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Improved Solutions for Inventory-Routing Problems through Valid Inequalities and Input Ordering

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Abstract. Inventory-routing problems (IRP) combine inventory control and vehicle routing, effectively optimizing inventory and replenishment decisions over several periods at an aggregated level. In this paper we provide an exact formulation which includes several well-known valid inequalities for some classes of IRPs. We then propose three new valid inequalities based on the relation between demand and available capacities. Then, following an idea proposed for the binary clustering and for the job scheduling problems, we also show how the order of the input data can have a major effect on the linear relaxation of the proposed model for the IRP. Extensive computational experiments confirm the success of our algorithm. We have used two available datasets with new solutions identified as recently as 2013. On one set of benchmark instances with 249 open instances, we have improved 98 lower bounds, we have computed 96 new best known solutions, and we have proved optimality for 11 instances. On the other dataset composed of larger instances, of which were 63 open, we have improved 32 lower bounds, we have obtained 20 new best known solutions, and we proved optimality for three instances.

Keywords. Inventory-routing, valid inequalities, symmetry breaking, input order, branch-and-cut.

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1 Introduction

Inventory control is one of the pillars of production economics. Its underlying theoretical basis is rooted in the ground breaking paper of Harris [15]. This seminal article formalizes the well-known Economic Order Quantity (EOQ) model which computes the quantity that minimizes total inventory holding and ordering costs in a constant demand environment. Several extensions and variations of this model have emerged over the years. For the case of non-constant demands, the classical reference is the paper of Wagner and Whitin [25] which generalizes the EOQ model to the dynamic lot sizing problem. This problem was solved exactly by dynamic programming by Wagner and Whitin [25], and heuristically by the well-known Silver-Meal algorithm [22]. The multi-product case was studied for more than 50 years, as was the problem of determining the lot size for the manufacturing of several products on the same machine. This problem, known as the Economic Lot Scheduling Problem, was introduced by Rogers [21] and was later extended by Elmaghraby [12].

Recently, the research community has focused its attention on joint decision making problems arising in several areas, thus removing some of the boundaries between some logistics activities. These include inventory problems arising in green and reverse logistics [14], robustness and resilience of inventory planning and control [18], safety stocks [24], as well as demand dynamism and stochasticity [23]. These problems extend to different supply chain activities such as production set up costs, inventory and transportations as in the production-routing problem [1, 6], and joint transportation and distribution issues as in inventory-routing [2, 11]. Inventory-Routing Problems (IRP), which are the focus of this paper, combine inventory management and vehicle routing decisions by jointly optimizing inventory levels and replenishment for several products over several periods with several vehicles. The optimization process takes place at the supplier's level. This is the case of vendor-managed inventory (VMI) applications in which the supplier is responsible for deciding when and how much to deliver to each of its customers.

The integration of inventory management and vehicle routing decisions dates back to the 1980s with the seminal paper of Bell et al. [5]. Since then, several technical contributions and applications have emerged. The survey paper of Andersson et al. [2] concentrates on the applications of the IRP, whereas that of Coelho et al. [11] focuses on the methodological aspects of the problem. In what follows, we review some of the most relevant recent algorithmic literature on the IRP.

The first exact algorithm for the IRP is due to Archetti et al. [3] who solved the single-vehicle case. The proposed model and algorithm yielded optimal solutions for instances with up to 30 customers and six periods, and with up to 50 customers and three periods. The authors also introduced the first testbed which contains benchmark instances used by most authors. Archetti et al. [4] have later developed a powerful matheuristic algorithm based on tabu search and on the solution of mixed-integer problems which was able of computing quasi-optimal solutions on the testbed instances within very short running time. These authors also introduced a second and larger set of instances, still considering a single vehicle. At the same time, Coelho et al. [9] proposed an ALNS heuristic in which subproblems were solved as minimum-cost network flow problems. This algorithm also provides quasi-optimal solutions. The algorithm of Coelho et al. [9] was later extended to [10] solve multi-vehicle instances. Finally, two similar exact algorithms, by Adulyasak et al. [1] and by Coelho and Laporte [7], were recently developed for multi-vehicle instances. The first solves the IRP by branch-and-cut as a special case of the production-routing problem, and was tested on the first set of instances. The second applies a branch-and-cut scheme enhanced by the exact solution of smaller mixed-integer linear programs, which constitutes a powerful upper bounding procedure and provides all best known solutions on the two sets of benchmark instances.

