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Worst-Case Analysis for New Online Bin Packing Problems

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Abstract. We consider two new online bin packing problems, the online Variable Cost and Size Bin Packing Problem (o-VCSBPP) and the online Generalized Bin Packing Problem (o-GBPP). We take two well-known bin packing algorithms to address them, the First Fit (FF) and the Best Fit (BF). We show that both algorithms have an asymptotic worst-case ratio bound equal to 2 for the o-VCSBPP and this bound is tight. When there are enough bins of a particular type to load all items, FF and BF also have an absolute worst-case ratio bound equal to 2 for the o-VCSBPP, and this bound is also tight. In addition, we prove that no worst-case ratio bound of FF and BF can be computed for the o-GBPP. Therefore, we consider a natural evolution of these algorithms, the First Fit with Rejection (FFR) and the Best Fit with Rejection (BFR), able to reject inconvenient bins at the end of the process. Similarly, we prove that no worst-case ratio of these algorithms can be computed for the o-GBPP. Finally, we give sufficient conditions under which algorithms do not admit any performance ratio, and conclude that the worst-case results obtained for the o-VCSBPP and the o-GBPP also hold for the offline variant of these two problems.

Keywords. Online bin packing problems asymptotic worst-case ratio absolute worst-case ratio.

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1 Introduction

The Bin Packing Problem (BPP), both online [11] and offline [12, 7, 4], is a widely studied problem. It consists of finding the minimum number of bins, with the same capacity, in order to accommodate a set of items satisfying capacity constraints.

Johnson [7] proposed the Next Fit (NF) algorithm for this problem and proved that its performance ratio is 2. Johnson et al. [8] showed that the First Fit (FF) and the Best Fit (BF) algorithms have both performance ratios of 17/10. Crainic et al. [3] conducted an asymptotic worst-case analysis on lower bounds for the BPP.

The BPP was exploited in many fields such as computer science and engineering, transportation, logistics, and telecommunications. Due to its theoretical and practical relevance, several variants and richer settings were proposed.

Li and Chen [10] studied the variant where all bins have the same capacity but are characterized by a nondecreasing concave cost function of the bin utilization. The authors proved that for this problem the FF and BF algorithms have asymptotic and absolute worst-case ratios equal to 2.

Friesen and Langston [6] proposed the Variable Sized Bin Packing Problem (VSBPP), where bins with different sizes are available and the goal is to minimize the wasted space. The authors provided one online and two offline algorithms and proved that their worst-case ratios are 2, 3/2, and 4/3, respectively.

Kang and Park [9] studied this problem by also considering the bin costs, but assuming that the bin unit cost does not increase as the bin size increases. They provided two offline algorithms and showed that their asymptotic worst-case ratio is equal to 3/2.

The most advanced variants of the BPP are the Variable Cost and Size Bin Packing Problem (VCSBPP) [5] and the Generalized Bin Packing Problem (GBPP) [2, 1].

In the VCSBPP, in addition to having different sizes, bins also have different fixed selection costs, which are not necessarily correlated to the bin sizes. Crainic et al. [5] proposed accurate bounds.

No perfomance ratio is available for the VCSBPP, either online or offline.

Finally, the GBPP generalizes several packing problems characterized by multiple item and bin attributes, and the presence of both compulsory and non-compulsory items. Exact and approximated methods for this problem were proposed in [1]. No performance ratio exists for this problem either.

Although the VCSBPP and the GBPP were studied in their offline variant, to the best of our knowledge, nobody has ever addressed their online variant. For this reason, we decided to consider the online VCSBPP (o-VCSBPP) and the online GBPP (o-GBPP), and perform a worst-case analysis of two well-known algorithms to address these problems, the FF and the BF.

In this paper, we show that, although the o-VCSBPP is a more general problem than the one studied by Li and Chen [10], we can still guarantee the same asymptotic worst-case ratio bound equal to 2 and this bound is tight. Furthermore, we prove that, when there are enough bins of a particular type to load all items, we can guarantee for the o-VCSBPP an absolute worst-case ratio bound equal to 2 of both algorithms and this bound is also tight. For the o-GBPP, we prove that no worst-case ratio bound of FF and BF can be computed.

We also show that one drawback of the FF and the BF algorithm when applied to the o-GBPP is that, at the end of the process, there might be some bins whose cost is greater than the total profit of the items loaded into them. For this reason, a natural evolution of these algorithms is considered: the First Fit with Rejection (FFR) and the Best Fit with Rejection (BFR). Both are able to reject inconvenient bins at the end of the process. We also prove that no worst-case ratio of these algorithms can be computed.

