3)$, is shown to provide faster results in practice. Potts and Whitehead (2007) also considered a coupled-operation scheduling problem with only $n/2$ time-lag constraints, and timing feasibility is checked in $O(n^2)$. The authors underlined the computational burden of such timing algorithms, which strongly degrades the performance of neighborhood searches or branch and bound procedures.

Hunsaker and Savelsbergh (2002) studied a case of $\{\emptyset|TL, TW\}$ timing in the context of dial-a-ride problems. Activities represent customer requests on pick-up and deliveries services, which occur by pairs, such that any pick-up always precedes its corresponding delivery in the sequence. Each such pair of activities is linked by a single positive time-lag constraint. The total number of time-lag constraints is thus $n/2$. The problem also involves time windows and maximum idle times for each activity. The authors claim that the resulting feasibility problem can be solved in three passes on the sequence of activities with linear complexity. Yet,

the algorithm presents a small flaw, which is straightforward to correct (Tang et al. 2010), but leads to a $O(n \log n)$ complexity (Haugland and Ho 2010). A $O(n)$ complexity is finally achieved in Firat and Woeginger (2011) by means of a reduction to a shortest path problem on a weighted interval graph.

The same setting is also addressed in Gschwind and Irnich (2012). The authors describe a labeling procedure based on $|M_i|$ resources for each pickups and each delivery activity a_i . M_i stands for the current number of open pickups at a_i . This labeling procedure provides another way to check the feasibility of a fixed activity sequence $\{\emptyset|TL, TW\}$ in $O(n^2)$. Also, it enables to evaluate the feasibility of an extended sequence with one additional activity a_{n+1} in $O(M_{n+1})$ given the information on the original sequence. Such forward extension function is critical when solving shortest path problems with underlying timing features.

Cordeau and Laporte (2003) and Berbeglia et al. (2011) consider a dial-a-ride setting with an additional duration constraint on the entire trip duration. The authors solve heuristically a Lagrangian relaxation of the problem with a hierarchical objective. Total trip duration infeasibility is minimized, then time-window infeasibility and, finally, time-lag infeasibility, that is the timing problem $\{DUR > D > TL|R\}$. The algorithm first minimizes duration and time-window infeasibility as in Section 6.1, then iteratively delays some pick-up services to reduce time-lag infeasibility without increasing any other violation. A computational complexity of $O(n^2)$ is achieved. It was observed (Private Communication 2010), however, that the previous approach only guarantees optimality under an additional assumption that we call *LIFO*, which requires that for any $1 \leq i < j < k < l \leq n$, no activities a_i, a_j, a_k, a_l present “entangled” time-lag constraints of the form $t_k - t_i \leq \delta_{ik}$ and $t_l - t_j \leq \delta_{jl}$. The LIFO assumption is frequently enforced in the vehicle routing literature, especially when transporting passengers, or in presence of complex loading constraints. In this case, the last object or customer received in the vehicle is the first one to leave. Without this assumption, the difficulty of many problems with time lags strongly increases, and no specialized efficient algorithm is actually known for $\{DUR > D > TL|R\}$ and similar problems.

6.6 Separable costs by pairs of variables

Separable costs by pairs of variables $\Sigma c_{ij}(t_i, t_j)$ generalize all problems combining single or two-dimensional features. The timing problem with this feature alone is NP-hard, as it includes $\{\Sigma c_i(t_i), \Sigma c_i(\Delta t_i)|\emptyset\}$ as special case (see Section 6.2). When the objective function is convex, the problem $\{\Sigma c_{ij}^{\text{cvx}}(t_j - t_i), \Sigma c_i^{\text{cvx}}(t_i)|\emptyset\}$ given in Equations (40-41), is equivalent to the *convex cost dual network flow problem* for which weakly polynomial algorithms are available (Ahuja et al. 2003).

$$\min_{\mathbf{t}=(t_1, \dots, t_n) \in \mathbb{R}^{n+}} \sum_{(i,j) \in Q} c_{ij}^{\text{cvx}}(t_j - t_i) + \sum_{1 \leq i \leq n} c_i^{\text{cvx}}(t_i) \quad (40)$$

$$s.t. \quad t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \quad (41)$$

7 Conclusion on “stand-alone” timing methods

In contrast to single-dimensional features, which appeared as fairly well addressed in Section 5 by means of a few algorithms and concepts, two-dimensional features lead to more diverse problem structures and algorithms. Several simple cases with duration minimization or time-dependent processing times can be solved in linear time, but other problems with time-lag

features actually require $O(n^3)$ algorithms to be solved exactly. Although polynomial, the latter methods can be impracticable in the context of local searches or branch-and-bound approaches.

Many practical timing settings result in models with linear constraints and linear or separable convex objectives. For these problems, the linear and convex programming theory ensures weakly polynomial solvability, and provides general solution methods (Khachiyan 1979, Karmarkar 1984, Hochbaum and Shanthikumar 1990). Some general problems, however, such as $\{\Sigma c_i(t_i), \Sigma c_i(\Delta t_i) | \emptyset\}$ with PiL functions, are NP-hard, while for other problems, such as $\{\Sigma c_i(t_i) | P(t)\}$ with general piecewise linear functions $P(t)$, the existence or non-existence of polynomial algorithm is still open. Timing settings thus lead to a rich variety of problem structures and complexities.

In all these cases, whether polynomial algorithms are available or not, research is still open to provide more efficient algorithms exploiting the particular structure of the features and problems at hand. The present paper contributed by building a formalism, a classification of features, timing problems and methods. We gathered the most efficient stand-alone timing approaches from various fields of research to tackle both specialized timing settings, and more general features. The focus can now be turned on filling the gaps that have been highlighted in this review, and which continue to appear, following the rich variety of application cases with time constraints emerging nowadays. Finally, important avenues of research target the efficient solving of *series* of timing problems, in the particular context of neighborhood searches and branch-and-bound. Such approaches are presented in the next part of this paper.

8 Timing re-optimization

In the first part of this article, we examined how to address timing problems as a stand-alone issue. Yet, most neighborhood-search-based heuristics, metaheuristics, and some exact methods require to solve iteratively a large number of closely related timing instances. In this case, solving each timing problem “from scratch”, without exploiting any knowledge on previous resolutions, can result in losses of information and redundant computations.

Most local searches for routing and scheduling problems (see Hoos and Stützle 2005 for a thorough presentation of LS) rely on a neighborhood based on a limited number of sequence changes. One or several timing subproblems are solved for each neighbor to estimate its feasibility and cost. Two classical neighborhoods, dedicated respectively to exchanging the tails of two vehicle routes in VRP context (2-opt* of Potvin and Rousseau 1995), and relocating a sequence of activities (Or-exchange of Or 1976), are illustrated in Figure 7. It is noticeable that large subsequences of activities (e.g. Seq.A...Seq.D on the figure) are shared by successive timing subproblems.

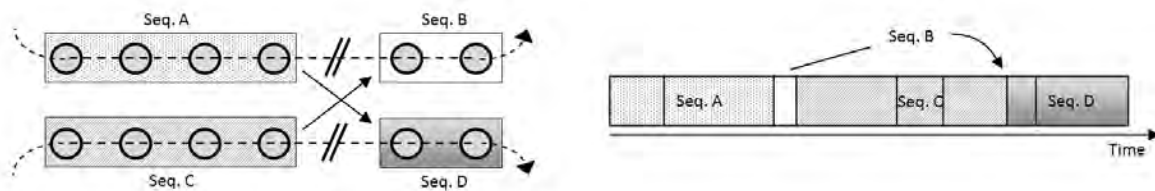


Figure 7: Sequence invariants in 2-opt* and relocate moves

Branch-and-bound procedures for problems with sequencing decisions can similarly involve timing subproblems at nodes of the search tree when evaluating sequences of activities, during lower bound computation and branch pruning (Hoogeveen and Van De Velde 1996, Sourd and Kedad-Sidhoum 2003). The search for improving columns, in columns generation approaches, also frequently involves elementary shortest paths with combined timing decisions and resource constraints (Desrochers et al. 1992, Prescott-Gagnon et al. 2009, Baldacci et al. 2011).