With respect to the existing literature, this paper makes three main contributions. First, we introduce new valid inequalities in the context of the multi-vehicle IRP, which are based on the demand and the capacities of the customers and of the vehicles. Second, we analyze the impact of changing the order of the input on the value of the linear relaxation, and thus, on the best lower bound value obtained after a given computing time. Third, we show how these first two contributions yield improved lower bounds and provide new best known solutions for large open benchmark instances of the IRP.

The remainder of the paper is organized as follows. In Section 2 we provide a formal statement of the problem as well as an exact mixed-integer linear formulation for it. We then detail in Section 3 the old and new valid inequalities which are incorporated in the model, as well as the notion of input ordering for the IRP is introduced. The exact branch-and-cut algorithm is briefly described in Section 4, followed by computational experiments Section 5. Conclusions are presented in Section 6.

2 Problem statement and mathematical formulation

We consider a multi-vehicle IRP in which routing costs are symmetric. The problem is defined on an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{0, ..., n\}$ is the vertex set and $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}, i < j\}$ is the edge set. Vertex 0 represents the supplier and the remaining vertices of $\mathcal{V}' = \mathcal{V} \setminus \{0\}$ represent n customers. A routing cost c_{ij} is associated with edge $(i, j) \in \mathcal{E}$. Both the supplier and customers incur unit inventory holding costs h_i per period ($i \in \mathcal{V}$), and each customer has a maximum inventory holding capacity C_i. The length of the planning horizon is p and, at each time period $t \in \mathcal{T} = \{1, ..., p\}$, the supplier holds a quantity r^t of a single product. We assume the supplier has sufficient inventory to meet the full customer demand during the planning horizon, and all demand also has to be satisfied, i.e., backlogging is not allowed. At the beginning of the planning horizon, the decision maker knows the current inventory level of the supplier and of the customers ($I_i^0, i \in \mathcal{V}$), and receives information on the demand d_i^t of each customer i for each time period t. Regarding timing issues, the quantity r^t held by the supplier in period t can be used for deliveries to customers in the same period, and the delivery amount received by customer i in period t can be used to meet the demand in that period. A set $\mathcal{K} = \{1, ..., K\}$ of vehicles are available. We denote by Q_k the capacity of vehicle k. Each vehicle can perform one route per time period, from the supplier to a subset of customers.

The aim is to determine vehicle routes and to compute delivery quantities for each period and each customer, such that all demand is satisfied, all capacities are respected, and the total cost is minimized.

We now formally describe the mathematical formulation of the IRP for a single product. The case of several products is conceptually similar, but requires an additional index [8]. Our MILP model works with routing variables x_{ij}^{kt} equal to the number of times edge (i, j) is used on the route of vehicle k in period t. We also use binary variables y_i^{kt} equal to one if and only if vertex i is visited by vehicle k in period t. Continuous integer variables I_i^t denote the inventory level at vertex $i \in \mathcal{V}$ at the end of period $t \in \mathcal{T}$, and we denote by q_i^{kt} the quantity of product delivered by vehicle k to customer i in period t. The problem can then be formulated as follows:

$$\text{minimize} \sum_{i\in\mathcal{V}} \sum_{t\in\mathcal{T}} h_i I_i^t + \sum_{(i,j)\in\mathcal{E}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} c_{ij} x_{ij}^{kt}, \tag{1}$$

subject to

$$I_0^t = I_0^{t-1} + r^t - \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{V}'} q_i^{kt} \quad t \in \mathcal{T} \tag{2}$$

$$I_i^t = I_i^{t-1} + \sum_{k\in\mathcal{K}} q_i^{kt} - d_i^t \quad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{3}$$

$$I_i^t \leq C_i \quad i \in \mathcal{V} \quad t \in \mathcal{T} \tag{4}$$

$$\sum_{k\in\mathcal{K}} q_i^{kt} \leq C_i - I_i^{t-1} \quad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{5}$$

$$q_i^{kt} \leq C_i y_i^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{6}$$

$$\sum_{i \in \mathcal{V}'} q_i^{kt} \leq Q_k y_0^{kt} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{7}$$

$$\sum_{j \in \mathcal{V}, i<j} x_{ij}^{kt} + \sum_{j \in \mathcal{V}, j<i} x_{ji}^{kt} = 2y_i^{kt} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{8}$$