Finally, we give sufficient conditions under which algorithms do not admit any performance ratio, and conclude that the worst-case results obtained for the o-VCSBPP and the o-GBPP also hold for the offline variant of these two problems.

This paper is organized as follows. In Section 2, we introduce the o-VCSBPP and the o-GBPP and, in Section 3, we present the worst-case analysis of FF and BF for the o-VCSBPP and the o-GBPP.

2 The new online bin packing problems

2.1 The o-VCSBPP

In the VCSBPP [5], a set of bins $\mathcal{J}$, with $|\mathcal{J}| = m$, and a set of items $\mathcal{I}$, with $|\mathcal{I}| = n$, are given. The bins are classified into types belonging to the set $\mathcal{T}$. Each item $i \in \mathcal{I}$ is characterized by volume w_i and each bin of type $t \in \mathcal{T}$ is characterized by cost $\mathcal{C}_t$ and capacity $\mathcal{W}_t$. Without loss of generality, when $|\mathcal{T}| > 1$ we assume that

$$\frac{\mathcal{C}_t}{\mathcal{W}_t} \leq \frac{\mathcal{C}_{t+1}}{\mathcal{W}_{t+1}}, \qquad \forall t \in \mathcal{T} \backslash \{|\mathcal{T}|\}. \tag{1}$$

The goal is to accommodate all items into proper bins in order to minimize the overall cost, given by the sum of the costs of the selected bins.

In the o-VCSBPP, items arrive online to a decision maker. When the decision maker receives an item, the item information is revealed. This information consists of the item volume and whether the incoming item is the last one.

2.2 The o-GBPP

In the GBPP [2], a set of bins $\mathcal{J}$, with $|\mathcal{J}| = m$, and a set of items $\mathcal{I}$, with $|\mathcal{I}| = n$, are given. The bins are classified into types belonging to the set $\mathcal{T}$. The set of items $\mathcal{I}$ is composed by two subsets, $\mathcal{I}^{\mathrm{C}}$ and $\mathcal{I}^{\mathrm{NC}}$. $\mathcal{I}^{\mathrm{C}}$ is the set of compulsory items, i.e., those items that are mandatory to load, whilst $\mathcal{I}^{\mathrm{NC}}$ is the set of non-compulsory items, i.e., those items that might not be loaded. Each item $i \in \mathcal{I}$ is characterized by volume w_i and profit p_i, and each bin of type $t \in \mathcal{T}$ is characterized by cost $\mathcal{C}_t$ and capacity $\mathcal{W}_t$. The goal is to minimize the net overall cost, i.e., the difference between the total cost of the selected bins and the total profit of the selected non-compulsory items. Note that, the total profit of the compulsory items is not taken into account because, since compulsory items must be loaded, their total profit would act as a constant within the objective function.

In the o-GBPP, items also arrive online to a decision maker. When the decision maker receives an item, the item information is revealed. This information consists of the item volume, the item profit, whether the item is compulsory or non-compulsory, and whether the incoming item is the last one.

If the given m bins are not enough to load all items for the o-VCSBPP and all compulsory items for the o-GBPP, then these two problems are clearly infeasible. Infeasibility can be handled as in [2] by introducing a dummy bin type. In particular, we add a special bin of type v with volume $\mathcal{W}_v = \sum_{i \in \mathcal{I}} w_i$ for the o-VCSBPP and $\mathcal{W}_v = \sum_{i \in \mathcal{I}^{\mathrm{C}}} w_i$ for the o-GBPP, and set its cost C_v to a value much higher than the costs of the other bins in order to discourage its use, e.g., $\mathcal{C}_v \gg \sum_{t \in \mathcal{T}} \mathcal{C}_t$.

3 Worst-case analysis of FF and BF for the o-VCSBPP and the o-GBPP

3.1 The asymptotic and the absolute worst-case ratios

Several denominations and definitions can be found in the literature for the asymptotic and the absolute worst-case ratio. Sometimes they are named *performance ratios* or *competitive ratios*. They measure the gap between the solution value found by an algorithm and the optimum in the worst-case.