It is noticeable that, in all these cases, numerous closely related timing problems must be solved, where long subsequences of consecutive activities remain unchanged, and only a minor proportion of problem parameters (reduced cost values for column generation) is impacted. Several authors thus propose to keep *meaningful data* on the global search process to save computations and solve more efficiently these series of similar timing problems. Neighborhood searches have actually largely benefited from these techniques, as move evaluations (and thus the resolution of timing problems) take the largest part of the computational effort. These *re-optimization* methodologies therefore can lead to dramatic reductions in the computational burden of algorithms.

We now formally define *serial* timing problems in Section 8.1, and present a general framework for re-optimization methods based on sequence concatenations in Sections 8.2-8.3. Links with related re-optimization methodologies are analyzed in Section 8.4, before reviewing or introducing efficient concatenation-based re-optimization methods for each major timing feature and related problems in Section 8.5.

8.1 Problem statement: Serial timing

This section formally defines serial timing problems. Sequence-dependent processing times are also considered here, in relation to a large range of vehicle routing applications, which heavily rely on re-optimization methods.

Definition 5 (Serial timing). *Let \mathcal{T} be an incumbent timing problem with n activities (a_1, \dots, a_n) , sequence-dependent processing-times p_{ij} , and additional features with their data and characteristic functions $f_i^x(\mathbf{t})$. Features are separated into two sets \mathcal{F}^{OBJ} and $\mathcal{F}^{\text{CONS}}$ following their role as objective or constraint (see Section 2). N permutation functions $\sigma^k : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ for $k \in \{1, \dots, N\}$, are also given. The serial timing problem involves to solve the timing subproblems \mathcal{T}^k of Equations (42-44), for $k \in \{1, \dots, N\}$.*

$$(\mathcal{T}^k) : \quad \min_{\mathbf{t}=(\sqcup_{\infty}, \dots, \sqcup_{\setminus}) \in \mathfrak{R}^{\setminus+}} \sum_{F^x \in \mathcal{F}^{\text{OBJ}}} \alpha_x \sum_{1 \leq y \leq m_x} f_y^x(\mathbf{t}) \quad (42)$$

$$s.t. \quad t_{\sigma^k(i)} + p_{\sigma^k(i)\sigma^k(i+1)} \leq t_{\sigma^k(i+1)} \quad 1 \leq i < n \quad (43)$$

$$f_y^x(\mathbf{t}) \leq 0 \quad F^x \in \mathcal{F}^{\text{CONS}}, \quad 1 \leq y \leq m_x \quad (44)$$

To efficiently solve the previous problem, several types of re-optimization approaches have been developed in the literature to take advantage of the information developed during the successive subproblem solving. One such approach involves re-arranging previously developed schedules in relation to the new settings. For network-flow or shortest-path formulations especially, re-optimization methods related to a change of arcs or costs in the network have been thoroughly studied (Goto and Sangiovanni-Vincentelli 1978, Pallottino 2003, Frangioni and Manca 2006, Miller-Hooks and Yang 2005). Most timing problems can also be formulated as linear programs, on which sensitivity analysis and warm start following a problem modification

have been studied for long, and can be tackled by means of a primal-dual simplex algorithm. Finally, a last methodology, on which we focus in the following, is based on the simple observation that a permutation of activities can be assimilated to a concatenation of some subsequences of consecutive activities. Hence, keeping information on subsequences (e.g., dynamic programming data) can lead to significant speed up in solving (Kindervater and Savelsbergh 1997, Cordone 2000). We introduce in the next sections a framework for these concatenation properties and the re-optimization approaches that follow.

8.2 Breakpoints and concatenations

We first introduce some vocabulary, and then emphasize the links between operations on sequences of activities, such as activity relocations or changes of order relations, and properties of the resulting permutation functions. These observations lead to efficient re-optimization approaches “by concatenation”.

Definition 6 (Permutation breakpoints). *Let $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be a permutation. Any integer b such that $\sigma(b) + 1 \neq \sigma(b + 1)$ and $1 \leq i < n$ is called a breakpoint of σ , and corresponds to non consecutive values in the permutation representation.*

Let $b(\sigma)$ denote the number of breakpoints of σ , and $b_1^\sigma, \dots, b_{b(\sigma)}^\sigma$ denote these breakpoints in increasing order. For instance, the permutation $\sigma_0 : \{1, 2, 3, 4, 5, 6\} \rightarrow \{4, \mathbf{5}, \mathbf{3}, 1, \mathbf{2}, 6\}$ has three breakpoints (indicated in boldface): $b_1^{\sigma_0} = 2$, $b_2^{\sigma_0} = 3$, and $b_3^{\sigma_0} = 5$. We now show the links between classical operations on activity sequences and the resulting permutation function properties in terms of breakpoints:

Lemma 1 (Order changes). *Let A' be an activity sequence obtained from A by changing l order relations and $\sigma_{A \rightarrow A'}$ the associated permutation function, then $b(\sigma_{A \rightarrow A'}) = l$.*

Lemma 2 (Activity relocations). *Let A' be an activity sequence obtained from A by relocating l activities and $\sigma_{A \rightarrow A'}$ the associated permutation function, then $b(\sigma_{A \rightarrow A'}) \leq 3l$.*

Any change of the order relationship results in exactly one breakpoint, while any relocation of activity can be assimilated to at most three changes of order relations, and thus yields three breakpoints. Situations where k order relations or activities are changed from one timing problem T to another problem T' occur frequently in the context of neighborhood searches working on sequences of activities and solving timing subproblems to evaluate cost or feasibility of each new sequence. The interest of breakpoints is highlighted in the following proposition. Although straightforward, it provides the basis of re-optimization methods working by concatenation.

Proposition 1. *Let $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be a permutation with breakpoints $b_1^\sigma, \dots, b_{b(\sigma)}^\sigma$. Let A be a sequence of n activities. Then, $A' = \sigma(A)$ corresponds to the concatenation of exactly $b(\sigma) + 1$ subsequences of consecutive activities in A , as presented in Equation 45. A dummy breakpoint $b_0^\sigma = 1$ stands for the beginning of the sequence.*

$$A' = \bigoplus_{l=0, \dots, b(\sigma)-1} (a_{\sigma(b_l^\sigma+1)}, \dots, a_{\sigma(b_{l+1}^\sigma)}) \quad (45)$$

Any bounded number of operations transforming an activity sequence A into A' (relocation of activities, or changes of order relations) thus involves a permutation function with a bounded

number of breakpoints, such that A' can be seen as a concatenation of a bounded number of subsequences of A . Pre-processed informations from a bounded number of subsequences can then be exploited to produce information on their concatenation, to solve the timing subproblems without browsing all activities of the sequence at play. The next section formalizes the re-optimization operations and algorithms that can be developed to this extent.

8.3 Re-optimization “by concatenation”

Re-optimization by concatenation can be formalized through a set of four basic re-optimization operators, given in Table 2, which are used for data acquisition and exploitation on subsequences of consecutive activities. As the goal of this methodology is to avoid redundant computations, it is critical for all these operators to maintain the integrity of the input data.

Table 2: Re-optimization operators

Initialization:	Initialize the data $\mathcal{D}(A)$ of a sequence containing a single activity.
Forward extension:	Given an activity a_k and a sequence $A = (a_i, \dots, a_j)$ with its data, determine the data $\mathcal{D}(A')$ for the sequence $A' = (a_k, a_i, \dots, a_j)$.
Backward extension:	Given an activity a_k and a sequence $A = (a_i, \dots, a_j)$ with its data, determine the data $\mathcal{D}(A')$ for the sequence $A' = (a_i, \dots, a_j, a_k)$.
Evaluate concatenation:	Given L sequences of activities $\mathcal{D}(A_l), l = 1 \dots l$ and their data, evaluate the feasibility and the optimal solution cost of the timing problem involving the concatenation of these sequences.

It should first be noted that the *forward extension* (respectively *backward extension*) operation of Table 2 directly derives from forward or backward dynamic programming concepts. Bi-directional dynamic programming approaches can also be assimilated to both forward and backward extension operations, as well as a single concatenation evaluation to provide the optimal solution (Righini and Salani 2006). The re-optimization approach “by concatenation”, illustrated in Algorithm 1, is also based on these operators. Data is built on subsequences of the incumbent timing problem \mathcal{T} , by means of the forward and backward extension operators. These data, developed during a preprocessing phase or during the search, are then used to solve repetitively all the derived subproblems using the *evaluate concatenation* operator.

Algorithm 1 Re-optimization

- 1: Build re-optimization data on subsequences of the *incumbent timing problem* \mathcal{T} , using *initialize*, and *forward extension* or *backward extension*.
 - 2: For each timing subproblem $\mathcal{T}^k, k \in \{1, \dots, N\}$;
 - 3: Determine the breakpoints involved in the permutation function σ^k ;
 - 4: Evaluate the optimal cost of \mathcal{T}^k , as the concatenation of $b(\sigma) + 1$ activity subsequences from \mathcal{T} (Equation 45).
-

The efficiency of Algorithm 1 directly relies on the potential to develop concatenation operations that are less computationally complex than stand-alone methods. One should also pay attention to the price to pay in terms of data computing on subsequences, and whether the resulting computational effort is dominated by the quantity of derived timing subproblems to solve. There is no unique way to apply this method; problem specific design choices arise, for

example, when determining the nature of the data, the subsequences on which it is computed, as well as the phases dedicated to data computation.

We therefore illustrate an application in neighborhood search for routing problems with time constraints on routes. In this context, a local search improvement procedure based on sequence changes leads to a number of timing subproblems proportional to the number of neighborhood solutions to explore. The number of derived timing subproblems is often $N = \Omega(n^2)$ in the VRP or scheduling literature, and the derived activity sequences do not involve the concatenation of more than $k = 2, 3$ or 4 subsequences of the original problem.

The overall complexity of a neighborhood exploration, when solving each timing subproblem independently, is $Nc(T)$, $c(T)$ being the computational complexity of one stand-alone timing solution procedure. A straightforward re-optimization approach consists in exhaustively computing the data for each of the $n(n-1)$ subsequences of consecutive activities from \mathcal{T} , and then use it to evaluate all moves. Data computation is straightforward to perform in $O(n^2c(I) + n^2c(F/B))$. $c(I)$ and $c(F/B)$ stand for the computational complexity of initialization, and forward or backward extension, respectively. The overall complexity of the new neighborhood exploration procedure is thus $O(n^2c(I) + n^2c(F/B) + Nc(EC))$, $c(EC)$ being the complexity for evaluating the concatenation of less than 4 subsequences. Assuming that $N = \Omega(n^2)$, the computational complexity of neighborhood evaluation becomes $O(N[c(I) + c(F/B) + c(EC)])$ for re-optimization methods instead of $Nc(T)$ for independent solving. Re-optimization operators being less computationally complex than stand-alone methods, the resulting approach is likely to lead to reduced computational effort.

Concatenation operators involving more than two sequences are not actually available or not computationally suitable for efficient re-optimization for some settings such as $\{\emptyset|MTW\}$ and most problems with two-dimensional features. In this case, forward and backward propagation may be used along with concatenations of two subsequences only to perform the timing subproblem evaluations. Such an example is given by the lexicographic search of Savelsbergh (1985), which enables to evaluate timing subproblems associated to well-known neighborhoods for vehicle routing problems exclusively by means of concatenation of two subsequences. Designing efficient evaluation orders in such contexts becomes a problem dependent issue, clearly impacted by the complexity of the operators at hand, and the nature of the permutations involved.

Finally, if the concatenation of many subsequences can be operated efficiently, but data creation constitutes the bottleneck in terms of computational effort, the subset of subsequences involved can be restricted to $O(n^{4/3})$ or $O(n^{8/7})$, using the hierarchical approach of Irnich (2008a).

The relevant data to compute can also be tailored relatively to the neighborhoods at play. For example, the 2-opt* move presented in Figure 7 involves only the concatenation of subsequences containing the first or last activity, which we call *prefix* or *suffix subsequences*. The number of such subsequences requiring data computation is thus reduced to $O(n)$.

8.4 General literature on the topic

A large range of vehicle routing and scheduling problems are addressed using local search on sequences. Serial timing issues thus frequently arise in these fields.

Following the seminal work of Savelsbergh (1985, 1992), Kindervater and Savelsbergh (1997) proposed a framework to manage several constraints on vehicle routes, such as precedence constraints, time windows, collection and deliveries under capacity constraints. Several of

these constraints are explicitly, or can be assimilated to, timing features. To perform efficient feasibility checking, the authors develop *global variables* on partial routes, which are used in concatenation operations to evaluate moves consisting of a constant number of edge exchanges. Move evaluations are performed in lexicographic order to allow calculation of the global variables through the search.

Cordone (2000) and Duhamel (2001) report similar concepts of global data management on subsequences and concatenation operators. Although these methodologies are essentially dedicated to the VRPTW, they explore different possibilities related to the concept of macro-nodes. Indeed, when the information developed on subsequences has the same structure as the problem data on activities, subsequences of activities can be replaced by equivalent single activities during timing resolution. Concatenation concepts in this case enable to temporarily reduce the problem size by collapsing nodes into “macro-nodes”, opening the way to algorithms based on aggregation of activities and multi-level approaches (Bean et al. 1987, Walshaw 2002, Elhallaoui et al. 2005).

Campbell and Savelsbergh (2004) presented a compilation of efficient insertion heuristics for many vehicle routing problems with additional characteristics such as shift time limits, variable delivery quantities, fixed and variable delivery times, and multiple routes per vehicle. These methods iteratively create solutions by adding customers to the routes. The authors show that by managing global data on the routes, the cost of feasibility of customer insertions can be evaluated in amortized $O(1)$ for many of these settings.

A rich body of dynamic programming-based timing algorithms is also presented in Ibaraki et al. (2005), Hashimoto et al. (2006, 2008), Hashimoto (2008), Ibaraki et al. (2008), and Hashimoto et al. (2010). Forward and backward propagation is used, with an additional “connect” operator to manage concatenation of two subsequences, thus leading to efficient re-optimization approaches by concatenation for several timing problems involving Piecewise Linear (PiL) functions.

The framework of Desaulniers et al. (1998) models many constraints and objectives on sequences of activities as resources that are subject to window constraints, and are extended from one activity to the next by means of resource extension functions (REFs). This framework proved extremely efficient to model many crew scheduling and routing problems and solve them by column generation (Desaulniers et al. 2005). It has also been recently extended by Irnich (2008a,b) to perform efficient neighborhood search under various constraints on routes, such as load dependent costs, simultaneous pickup and deliveries, maximum waiting and duty times. To that extent, REFs “generalized to segments” are built on subsequences to characterize their resource consumption. Inverse REFs are defined to give an upper bound on the resource consumptions that allow the sequence to be processed. Developing this data on subsequences enables then to evaluate efficiently the cost or feasibility of local search moves. This framework, however, requires rather restrictive conditions: the existence of REFs that can be generalized to subsequences and inversed. These conditions are satisfied only by a limited subset of the timing problems and features introduced previously, such as $\{\emptyset|TW\}$, $\{\emptyset|MTW\}$ or $\{DUR|TW\}$.