Formally, given a *minimization* problem Π, an instance $I \in \Pi$ of the problem, and an algorithm $\mathcal{A}$, the value of the solution yielded by the algorithm is $\mathcal{A}(I)$ and the optimum is $\mathrm{OPT}(I)$. The *asymptotic worst-case ratio* is the smallest positive $\mathcal{R}$ such that the following relation holds *for any instance* of the problem

$$\mathcal{A}(I) \leq \mathcal{R} \cdot \mathrm{OPT}(I) + \mathcal{O}(1), \qquad \forall\, I \in \Pi \tag{2}$$

The *absolute worst-case ratio* is the smallest positive ρ such that the following relation holds *for any instance* of the problem

$$\mathcal{A}(I) \leq \rho \cdot \mathrm{OPT}(I), \qquad \forall\, I \in \Pi \tag{3}$$

3.2 The FF and the BF algorithms

For the o-VCSBPP and the o-GBPP, FF works as follows. Each time an item arrives online, the decision maker tries to load it into the *first* open bin which can contain it. If none among the open bins has enough residual space to accommodate the new item, then a new bin is opened. The type of this new bin is the one with the smallest cost over volume ratio among the available bin types (for the sake of simplicity, we assume this type is unique). According to (1), this also means opening all type 1 bins first. If there are not enough bins to accommodate all items, then type 2 bins will be opened, and so on.

Similarly, BF works as FF for both problems with the only difference that, each time an item arrives online, the decision maker tries to load it into the *best* open bin, i.e., the one with the smallest residual space after depositing the item. If no open bins have enough residual space to accommodate the new item, then a new bin is opened as done in FF.

Given any instance I of problem Π, with $\Pi \in \{o-VCSBPP,\ o-GBPP\}$, and any algorithm $\mathcal{A}$ applied to instance $I \in \Pi$, we name

- $\mathcal{T}^{\mathcal{A}} \subseteq \mathcal{T}$ the set of bin types selected by algorithm $\mathcal{A}$
- $\mathcal{J}^{\mathcal{A}} \subseteq \mathcal{J}$ the set of bins selected by algorithm $\mathcal{A}$, with $p = |\mathcal{J}^{\mathcal{A}}|$
- $\mathcal{J}_t^{\mathcal{A}} \subseteq \mathcal{J}^{\mathcal{A}}$ the set of bins of type $t \in \mathcal{T}^{\mathcal{A}}$ selected by algorithm $\mathcal{A}$, with $p_t = |\mathcal{J}_t^{\mathcal{A}}|$. These sets are clearly a partition of $\mathcal{J}^{\mathcal{A}}$, i.e., $\cup_{t \in \mathcal{T}^{\mathcal{A}}} \mathcal{J}_t^{\mathcal{A}} = \mathcal{J}^{\mathcal{A}}$, $\cap_{t \in \mathcal{T}^{\mathcal{A}}} \mathcal{J}_t^{\mathcal{A}} = \emptyset$, and $\sum_{t \in \mathcal{T}^{\mathcal{A}}} p_t = p$
- $\mathcal{J}^* \subseteq \mathcal{J}$ the set of bins in an optimal solution of instance $I \in \Pi$, with $q = |\mathcal{J}^*|$
- $\sigma : \mathcal{J} \rightarrow \mathcal{T}$ an indicator function such that, given bin $j \in \mathcal{J}$, $\sigma(j)$ is its type $t \in \mathcal{T}$
- $\beta(j)$ the *level* of bin $j \in \mathcal{J}^{\mathcal{A}} \cup \mathcal{J}^*$, i.e., the total volume of the items loaded into bin j
- *open* bin any used bin.

3.3 Worst-case analysis of FF and BF for the o-VCSBPP

Theorem 1 *The asymptotic worst-case ratio of the* FF *and the* BF *algorithms for the o-VCSBPP has a bound equal to 2 and this bound is tight.*

Proof. Let $\mathcal{A} \in \{\text{FF, BF}\}$ be either the FF or the BF algorithms. In general, there might not be enough type 1 bins to accommodate all items and more than one bin type is necessary. For each set $\mathcal{J}_t^{\mathcal{A}}$, there is at most one bin $i \in \mathcal{J}_t^{\mathcal{A}}$ such that $\beta(i) \leq \frac{\mathcal{W}_t}{2}$. If, by contradiction, another bin $j \in \mathcal{J}_t^{\mathcal{A}}$ existed such that $\beta(j) \leq \frac{\mathcal{W}_t}{2}$, then the items in bins i and j could be merged together into a unique bin.

Therefore, two cases hold

1. all bins in $\mathcal{J}_t^{\mathcal{A}}$ have a level greater than half of their capacity
2. all bins but one in $\mathcal{J}_t^{\mathcal{A}}$ have a level greater than half of their capacity.