Several general methodologies thus exist in the literature to tackle timing problems within a neighborhood search context. However, these approaches are restricted by the types of concatenations allowed, the assumptions made on features, or the applicability of the models to a wide range of constraints. The timing formalism we propose and its generalization to re-optimization procedures by concatenation, following the concepts of Kindervater and Savelsbergh (1997), is less specialized and thus can encompass a wider range of timing settings and methods. As

shown in the next sections, this framework unifies previous successful concepts, and provides a line of thought for the development of efficient algorithms for various timing problems. It can also be viewed as a generalization of Irnich (2008a,b) concepts applied to timing problems. Indeed, generalized REFs and their inverse provide, when they exist, the suitable data for re-optimization.

8.5 Re-optimization algorithms

We now review and analyze re-optimization approaches for main timing features and problems. As in the first part of this paper, the analysis is organized by increasing order of complexity and feature dimensions, starting with single-dimensional features in Sections 8.5.1 to 8.5.5, and finishing with two-dimensional features in Sections 8.5.6 to 8.5.8. For each timing setting we describe the re-optimization data, as well as the operators that can be used for forward and backward data computation and for evaluating the concatenation of several subsequences. These operators can be used within Algorithm 1 to develop efficient re-optimization approaches by concatenation.

8.5.1 Constant activity costs & cumulative resources.

In presence of constant activity costs or, more generally, in presence of a cumulated resource such as distance, load or time with a global constraint, evaluating the total cost of a solution from scratch would involve to browse each activity and cumulate the costs, or the resource consumption, resulting in a $O(n)$ complexity. To perform more efficient evaluations, a simple re-optimization approach involves applying Algorithm 1 with the following data and operators.

DATA. Partial cost $C(A_i)$ (or resource consumed) by each subsequence A_i .

DATA COMPUTATION. Cumulating the cost (or resource consumption) on the subsequence.

EVALUATE CONCATENATION. Cumulating the partial costs (or resource consumptions) on subsequences: $C(A_1 \oplus \dots \oplus A_k) = \sum C(A_i)$.

The evaluation of the concatenation of a bounded number of subsequences can thus be performed in a bounded number of operations, leading to $O(1)$ complexity for move evaluation when the data is available. Data can be processed in amortized constant time for each move during a local search procedure for many classic neighborhoods, using a *lexicographic order* (Kindervater and Savelsbergh 1997) for move evaluation, or developed in a preprocessing phase for a complexity of $O(n^2)$, which is often dominated by the neighborhood size.

8.5.2 Weighted execution dates & non-decreasing linear costs.

The feature W and, in general, non-decreasing linear time-dependent costs of the form $c_i(t_i) = w_i t_i + c_i$ with $w_i \geq 0$ for $i \in \{1, \dots, n\}$ can be addressed with the following re-optimization data and operators.

DATA. Total processing time $T(A_i)$ for all the activities of the sequence A_i but the last one. Waiting cost $W(A_i)$ related to a delay of one time unit in the sequence processing, and sequence cost $C(A_i)$ when started at time 0.

DATA COMPUTATION AND EVALUATE CONCATENATION. For a sequence A with a single activity, $T(A) = 0$, $W(A) = w_{A(1)}$ and $C(A) = c_{A(1)}$. Equations (46-48) enable both to evaluate the cost of concatenation and compute by induction the data for sequences with more activities.

$$W(A_1 \oplus A_2) = W(A_1) + W(A_2) \quad (46)$$

$$C(A_1 \oplus A_2) = C(A_1) + W(A_2)(T(A_1) + p_{A_1(|A_1|)A_2(1)}) + C(A_2) \quad (47)$$

$$T(A_1 \oplus A_2) = T(A_1) + p_{A_1(|A_1|)A_2(1)} + T(A_2) \quad (48)$$

The previous equations have been formulated to also manage sequence-dependent processing times. It is remarkable that, in this case, the re-optimization data is a simple generalization of single-activity characteristics to sequences of activities, similar to the *generalization to segments* concepts of Irnich (2008b).

8.5.3 Time-windows feasibility check.

Savelsbergh (1985) opened the way to efficient feasibility checking with regards to time windows in the context of local search. In the subsequent work of Kindervater and Savelsbergh (1997), the following re-optimization data and operators are introduced.

DATA. Total processing time $T(A_i)$ for all the activities of the sequence A_i but the last one. Earliest execution date $E(A_i)$ of the last activity in any feasible schedule for A_i . Latest execution date $L(A_i)$ of the first activity in any feasible schedule for A_i . A record $isFeas(A_i)$ valuated to true if and only if a feasible schedule for A_i exists.

DATA COMPUTATION AND EVALUATE CONCATENATION. For a sequence A with a single activity, $T(A) = 0$, $E(A) = r_{A(1)}$, $L(A) = d_{A(1)}$ and $isFeas(A) = true$. Equations (49-52) enable to evaluate the cost of concatenation and compute the data for sequences with more activities in $O(1)$.

$$T(A_1 \oplus A_2) = T(A_1) + p_{A_1(|A_1|)A_2(1)} + T(A_2) \quad (49)$$

$$E(A_1 \oplus A_2) = \max\{E(A_1) + p_{A_1(|A_1|)A_2(1)} + T(A_2), E(A_2)\} \quad (50)$$

$$L(A_1 \oplus A_2) = \min\{L(A_1), L(A_2) - p_{A_1(|A_1|)A_2(1)} - T(A_1)\} \quad (51)$$

$$isFeas(A_1 \oplus A_2) \equiv isFeas(A_1) \wedge isFeas(A_2) \wedge (E(A_1) + p_{A_1(|A_1|)A_2(1)} \leq L(A_2)) \quad (52)$$

It should also be noted that the total duration to process the sequence is given by $DUR(A_i) = \max\{E(A_i) - L(A_i), T(A_i)\}$, and thus this approach also enables to solve $\{\emptyset|DUR, TW\}$ and $\{DUR|TW\}$ serial timing problems in $O(1)$.

8.5.4 Earliness, tardiness, soft time-windows, and separable convex costs.

Timing problems with tardiness $\{D|\emptyset\}$, earliness and tardiness $\{R, D(r_i = d_i)|\emptyset\}$, or with soft time windows $\{TW|\emptyset\}$ can be solved in a stand-alone way by means of a minimum idle time policy in $O(n)$, or variants of the PAV algorithm (Section 5.3) in $O(n \log n)$, respectively. Nevertheless, a better complexity can be achieved by means of re-optimization approaches.

Ibaraki et al. (2008) considers the efficient resolution of the serial timing problem $\{\Sigma c^{cvx}(t)|\emptyset\}$, and attains an amortized logarithmic complexity per sub-problem when the activity costs are PiL non-negative, lower semicontinuous and convex. The re-optimization data corresponds to the dynamic programming functions described in Section 5.4. These functions are represented as segment pieces within a tree data structure, and computed only on *prefix* and *suffix* subsequences, containing the first or the last activity of the incumbent timing problem.

DATA. Total duty time $T(A_i)$ on a partial sequence A_i . Optimal cost $\bar{F}(A_i)(t)$ of a schedule for A_i , when the first activity is executed before t ($t_{A_i(1)} \leq t$). Optimal cost $\bar{B}(A_i)(t)$ of a schedule for A_i , when the last activity is executed after t .

DATA COMPUTATION. $\bar{F}(A_i)(t)$ and $\bar{B}(A_i)(t)$ are computed by means of forward and backward dynamic programming (Equations 19-20), respectively. The use of tree structures for function representations allows for data computations in $O(\varphi_c \log \varphi_c)$ for each subsequence, where φ_c is the total number of pieces in the cost functions of the timing subproblem.

EVALUATE CONCATENATION. Equation (53) returns the optimal cost $Z^*(A_1 \oplus A_2)$ of the timing problem related to the concatenation of two sequences. This value can be computed in $O(\log \varphi_c)$. When the number of pieces of cost functions is linear in the number of activities, as in $\{D|\emptyset\}$, $\{R, D(r_i = d_i)|\emptyset\}$ or $\{TW|\emptyset\}$ settings, a $O(\log n)$ complexity is attained. The concatenation of a bounded number of subsequences can be also efficiently evaluated (see Ibaraki et al. 2008).

$$Z^*(A_1 \oplus A_2) = \min_{t \geq 0} \{F(A_1)(t) + B(A_2)(t + p_{A_1(|A_1|)A_2(1)})\} \quad (53)$$

Only Ergun and Orlin (2006) and Kedad-Sidhoum and Sourd (2010) have actually reported methods with an *evaluate concatenation* operator working in amortized $O(1)$ for $\{D|\emptyset\}$ and $\{D, R(d_i = r_i)|NWT\}$ serial timing, respectively, in presence of particular types of permutation functions (neighborhood search based on swap, insert as well as compound moves), and for sequence-independent processing times. The cornerstone of these two approaches is that they call the functions $B(A_2)(t)$ associated to subsequences by series of $O(n)$ increasing values of t . The methodology requires some orderings, which are performed in a pre-processing phase in $O(n \log n)$. This complexity is generally dominated by the number N of timing subproblems. It is actually an open question whereas this type of approach can be extended to more general problems. This approach is likely to become far more complex when tackling sequence-dependent processing times, in particular. Approximate serial timing procedures in $O(1)$ may also be used when computational time is critical (Taillard et al. 1997).

8.5.5 Separable cost functions and multiple time-windows.

Ibaraki et al. (2005) and Hendel and Sourd (2006) simultaneously introduced a re-optimization approach in $O(\varphi_c)$ for timing problems with non-convex PiL cost functions. This approach brings into play the same quantities and functions $T(A_i)$, $\bar{F}(A_i)(t)$ and $\bar{B}(A_i)(t)$ as in Section 8.5.4. In this case, simple linked lists are sufficient for function representation. Evaluating the concatenations is also performed by means of Equation (53), resulting in $O(\varphi_c)$ complexity.

8.5.6 Flexible processing times: a special case.

The flexible processing time feature involves separable functions of successive activity execution dates. Complex problems are raised when this feature is coupled with time-window constraints as in $\{\Sigma c_i(\Delta t_i)|TW\}$ (Equations 54-55), or with separable activity execution costs. The total order constraints can be directly taken into account in the objective, with cost functions c_{ij} such that for $\Delta t < 0$, $c_{ij}(\Delta t) = +\infty$.

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^{n+}} \sum_{i=1}^n c'_{\sigma_i \sigma_{i+1}} (t_{\sigma_{i+1}} - t_{\sigma_i}) \quad (54)$$

$$s.t. \quad r_i \leq t_i \leq d_i \quad 1 \leq i \leq n \quad (55)$$

We first present a re-optimization approach for a particular shape of cost functions c_{ij} , introduced by Nagata (2007) in the context of VRP with time-windows as an alternative time relaxation. Instead of allowing earliness or lateness with respect to time-window constraints, penalized processing time reductions are allowed. The resulting cost functions are given in Equation (56). In this case, no limit is fixed on the amount of processing time reduction, thus allowing negative processing times and non infinite c_{ij} values on \mathbb{R}^{*-} .

$$c'_{ij}(\Delta t) = \begin{cases} 0 & \text{if } \Delta t \geq p_{ij} \\ \alpha(p_{ij} - \Delta t) & \text{otherwise} \end{cases} \quad (56)$$

Nagata et al. (2010) and Hashimoto et al. (2008) introduced forward and backward dynamic programming functions to efficiently evaluate the merge of two sequences as a result of some particular VRP neighborhoods. We describe here the re-optimization approach of Vidal et al. (2013), which enables to work with any number of concatenations, and accounts for duration features. This approach is closely related to the method of Kindervater and Savelsbergh (1997) for $\{\emptyset|DUR\}$ and $\{\emptyset|DUR, TW\}$.

DATA. Minimum duration $D(A_i)$ to perform all the activities of A_i but the last one. Earliest execution date $E(A_i)$ of the first activity in any feasible schedule with minimum idle time for A_i . Latest possible execution date $L(A_i)$ of the first activity in any feasible schedule with minimum processing time reduction for A_i . Minimum processing time reduction $TW(A_i)$ in any feasible schedule for A_i .

DATA COMPUTATION AND EVALUATE CONCATENATION. For a sequence A with a single activity, $D(A) = TW(A) = 0$, $E(A) = r_{A(1)}$ and $L(A) = d_{A(1)}$. Equations (57-62) enable then to compute the re-optimization data and cost for a concatenation of two sequences.

$$E(A_1 \oplus A_2) = \max\{E(A_2) - D(A_1) + TW(A_1) - p_{A_1(|A_1|)A_2(1)}, E(A_1)\} - \delta_{WT} \quad (57)$$

$$L(A_1 \oplus A_2) = \min\{L(A_2) - D(A_1) + TW(A_1) - p_{A_1(|A_1|)A_2(1)}, L(A_1)\} + \delta_{TW} \quad (58)$$

$$D(A_1 \oplus A_2) = D(A_1) + D(A_2) + p_{A_1(|A_1|)A_2(1)} + \delta_{WT} \quad (59)$$

$$TW(A_1 \oplus A_2) = TW(A_1) + TW(A_2) + \delta_{TW} \quad (60)$$

$$\text{with } \delta_{WT} = \max\{E(A_2) - D(A_1) + TW(A_1) - p_{A_1(|A_1|)A_2(1)} - L(A_1), 0\} \quad (61)$$

$$\text{and } \delta_{TW} = \max\{E(A_1) + D(A_1) - TW(A_1) + p_{A_1(|A_1|)A_2(1)} - L(A_2), 0\} \quad (62)$$

This approach enables to evaluate in $O(1)$ the minimum amount of processing time reduction for any constant number of subsequence concatenations. In terms of computational complexity, the flexible processing time relaxation is a good option for local search methods. In contrast, the best re-optimization methods for soft time-window relaxations $\{TW|P\}$ or $\{D|R, P\}$ attain a complexity of $O(\log n)$ per sub-problem (Section 8.5.4).

8.5.7 General flexible processing-times.

Sourd (2005) and Hashimoto et al. (2006) addressed a general serial timing problem $\{\Sigma c_i(\Delta t_i), \Sigma c_i(t_i)|\emptyset\}$ with flexible and sequence-dependent processing times and costs on activity execution dates. This problem is formulated in Equation (63). Functions $c_i(t)$ and $c'_{ij}(t)$ are assumed to be PiL, lower semicontinuous, non-negative and take infinite value for $t < 0$. Functions $c'_{\sigma_i \sigma_{i+1}}(\Delta t)$ are convex.

$$\min_{(t_1, \dots, t_n) \in \mathbb{R}^n} \sum_{i=1}^n c_i(t_i) + \sum_{i=1}^{n-1} c'_{\sigma_i \sigma_{i+1}}(t_{\sigma_{i+1}} - t_{\sigma_i}) \quad (63)$$

DATA. Optimal schedule cost $F(A_i)(t)$ when the last activity is executed exactly at time t . Optimal cost $B(A_i)(t)$ when the first activity is executed exactly at time t .