Case 1

We have that

$$\beta(j) > \frac{\mathcal{W}_t}{2} \qquad \forall j \in \mathcal{J}_t^{\mathcal{A}},\ \forall t \in \mathcal{T}^{\mathcal{A}} \tag{4}$$

By (4) we have

$$\sum_{j \in \mathcal{J}_t^{\mathcal{A}}} \beta(j) = \sum_{j=1}^{p_t} \beta(j) > \sum_{j=1}^{p_t} \frac{\mathcal{W}_t}{2} = \frac{\mathcal{W}_t}{2} p_t \tag{5}$$

Case 2

There exists a bin $i \in \mathcal{J}_t^{\mathcal{A}}$ with $\beta(i) \le \frac{\mathcal{W}_t}{2}$ and $\beta(j) > \frac{\mathcal{W}_t}{2}, \forall j \in \mathcal{J}_t^{\mathcal{A}} \backslash \{i\}$, $\forall t \in \mathcal{T}^{\mathcal{A}}$. Moreover, for each bin $j \in \mathcal{J}_t^{\mathcal{A}} \backslash \{i\}$ it must be

$$\beta(j) + \beta(i) > \mathcal{W}_t \qquad \forall j \in \mathcal{J}_t^{\mathcal{A}} \backslash \{i\},\ \forall t \in \mathcal{T}^{\mathcal{A}} \tag{6}$$

otherwise the items of bin i could be merged with the items of another open bin in $\mathcal{J}_t^{\mathcal{A}}$.

We have

$$\begin{aligned}
\sum_{j \in \mathcal{J}_t^{\mathcal{A}}} \beta(j) &= \sum_{j \in \mathcal{J}_t^{\mathcal{A}} \backslash \{i\}} \beta(j) + \beta(i) > \\
&> \sum_{j \in \mathcal{J}_t^{\mathcal{A}} \backslash \{i\}} (\mathcal{W}_t - \beta(i)) + \beta(i) = \\
&= (p_t - 1)(\mathcal{W}_t - \beta(i)) + \beta(i) = \\
&= (p_t - 2)(\mathcal{W}_t - \beta(i)) + (\mathcal{W}_t - \beta(i)) + \beta(i) = \\
&= \mathcal{W}_t + (p_t - 2)(\mathcal{W}_t - \beta(i)) \ge \\
&\ge \mathcal{W}_t + (p_t - 2)\left(\mathcal{W}_t - \frac{\mathcal{W}_t}{2}\right) = \frac{\mathcal{W}_t}{2} p_t
\end{aligned} \tag{7}$$

which is like (5) in Case 1.

In both cases, we have

$$\sum_{j \in \mathcal{J}^{\mathcal{A}}} \beta(j) = \sum_{t \in \mathcal{T}^{\mathcal{A}}} \sum_{j \in \mathcal{J}_t^{\mathcal{A}}} \beta(j) > \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} \tag{8}$$

Therefore, for a general instance I

$$\mathcal{A}(I) = \sum_{t \in \mathcal{T}^{\mathcal{A}}} p_t \mathcal{C}_t = 2 \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} \frac{\mathcal{C}_t}{\mathcal{W}_t} \tag{9}$$

Note that, by (1), $\frac{\mathcal{C}_t}{\mathcal{W}_t} \leq \max_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{C}_t}{\mathcal{W}_t} = \frac{\mathcal{C}_{|\mathcal{T}^{\mathcal{A}}|}}{\mathcal{W}_{|\mathcal{T}^{\mathcal{A}}|}}$. Consequently, (9) becomes

$$\mathcal{A}(I) = 2 \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} \frac{\mathcal{C}_t}{\mathcal{W}_t} < 2 \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} \frac{\mathcal{C}_{|\mathcal{T}^{\mathcal{A}}|}}{\mathcal{W}_{|\mathcal{T}^{\mathcal{A}}|}} \tag{10}$$

Moreover, there must be a proper $\Delta \geq 0$ such that

$$\frac{\mathcal{C}_{|\mathcal{T}^{\mathcal{A}}|}}{\mathcal{W}_{|\mathcal{T}^{\mathcal{A}}|}} = \frac{\mathcal{C}_1}{\mathcal{W}_1} + \Delta \tag{11}$$

We have

$$\begin{aligned}
\mathcal{A}(I) &< 2 \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} \frac{\mathcal{C}_{|\mathcal{T}^{\mathcal{A}}|}}{\mathcal{W}_{|\mathcal{T}^{\mathcal{A}}|}} = \\
&= 2 \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} \left(\frac{\mathcal{C}_1}{\mathcal{W}_1} + \Delta \right) = \\
&= 2 \left(\frac{\mathcal{C}_1}{\mathcal{W}_1} + \Delta \right) \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2}
\end{aligned} \tag{12}$$