DATA COMPUTATION. For a sequence A with a single activity, $F(A)(t) = B(A)(t) = c_{A(1)}(t)$. Equations (64-65) can then be used to compute the $F(A_i)(t)$ and $B(A_i)(t)$ functions, respectively, on prefix and suffix subsequences of the incumbent timing problem.

$$F(A_i \oplus A)(t) = \min_{0 \leq x \leq t} \{F(A_i)(x) + c'_{A_i(|A_i|)A(1)}(t-x)\} + c_{A(1)}(t) \quad (64)$$

$$B(A \oplus A_i)(t) = c_{A(1)}(t) + \min_{x \geq t} \{c'_{A(1)A_i(|A_i|)}(x-t) + B(A_i)(x)\} \quad (65)$$

EVALUATE CONCATENATION. Equation (66) returns the optimal cost $Z^*(A_1 \oplus A_2)$ of the timing problem related to the concatenation of a prefix and a suffix subsequence in $O(\varphi_c + \widehat{\varphi}_c \times \varphi'_c)$ (Hashimoto et al. 2006). φ_c and φ'_c are respectively the total number of pieces in cost functions c_i and $c'_{\sigma_i \sigma_{i+1}}$. $\widehat{\varphi}_c$ represents the number of convex pieces in $c'_{\sigma_i \sigma_{i+1}}$.

$$Z^*(A_1 \oplus A_2) = \min_{t \geq 0} \{F(A_1)(t) + \min_{t' \geq t} (c'_{A_1(|A_1|)A_2(1)}(t'-t) + B(A_2)(t'))\} \quad (66)$$

Finally, it was noted in Section 6.3 that $\{DUR|MTW\}$ and $\{\emptyset|DUR, MTW\}$ constitute special cases of $\{\Sigma c_i(\Delta t_i), \Sigma c_i(t_i)|\emptyset\}$. Hence, the previous re-optimization approach applies, leading to an amortized complexity of $O(1 + \varphi_{MTW})$ for solving successive $\{DUR|MTW\}$ or $\{\emptyset|DUR, MTW\}$ subproblems, where φ_{MTW} denotes the total number of time windows in the problem. This is an improvement over the previous procedures in $O(n + n\varphi_{MTW})$ for these settings.

8.5.8 Time-dependent processing times.

Donati et al. (2008) solve the feasibility problem $\{\emptyset|P(t), TW\}$ using an extension of the method of Savelsbergh (1985). Let $g_{ij}(t) = t + p_{ij}(t)$ be the completion date of an activity i started at t , and followed by activity j . Under the assumption that all $g_{ij}(t)$ are continuous and strictly increasing (any activity started strictly later will finish strictly later), the inverse function $g_{ij}^{-1}(t)$ can be defined and the following re-optimization data and operators enable to perform efficient feasibility checks.

DATA. Earliest possible execution date $E(A_i)$ of the last activity in any feasible schedule for A_i . Latest possible execution date $L(A_i)$ of the first activity in any feasible schedule for A_i .

DATA COMPUTATION. For a sequence A with a single activity, $E(A) = r_{A(1)}$, and $L(A) = d_{A(1)}$. Equations (67-70) then enables to determine the re-optimization data on prefix and suffix subsequences by forward and backward dynamic programming.

$$E(A_1 \oplus A) = \max\{r_{A(1)}, g_{A_1(|A_1|)A(1)}(E(A_1))\} \quad (67)$$

$$isFeas(A_1 \oplus A) \equiv isFeas(A_1) \wedge \{E(A_1 \oplus A) \leq d_{A(1)}\} \quad (68)$$

$$L(A \oplus A_2) = \min\{d_{A(1)}, g_{A(1)A_2(1)}^{-1}(L(A_2))\} \quad (69)$$

$$isFeas(A \oplus A_2) \equiv isFeas(A_2) \wedge \{L(A \oplus A_2) \geq r_{A(1)}\} \quad (70)$$

EVALUATE CONCATENATION. Equation (71) enables to state on the feasibility of any concatenation of a pair of prefix and suffix subsequences.

$$isFeas(A_1 \oplus A_2) \equiv isFeas(A_1) \wedge isFeas(A_2) \wedge \{E(A_1) + p_{A_1(|A_1|)A_2(1)}(E(A_1)) \leq L(A_2)\} \quad (71)$$

Under the assumption that function $g^{-1}(t)$ is evaluated in $O(1)$, the previous re-optimization framework enables to evaluate the feasibility of a concatenation of two subsequences in $O(1)$. However, it does not allow to concatenate more than two subsequences, as opposed to the fixed processing time setting treated in Section 8.5.3, thus limiting the range of local search moves that can be evaluated efficiently without relying on a lexicographic search order.

Finally, Hashimoto et al. (2008) proposed a dynamic programming approach similar to the one of Sections 8.5.4 and 8.5.6 to manage the general case with time-dependent (and sequence-dependent) processing times with separable execution costs $\{\Sigma c_i(t_i)|P(t)\}$. Evaluations of concatenations of pairs of subsequences A_1 and A_2 are performed in $O(\varphi_c + \varphi_p)$, where φ_c and φ_p denote respectively the total number of pieces in the cost and processing-time functions.

8.6 Summary and perspectives on re-optimization

Table 3 summarizes the complexity of stand-alone resolution and re-optimization operations for the main timing settings in the literature. The leftmost column lists the problems, while the next block of columns present stand-alone approaches and their complexity. Following to the right, the next columns are dedicated to summarize re-optimization approaches, the complexity of forward and backward data construction, “F/B”, the complexity of concatenation of two, “C2”, and more than two subsequences, “C3+”, if available. Column “Sd” finally indicates whether the re-optimization approach enables to tackle problems with sequence-dependent parameters. Additional assumptions on the problems are listed in the last column.

Re-optimization has clearly proven useful to address more efficiently several timing settings, reducing in many case the computational complexity by a factor of n or φ_c . Still, the complexity gain remains smaller in some particular cases. Solving the timing problem $\{D|R\}$ from scratch, for example, can be done in $O(n)$ but the best re-optimization approach attains only a complexity of $O(\log n)$. This latter timing problem takes a prominent role in many VRP heuristics that allow time-constraints relaxations. Whether a $O(1)$ re-optimization algorithm exists for $\{D|R\}$ remains an open question. In addition, no re-optimization algorithm is available, to our knowledge, for several problems such as $\{DUR|MTW\}$, $\{\emptyset|DUR, MTW\}$, $\{DUR|TW, P(t)\}$, $\{\emptyset|TL, TW\}$ and $\{DUR > D > TL|R\}$. Finally, another promising research path concerns the study of re-optimization data and concatenation operators involving more than two subsequences and their application to sequence-dependent problems. Such operators are still missing in several cases such as $\{\Sigma c_i(t_i)|P(t)\}$ or $\{\Sigma c_i^{cvx}(\Delta t_i), \Sigma c_i(t_i)|\emptyset\}$, but are necessary to achieve high-performance local searches in many combinatorial optimization problems with time features.

9 General conclusion

In this paper, a rich body of problems with time characteristics and totally ordered variables has been identified and classified. Many algorithms from different fields of research were analyzed to identify key problem features and main solution concepts. As timing subproblems frequently arise in the context of local search, both the stand-alone resolution of problems, and the efficient resolution of series of problems by means of re-optimization approaches were analyzed. A general re-optimization framework based on decomposition and recombination of sequences was introduced, and links to other re-optimization approaches were highlighted.