Applying (8) to (12), we get

$$\mathcal{A}(I) < 2 \left(\frac{\mathcal{C}_1}{\mathcal{W}_1} + \Delta \right) \sum_{t \in \mathcal{T}^{\mathcal{A}}} \frac{\mathcal{W}_t p_t}{2} < 2 \left(\frac{\mathcal{C}_1}{\mathcal{W}_1} + \Delta \right) \sum_{j \in \mathcal{J}^{\mathcal{A}}} \beta(j) \tag{13}$$

Since all items are loaded, the sum of the levels of the bins in the solution yielded by algorithm $\mathcal{A}$ is equal to the sum of the levels of the bins in an optimal solution and so to the total item volume, i.e. $\sum_{j \in \mathcal{J}^{\mathcal{A}}} \beta(j) = \sum_{j \in \mathcal{J}^*} \beta(j) = \sum_{i \in \mathcal{I}} w_i$. Therefore, (13) becomes

$$\begin{aligned}
\mathcal{A}(I) &< 2\left(\frac{\mathcal{C}_1}{\mathcal{W}_1}+\Delta\right)\sum_{j\in\mathcal{J}^{\mathcal{A}}}\beta(j) = \\
&= 2\left(\frac{\mathcal{C}_1}{\mathcal{W}_1}+\Delta\right)\sum_{j\in\mathcal{J}^{*}}\beta(j) = \\
&= 2\frac{\mathcal{C}_1}{\mathcal{W}_1}\sum_{j\in\mathcal{J}^{*}}\beta(j) + 2\Delta\sum_{j\in\mathcal{J}^{*}}\beta(j) = \\
&= 2\sum_{j\in\mathcal{J}^{*}}\frac{\mathcal{C}_1}{\mathcal{W}_1}\beta(j) + 2\Delta\sum_{i\in\mathcal{I}} w_i \qquad (14)
\end{aligned}$$

Due to (1), we have $\frac{\mathcal{C}_1}{\mathcal{W}_1} \leq \frac{\mathcal{C}_{\sigma(j)}}{\mathcal{W}_{\sigma(j)}}, \forall j \in \mathcal{J}^*$. Then

$$\mathcal{A}(I) < 2\sum_{j\in\mathcal{J}^{*}}\frac{\mathcal{C}_1}{\mathcal{W}_1}\beta(j) + 2\Delta\sum_{i\in\mathcal{I}} w_i \leq 2\sum_{j\in\mathcal{J}^{*}}\frac{\mathcal{C}_{\sigma(j)}}{\mathcal{W}_{\sigma(j)}}\beta(j) + 2\Delta\sum_{i\in\mathcal{I}} w_i \qquad (15)$$

As $\beta(j) \leq \mathcal{W}_{\sigma(j)}, \forall j \in \mathcal{J}^*$, we get

$$\mathcal{A}(I) < 2\sum_{j\in\mathcal{J}^{*}}\frac{\mathcal{C}_{\sigma(j)}}{\mathcal{W}_{\sigma(j)}}\beta(j)+2\Delta\sum_{i\in\mathcal{I}} w_i \leq 2\sum_{j\in\mathcal{J}^{*}}\frac{\mathcal{C}_{\sigma(j)}}{\mathcal{W}_{\sigma(j)}}\mathcal{W}_{\sigma(j)}+2\Delta\sum_{i\in\mathcal{I}} w_i = 2\,\mathrm{OPT}(I)+2\Delta\sum_{i\in\mathcal{I}} w_i \qquad (16)$$

which means that algorithm $\mathcal{A}$ has an asymptotic worst-case ratio bound equal to 2.

To prove that this bound is tight, let us consider instance I composed by n items with volume $w = \frac{1}{2} + \epsilon$, with $\epsilon > 0$, $k \leq n$ type 1 bins with $\mathcal{C}_1 = 1$, $\mathcal{W}_1 = 1 + \epsilon$, and n type 2 bins with $\mathcal{C}_2 = \frac{1}{2} + \epsilon$, $\mathcal{W}_2 = \frac{1}{2} + \epsilon$.

Note that

$$\frac{\mathcal{C}_1}{\mathcal{W}_1} = \frac{1}{1+\epsilon} < 1 = \frac{\mathcal{C}_2}{\mathcal{W}_2} \qquad (17)$$

Therefore, $\mathcal{A}$ will select k bins of type 1 first and then $n-k$ type 2 bins. Note that each type 1 bin is not big enough to accommodate two items because $2w = 1+2\epsilon > 1+\epsilon = \mathcal{W}_1$. Thus we have

$$\mathcal{A}(I) = k\,\mathcal{C}_1 + (n-k)\mathcal{C}_2 = k + (n-k)\left(\frac{1}{2}+\epsilon\right) \qquad (18)$$

On the contrary, the optimal solution consists of accommodating each of the n items into a type 2 bin because, for small values of ϵ, $\mathcal{C}_2 < \mathcal{C}_1$. Then

$$\mathrm{OPT}(I) = n\,\mathcal{C}_2 = n\left(\frac{1}{2} + \epsilon\right) \tag{19}$$

For the approximation ratio, we get

$$\frac{\mathcal{A}(I)}{\mathrm{OPT}(I)} = \frac{k + (n-k)\left(\frac{1}{2} + \epsilon\right)}{n\left(\frac{1}{2} + \epsilon\right)} = \frac{2k + (n-k)(1+2\epsilon)}{n(1+2\epsilon)} \tag{20}$$

Computing the limit for $\epsilon \to 0$, we obtain

$$\lim_{\epsilon \to 0} \frac{\mathcal{A}(I)}{\mathrm{OPT}(I)} = \lim_{\epsilon \to 0} \frac{2k + (n-k)(1+2\epsilon)}{n(1+2\epsilon)} = \frac{n+k}{n} \tag{21}$$

When $k < n$, there are not enough type 1 bins to accommodate all items and $\mathcal{A}$ yields $n-k$ bins in common with those in the optimal solution. Therefore, the smaller k is, the more the approximation ratio approaches a value of 1. The worst-case ratio is met when there are enough type 1 bins to accommodate all items, i.e., $k = n$. In this case, the approximation ratio is equal to 2, which proves the bound tightness.

□

Corollary 1 *The absolute worst-case ratio of the* FF *and the* BF *algorithms for the o-VCSBPP, when there are enough type 1 bins to accommodate all items, has a bound equal to 2 and this bound is tight.*

Proof. As in Theorem 1, let $\mathcal{A} \in \{\mathrm{FF}, \mathrm{BF}\}$ be either the FF or the BF algorithm. When there are enough type 1 bins to accommodate all items, $\mathcal{A}$ will just select these bins. Therefore $\left|\mathcal{T}^{\mathcal{A}}\right| = 1$, $\dfrac{\mathcal{C}_1}{\mathcal{W}_1} = \dfrac{\mathcal{C}_{|\mathcal{T}^{\mathcal{A}}|}}{\mathcal{W}_{|\mathcal{T}^{\mathcal{A}}|}}$, and, because $\Delta = 0$, (16) becomes

$$\mathcal{A}(I) < 2\,\mathrm{OPT}(I). \tag{22}$$

The proof of the bound tightness is the same of that of Theorem 1.

□

3.4 Worst-case analysis of FF and BF for the o-GBPP

Whilst for the o-VCSBPP we can guarantee performance ratios of the FF and the BF algorithms, this is not true for the o-GBPP.

Theorem 2 *It is impossible to compute the asymptotic and absolute worst-case ratios of the* FF *and the* BF *algorithms for the o-GBPP.*

Proof. Let us consider instance $I(\nu_A, \nu_B, \nu_C)$, composed of one bin type ($|\mathcal{T}| = 1$ and $t = 1$) with $\mathcal{W}_1 = \mathcal{W}$, $\mathcal{C}_1 = \mathcal{C}$, and the set of items $\mathcal{I}$ is split into three subsets, A, B, and C, with $|A| = \nu_A$, $|B| = \nu_B$, and $|C| = \nu_C$. An item which belongs to subset $X \in \{A, B, C\}$ is called a type X item. Let type A items be compulsory with $w_A = \mathcal{W}$, type B items be non-compulsory with $w_B = \mathcal{W}$, $p_B = \mathcal{C} + \epsilon$, and type C items be non-compulsory with $w_C = \mathcal{W}$, and $p_C = \mathcal{C} - \epsilon$, with $\epsilon > 0$ (note that the profit p_A is not defined because type A items are compulsory). As before, let $\mathcal{A} \in \{\text{FF}, \text{BF}\}$ be either the FF or the BF algorithm.