A subset of timing methods, originating from various research fields, and constituting the

Table 3: Complexity of algorithms and re-optimization operators for common timing problems

Problem	From Scratch	Re-opt. by concat.	F/B	C2	C3+	Sd	Assumptions
Const. act. costs $\{W \emptyset\}$ $\{\emptyset TW\}$	— Min idle time Min idle time	— — Savelsbergh (1985) & Kind. and Sav. (1997)	$O(1)$ $O(1)$ $O(1)$	$O(1)$ $O(1)$ $O(1)$	$O(1)$ $O(1)$ $O(1)$	✓ ✓ ✓	penalty coefficient depending upon act. penalty coefficient depending upon act.
$\{D \emptyset\}$	Min idle time	Ergun and Orlin (2006)	$O(\log n)$	$O(1)^*$	—	✓	penalty coefficient depending upon act.
$\{D, R(d_i = r_i) NWWT\}$	Min idle time	Kedad-Sidhoum and Sourd (2010)	$O(\log n)$	$O(1)^*$	—	✓	penalty coefficient depending upon act.
$\{D, R(d_i = r_i) \emptyset\}$	Garey et al. (1988) & Ahuja and Orlin (2001)	Ibaraki et al. (2008)	$O(\log n)$	$O(\log n)$	$O(\log n)$	✓	
$\{D R\}$	Min idle time	Ibaraki et al. (2008)	$O(\log n)$	$O(\log n)$	$O(\log n)$	✓	
$\{\Sigma c_i^{\text{cvx}}(t_i) \emptyset\}$	Ibaraki et al. (2008)	Ibaraki et al. (2008)	$O(\log \varphi_c)$	$O(\log \varphi_c)$	$O(\log \varphi_c)$	✓	cost f. ≥ 0 , PiL & LSC
$\{\Sigma c_i(t_i) \emptyset\}$	Ibaraki et al. (2005)	Ibaraki et al. (2005)	$O(n\varphi_c)$	$O(\varphi_c)$	$O(\varphi_c)$	✓	cost f. ≥ 0 , PiL & LSC
$\{\emptyset MTW\}$	Min idle time	Ibaraki et al. (2005)	$O(n + \varphi_{MTW})$	$O(\log \varphi_{MTW})$	—	✓	
$\{DUR TW\}$, $\{\emptyset DUR, TW\}$	Malcolm et al. (1959)	Savelsbergh (1992) & Kind. and Sav. (1997)	$O(n)$	$O(1)$	$O(1)$	✓	
$\{DUR MTW\}$, $\{\emptyset DUR, MTW\}$	Tricoire et al. (2010)	Hashimoto et al. (2006)	$O(n\varphi_{MTW})$	$O(\varphi_{MTW})$	—	✓	
$\{\emptyset IDL, TW\}$	Hunsaker and S. (2002)	—	$O(n)$	—	—	—	with ϵ precision
$\{\Sigma c_i^{\text{cvx}}(\Delta t_i) DUR\}$	Hochbaum (1994)	—	$O(n \log n \log(d_{max}/\epsilon n))$	—	—	—	cost f. ≥ 0 , PiL & LSC
$\{\Sigma c_i^{\text{cvx}}(\Delta t_i), \Sigma c_i(t_i) \emptyset\}$	Sourd (2005) & Hashimoto et al. (2006)	Sourd (2005) & Hashimoto et al. (2006)	$O(n(\varphi_c + \widehat{\varphi}_c \times \varphi'_c))$	$O(\varphi_c + \widehat{\varphi}_c \times \varphi'_c)$	—	✓	
$\{D R, P(t)\}$	Min idle time	—	$O(n)$	—	—	—	FIFO assumption
$\{\emptyset TW, P(t)\}$	Donati et al. (2008)	Donati et al. (2008)	$O(n)$	$O(1)$	—	✓	FIFO assumption
$\{\Sigma c_i(t_i) P(t)\}$	Hashimoto et al. (2008)	Hashimoto et al. (2008)	$O(n(\varphi_c + \varphi_p))$	$O(\varphi_c + \varphi_p)$	—	✓	cost f. ≥ 0 , PiL & LSC & HYI assumption
$\{\emptyset TL, TW\}$	Hurink and Keuchel (2001)	—	$O(n^3)$	—	—	—	$O(n)$ TL constraints, $M \Leftrightarrow$ nb. open deliv.
$\{\emptyset TL, TW\}$	Firat and Woeginger (2011)	Gschwind and Irnich (2012)	$O(n)$	$O(M)$	—	✓	$O(n)$ TL constraints & LIFO assumption
$\{DUR > D > TL R\}$	Cordeau and Laporte (2003)	—	$O(n^2)$	—	—	—	U is an upper bound of execution dates
$\{\Sigma c_{ij}^{\text{cvx}}(t_j - t_i), \Sigma c_i^{\text{cvx}}(t_i) \emptyset\}$	Aluja et al. (2003)	—	$O(n^3 \log n \log(nU))$	—	—	—	

actual state-of-the-art for timing problems with different features, has been identified. These algorithms are a keystone for addressing many rich combinatorial optimization problems with time characteristics, for which the timing sub-problem represents the core of the originality and difficulty. Having a library of timing algorithms at hand opens also the door to further developments on more generic solvers that relegate the problem difficulties to known subproblems and methods.

Many interesting avenues of research arise as a result of this work. First of all, for several timing features studied in this article, more efficient stand-alone, re-optimization methods and complexity lower bounds should be investigated (Section 8.6). The impact of sequence-dependency on re-optimization is also another interesting concern, as sequence-dependency constitutes a fundamental delimitation between routing-related problems and scheduling settings. Identifying precisely its impact on re-optimization procedures would lead to better insights on local search-based methods for these two important classes of problems. Finally, even if the focus of this paper was on time characteristics, other cumulative resources such as loads, energy or workforce, lead to similar features and models. The work performed on timing can thus prove useful for an even broader range of applications.

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Appendix

A Reduction of LISP to $\{TW(\text{unit})|P\}$

Given a vector $\mathbf{N} = (N_1, \dots, N_n)$ of n real numbers, LISP aims to find the maximum length L of a non-decreasing subsequence of numbers: $L = \max\{k | 1 \leq i_1 < \dots < i_k \leq n \text{ and } N_{i_1} \leq \dots \leq N_{i_k}\}$.

From a LISP instance, we construct the following instance \mathcal{T} of $\{TW(\text{unit})|\emptyset\}$, with n activities such that for $i \in \{1, \dots, n\}$, $r_i = d_i = N_i$ and $p_i = 0$. This instance is created in n elementary algorithmic operations.

Let $z^*(\mathcal{T})$ be the optimal solution cost of \mathcal{T} . This solution naturally initiates as many activities as possible without penalties, within a non-decreasing order of execution dates. Hence, the activities achieved without penalty correspond to the LISP subsequence sought in the original problem, whose length is $L^* = n - z^*(\mathcal{T})$. Hence, LISP admits a many-one reduction to $\{TW(\text{unit})|\emptyset\}$. Conversely, $\{TW(\text{unit})|\emptyset\}$ generalizes LISP.

B Proof of Theorem 1: Block optimality conditions of $\{\sum c_i^{\text{cvx}}(t_i)|\emptyset\}$

We first recall the corresponding timing problem, stated in Equations (72-73).

$$\min_{(t_1, \dots, t_n) \in \mathfrak{R}^n} \sum_{i=1}^n c_i(t_i) \quad (72)$$

$$\text{s.t. } t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \quad (73)$$

The functions $c_i(t) : \mathfrak{R} \rightarrow \mathfrak{R}$ are assumed to be convex, but not necessarily smooth. We note $\delta c_i(t)$ the subdifferential of c_i at t , which is necessarily a non-empty interval, as a byproduct of the convexity assumption and the space of definition. We first recall two useful properties on subdifferentials from Rockafellar (1970), to follow with the proof of Theorem (1).