Since all items have a volume equal to $\mathcal{W}$, each bin can accommodate only one item. Consequently, we must open ν_A bins to accommodate all ν_A type A compulsory items and ν_B bins to accommodate all ν_B type B non-compulsory items, which are taken because $p_B > \mathcal{C}$. Since no type C item is profitable and cannot be loaded with any other item, none of them will be loaded. We have

$$\text{OPT}(I(\nu_A, \nu_B, \nu_C)) = \nu_A\mathcal{C} + \nu_B(\mathcal{C} - p_B) = \nu_A\mathcal{C} - \nu_B\epsilon \tag{23}$$

Let us call i_X an item i which belongs to the subset X and consider the following online item sequence

$$\overbrace{i_A \quad \ldots \quad i_A}^{\nu_A \text{ times}} \; \overbrace{i_B \quad \ldots \quad i_B}^{\nu_B \text{ times}} \; \overbrace{i_C \quad \ldots \quad i_C}^{\nu_C \text{ times}}$$

Applying $\mathcal{A}$ to the above sequence, we have

$$\mathcal{A}(I(\nu_A, \nu_B, \nu_C)) = \nu_A\mathcal{C} + \nu_B(\mathcal{C} - p_B) + \nu_C(\mathcal{C} - p_C) = \nu_A\mathcal{C} - \nu_B\epsilon + \nu_C\epsilon \tag{24}$$

According to (2), in order to compute the asymptotic worst-case ratio, we have to find a proper constant $\mathcal{O}(1)$ and the smallest positive $\mathcal{R}$ such that

$$\nu_A\mathcal{C} - \nu_B\epsilon + \nu_C\epsilon \leq \mathcal{R}(\nu_A\mathcal{C} - \nu_B\epsilon) + \mathcal{O}(1) \tag{25}$$

If we consider instance $I(0,\, 0,\, \nu_C)$, (25) becomes

$$\nu_C \epsilon \leq \mathcal{O}(1) \tag{26}$$

which is impossible because a constant cannot be greater than a linear ($\mathcal{O}(\nu)$) term. Therefore, the asymptotic worst-case ratio cannot be computed.

According to (3), in order to compute the absolute worst-case ratio, we have to find the smallest positive ρ such that

$$\nu_A \mathcal{C} - \nu_B \epsilon + \nu_C \epsilon \leq \rho(\nu_A \mathcal{C} - \nu_B \epsilon) \tag{27}$$

If we consider instance $I(1,\, 0,\, \nu_C)$, (27) becomes

$$\mathcal{C} + \nu_C \epsilon \leq \rho \mathcal{C} \tag{28}$$

which implies $\rho \to +\infty$, since ν_C can be arbitrarily large. On the contrary, if we consider instance $I(0,\, 1,\, \nu_C)$, (27) becomes

$$-\epsilon + \nu_C \epsilon \leq -\rho \epsilon \tag{29}$$

which implies $\rho \to -\infty$, since ν_C can be arbitrarily large, and contradicts the previous assumption on ρ. Therefore, the absolute worst-case ratio cannot be computed.

$\square$

One drawback of the FF and the BF algorithm when applied to the o-GBPP is that they accept every incoming item, independently of its profit. Therefore, at the end of the process, there might be some bins containing non-compulsory items only, whose cost is greater than the total profit of the items loaded into them. Using these bins is clearly wrong. A natural improvement of FF and BF consists of rejecting, at the end of the process, bins of this kind. These algorithms are named FFR and BFR, where R stands for Rejection.

We also prove that for the FFR and the BFR algorithms the asymptotic and absolute worst-case ratios cannot be computed.

Theorem 3 *It is impossible to compute the asymptotic and absolute worst-case ratios of the* FFR *and* BFR *algorithms for the o-GBPP.*

Proof. Let us consider instance $I(\mu, \nu)$ composed of one bin type ($|\mathcal{T}| = 1$ and $t = 1$), with $\mathcal{W}_1 = \mathcal{W}$, $\mathcal{C}_1 = \mathcal{C}$, and where the set of items $\mathcal{I}$ is split into three subsets, A, B, and C, with $|A| = \mu$, and $|B| = |C| = 2\nu$. Let type A items be compulsory with $w_A = \mathcal{W}$, let type B items be non-compulsory with $w_B = \frac{\mathcal{W}}{2}$ and $p_B = \frac{\mathcal{C}}{2} + \epsilon$, and let type C items be non-compulsory with $w_C = \frac{\mathcal{W}}{2}$ and $p_C = \frac{\mathcal{C}}{2} - 2\epsilon$, for small positive values of ϵ. Finally, let $\mathcal{A} \in \{\text{FFR, BFR}\}$ be either the FFR or the BFR algorithm.