Proposition 2. *Let f_1, \dots, f_m be subdifferentiable functions on \mathfrak{R}^n , then:*

$$\delta(f_1(x) + \dots + f_m(x)) \supset \delta f_1(x) + \dots + \delta f_m(x) \quad \forall x \quad (74)$$

Theorem 2. *Let f_1, \dots, f_m be a set of proper convex functions on \mathfrak{R}^n having at least one common point in the relative interior of their domains $ri(\text{dom}(f_i))$, then:*

$$\delta(f_1(x) + \dots + f_m(x)) = \delta f_1(x) + \dots + \delta f_m(x) \quad \forall x \in \bigcap ri(\text{dom}(f_i)) \quad (75)$$

Constraint qualifications hold as the constraints are linear, and any solution with idle time between each activity is feasible, thus taking place in the relative interior of the polytope. Hence, strong duality applies, leading to the following necessary and sufficient optimality conditions. A solution $\mathbf{t}^* = (t_1^*, \dots, t_n^*)$ of Problem (72-73) is optimal if and only if a set of Lagrangian multipliers $(\lambda_1^*, \dots, \lambda_{n-1}^*)$ exists, such that conditions (76) are satisfied (Bertsekas et al. 2003, Propositions 5.7.1 and 6.4.2).

$$\left\{ \begin{array}{ll} t_{i-1}^* + p_{i-1} \leq t_i^* & i = 2, \dots, n \\ 0 \in \delta c_1(t_1^*) + \lambda_1^* & \\ 0 \in \delta c_i(t_i^*) + \lambda_i^* - \lambda_{i-1}^* & i = 2, \dots, n-1 \\ 0 \in \delta c_n(t_n^*) - \lambda_{n-1}^* & \\ \lambda_{i-1}^*(t_{i-1}^* + p_{i-1} - t_i^*) = 0 & i = 2, \dots, n \\ \lambda_i^* \geq 0 & i = 1, \dots, n \end{array} \right. \quad (76)$$

Solution \mathbf{t}^* can be represented as a succession of blocks of activities (B_1, \dots, B_m) , such that activities within each block are processed without idle time, and the last activities of blocks are followed by non-zero idle time. The previous definition, combined with primal feasibility, yields $t_{B_j(|B_j|)}^* + p_{B_j(|B_j|)} < t_{B_j(|B_j|)+1}^*$, and thus $\lambda_{B_j(|B_j|)}^* = 0$ for $j \in \{1, \dots, m-1\}$. Conditions (76) are thus equivalent to primal feasibility (equivalent to the first condition of (1) combined with the definition of blocks) and the following independent systems of equations for each block:

$$\forall j \in \{1, \dots, m\} \left\{ \begin{array}{ll} \text{i)} & 0 \in \delta c_{B_j(1)}(t_{B_j(1)}^*) + \lambda_{B_j(1)}^* \\ \text{ii)} & 0 \in \delta c_i(t_{B_j(1)}^* + p_{B_j(1)i-1}) + \lambda_i^* - \lambda_{i-1}^* \quad i = B_j(1) + 1, \dots, B_j(|B_j|) - 1 \\ \text{iii)} & 0 \in \delta c_{B_j(|B_j|)}(t_{B_j(1)}^* + p_{B_j(1)B_j(|B_j|-1)}) - \lambda_{B_j(|B_j|-1)}^* \\ \text{iv)} & \lambda_i^* \geq 0 \quad i = B_j(1), \dots, B_j(|B_j|) \end{array} \right. \quad (77)$$

Necessary condition proof. For $1 \leq i \leq j \leq n$, let p_{ij} be the cumulative processing duration of activities a_i to a_j . Relying on Proposition (2), one can sum i) , ii) and iii) of (77), leading to:

$$0 \in \delta c_{B_j(1)}(t_{B_j(1)}^*) + \sum_{i=B_j(1)+1}^{B_j(|B_j|)} \delta c_i(t_{B_j(1)}^* + p_{B_j(1)i-1}) \Rightarrow 0 \in \delta C_{B_j}(t_{B_j(1)}^*), \quad (78)$$

and thus, following the definition of the optimal block execution cost of Equation (18), $t_{B_i(1)}^* \in [T_{B_i}^-, T_{B_i}^{+*}]$. Any optimal solution thus verifies the first statement of Theorem (1). Finally, for any block B_j and prefix block B_j^k , summing i) , ii) and iii) for $j \in \{B_j(1) + 1, \dots, k\}$ leads to:

$$-\lambda_k^* \in \delta c_{B_j(1)}(t_{B_j(1)}^*) + \sum_{i=B_j(1)+1}^k \delta c_i(t_{B_j(1)}^* + p_{B_j(1)i-1}) \Rightarrow -\lambda_k^* \in \delta C_{B_j}(t_{B_j^k(1)}^*), \quad (79)$$

which can be reformulated as $T_{B_j^k}^{+*} \geq t_{B_j(1)}^*$ and implies the last statement of Theorem (1).

Sufficient condition proof. Consider a solution $\mathbf{t} = (t_1, \dots, t_n)$ with its blocks (B_1, \dots, B_m) , respecting conditions of Theorem (1). Following block definitions and primal feasibility, it only remains to prove that Conditions (77) are respected for each block. We choose the following Lagrangian multipliers, which are non-negative as $T_{B_i}^{+*} \geq t_{B_i(1)} \Rightarrow \exists x \leq 0 \in \delta C_{B_j^i}(t_{B_j(1)})$:

$$\forall j \in \{1, \dots, m\} \left\{ \begin{array}{ll} \lambda_i^* = -\min(x \in \delta C_{B_j^i}(t_{B_j(1)})) & i = B_j(1) + 1, \dots, B_j(|B_j|) - 1 \\ \lambda_{B_j(|B_j|)}^* = 0 & \end{array} \right. \quad (80)$$

Proposition (2) then involves that for $i \in \{1, \dots, m\}$ and $j \in \{B_j(1) + 1, \dots, B_j(|B_j|) - 1\}$;

$$-\lambda_i^* \in \delta C_{B_j^i}(t_{B_j(1)}) = \delta c_{B_j(1)}(t_{B_j(1)}) + \sum_{k=B_j(1)+1}^i \delta c_k(t_{B_j(1)} + p_{B_j(1)k-1}) \quad (81)$$

In the case where $i = B_j(1)$, Equation (81) proves statement i) of (77). Also, $\lambda_{B_j(1)}^* \in \delta(-c_{B_j(1)})(t_{B_j(1)})$, and we can combine this statement with Equation (81) for $i = B_j(1) + 1$, using Proposition 2), leading to:

$$\begin{aligned} \lambda_{B_j(1)+1}^* - \lambda_{B_j(1)}^* &\in \delta(-c_{B_j(1)})(t_{B_j(1)}) + \delta c_{B_j(1)}(t_{B_j(1)}) + \delta c_{B_j(1)+1}(t_{B_j(1)} + p_{B_j(1)}) \\ &= \delta c_{B_j(1)+1}(t_{B_j(1)} + p_{B_j(1)}) \end{aligned} \quad (82)$$

The remaining statements of Equation ii) and Equation iii) in (77) are proven by recurrence. Assuming that for a given $i \in \{B_j(1) + 1, \dots, B_j(|B_j|) - 1\}$, $\lambda_{i-1}^* - \lambda_i^* \in \delta c_i(t_{B_j(1)} + p_{B_j(1)i})$, then

$$\begin{aligned} \lambda_i^* - \lambda_{i+1}^* &= -\lambda_{i+1}^* + \lambda_{i-1}^* - (\lambda_{i-1}^* - \lambda_i^*) \\ &\in \delta c_{B_j(1)}(t_{B_j(1)}) + \sum_{k=B_j(1)}^{i+1} \delta c_k(t_{B_j(1)} + p_{B_j(1)k-1}) + \delta(-c_{B_j(1)})(t_{B_j(1)}) \\ &\quad + \sum_{k=B_j(1)}^{i-1} \delta(-c_k)(t_{B_j(1)} + p_{B_j(1)k-1}) + \delta(-c_i)(t_{B_j(1)} + p_{B_j(1)i-1}) \\ &\subset \delta c_{i+1}(t_{B_j(1)} + p_{B_j(1)i}) \end{aligned} \quad (83)$$

All the sufficient optimality conditions are thus satisfied by solution $\mathbf{t} = (t_1, \dots, t_n)$. \square