It can be easily verified that an optimal solution consists of μ bins each containing one type A compulsory item, and ν bins each containing two type B non-compulsory items. Type C non-compulsory items are not selected because they are not profitable. Thus

$$\text{OPT}(I(\mu, \nu)) = \mu\mathcal{C} + \nu(\mathcal{C} - 2p_B) = \mu\mathcal{C} - 2\nu\epsilon \tag{30}$$

Let us consider the following online item sequence

$$\overbrace{i_A \dots i_A}^{\mu \text{ times}} \quad \overbrace{i_B \quad i_C \quad \dots \quad i_B \quad i_C}^{2\nu \text{ times}}$$

which, for both FFR and BFR, yields μ bins each containing one type A compulsory item and ν bins each containing one type B and one type C non-compulsory item. However, the ν bins containing non-compulsory items will be discarded because $p_B + p_C = \mathcal{C} - \epsilon < \mathcal{C}$. Therefore

$$\mathcal{A}(I(\mu, \nu)) = \mu\mathcal{C} \tag{31}$$

We show that, according to definition (2), it is impossible to find $\mathcal{R}$ and $\mathcal{O}(1)$ such that

$$\mathcal{A}(I(\mu, \nu)) \leq \mathcal{R} \cdot \text{OPT}(I(\mu, \nu)) + \mathcal{O}(1) \tag{32}$$

If we substitute (30) and (31) into (32), we have

$$\mu\mathcal{C} \leq \mathcal{R}(\mu\mathcal{C} - 2\nu\epsilon) + \mathcal{O}(1) \tag{33}$$

If we consider instance $I(0, \nu)$, (33) becomes

$$0 \leq -2\mathcal{R}\nu\epsilon + \mathcal{O}(1) \tag{34}$$

Since ν can be arbitrarily large, then it implies that $\mathcal{R} \leq 0$, independently of the constant $\mathcal{O}(1)$.

If we consider instance $I(\mu,\, 0)$, (33) becomes

$$\mu\mathcal{C} \leq \mathcal{R}\mu\mathcal{C} + \mathcal{O}(1) \tag{35}$$

which implies that $\mathcal{R} \geq 1$, independently of constant $\mathcal{O}(1)$, in contrast with the previous requirement that $\mathcal{R} \leq 0$. Therefore, it is impossible to compute the asymptotic worst-case ratio of $\mathcal{A}$.

This result holds independently of constant $\mathcal{O}(1)$ and, according to (2) and (3), the absolute worst-case ratio is the particular case when $\mathcal{O}(1) = 0$. As a result, the absolute worst-case ratio of $\mathcal{A}$ cannot be computed.

□

The results of Theorem 3 can easily be generalized as follows

Corollary 2 *Given a minimization problem* Π *and an algorithm* $\mathcal{A}$*, let* $I(\mu,\, \nu) \in \Pi$ *be an instance with* $\mu,\, \nu \in \mathbb{N}$*, such that* $\mathcal{A}(I(\mu,\, \nu)) = \alpha\mu$*, and* $\mathrm{OPT}(I(\mu,\, \nu)) = \beta\mu - \gamma\nu$*, with* $\alpha,\, \beta,\, \gamma > 0$*. Then, it is impossible to compute the asymptotic and absolute worst-case ratios of algorithm* $\mathcal{A}$*.*

Proof. Trivial, similar to that of Theorem 3.

□

We observe that FF and BF for the o-VCSBPP and the o-GBPP, and FFR and BFR for the o-GBPP, act in the same way when they address the offline variant of these two problems. We can conclude therefore, that the worst-case results obtained in this paper for the online Variable Cost and Size Bin Packing Problem and the online Generalized Bin Packing Problem also hold for the offline variant of these two problems.

4 Conclusions

We introduced two new online bin packing problems, the online Variable Cost and Size Bin Packing Problem and the online Generalized Bin Packing Problem, of great interest to the fields of computer science and engineering as well as transportation, logistics, and telecommunications.

The contribution of the paper to the literature is of two orders. First, it yields an asymptotic and absolute worst-case ratio bound of First Fit and Best Fit equal to 2 for the o-VCSBPP and this bound is tight. Second, it proves that no worst-case ratio bound of these algorithms can be computed for the o-GBPP, even considering their natural evolution, i.e., First Fit with Rejection and Best Fit with Rejection. Moreover, the paper gives an interesting generalization of these results which consists of sufficient conditions under which algorithms do not admit any performance ratio. Finally, we showed that the worst-case results obtained for the online Variable Cost and Size Bin Packing Problem and the online Generalized Bin Packing Problem also hold for the offline variant of these two problems.

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