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}, i<j} x_{ij}^{kt} \leq \sum_{i \in \mathcal{S}} y_i^{kt} - y_m^{kt} \quad \mathcal{S} \subseteq \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \quad m \in \mathcal{S} \tag{9}$$

$$\sum_{k \in \mathcal{K}} y_i^{kt} \leq 1 \quad i \in \mathcal{V}' \quad t \in \mathcal{T} \tag{10}$$

$$I_i^t, q_j^{kt} \geq 0 \quad i \in \mathcal{V} \quad j \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{11}$$

$$x_{i0}^{kt} \in \{0, 1, 2\} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{12}$$

$$x_{ij}^{kt} \in \{0, 1\} \quad i, j \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{13}$$

$$y_i^{kt} \in \{0, 1\} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T}. \tag{14}$$

Constraints (2) and (3) define the inventory conservation at the supplier and at the customers. Constraints (4) impose maximal inventory level at the customers. Constraints (5) and (6) link the quantities delivered to the routing variables. In particular, they only allow a vehicle to deliver products to a customer if the customer is visited by this vehicle. Constraints (7) ensure the vehicle capacities are respected while constraints (8) and (9) are degree constraints and subtour elimination constraints, respectively. Constraints (10) guarantee that split deliveries are not allowed. Constraints (11)−(14) enforce integrality and non-negativity conditions on the variables.

This model is very difficult to solve since it encompasses the vehicle routing problem, an NP-hard problem [19].

3 Valid inequalities and input ordering

In this section we present valid inequalities that strengthen the formulation as well as an innovative way of considering the input data for the problem. In Section 3.1 we describe known valid inequalities for the IRP, in Section 3.2 we introduce new classes of valid inequalities for the single-vehicle IRP, which are then extended to the multi-vehicle case in Section 3.3. Input ordering considerations are presented in Section 3.4.

3.1 Known valid inequalities

Archetti et al. [3] have introduced several classes of inequalities for the IRP, some of which have been extended to the multi-vehicle IRP by Coelho et al. [10]. They are as follows:

$$x_{i0}^{kt} \leq 2y_i^{kt} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{15}$$

$$x_{ij}^{kt} \leq y_i^{kt} \quad i,j \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{16}$$

$$y_i^{kt} \leq y_0^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{17}$$

$$\sum_{k \in \mathcal{K}} \sum_{l=1}^{t} y_i^{kl} \geq \left\lceil \left(\sum_{l=1}^{t-1} d_i^l - I_i^0 \right) / C_i \right\rceil \quad i \in \mathcal{V} \quad t \in \mathcal{T}. \tag{18}$$

Constraints (15) and (16) are referred to as logical inequalities. They enforce the condition that if the supplier is the successor of a customer in the route of vehicle k in period t, i.e., $x_{i0}^{kt} = 1$ or 2, then i must be visited by the same vehicle, i.e., $y_i^{kt} = 1$. A similar reasoning is applied to customer j in inequalities (16). Constraints (17) include the supplier in the route of vehicle k if any customer is visited by that vehicle in that period. Constraints (18) ensure that customer i is visited at least the number of times corresponding to the right-hand side of the inequality. This inequality is only valid if the fleet is homogeneous.

Coelho et al. [10] have also added symmetry breaking constraints with respect to the vehicles capacities, for the case when the fleet is homogeneous:

$$y_0^{kt} \leq y_0^{k-1,t} \quad k \in \mathcal{K}\backslash\{1\} \quad t \in \mathcal{T} \tag{19}$$

$$y_i^{kt} \leq \sum_{j<i} y_j^{k-1,t} \quad i \in \mathcal{V} \quad k \in \mathcal{K}\backslash\{1\} \quad t \in \mathcal{T}. \tag{20}$$

Constraints (19) ensure that vehicle k cannot leave the depot if vehicle $k-1$ is not used. This symmetry breaking rule is then extended to the customer vertices by constraints (20) which state that if customer i is assigned to vehicle k in period t, then vehicle $k-1$ must serve a customer with an index smaller than i in the same period.

We also introduce three new classes of valid inequalities. These new cuts are computed based on the instance data and take the demand and the capacities as a way to strengthen the relaxation of the y_i^{kt} variables. We first introduce them for the single-vehicle IRP in Section 3.2 and we then extend their scope to the multi-vehicle IRP in Section 3.3

3.2 New valid inequalities for the single-vehicle IRP

If the sum of the demands over $[t_1, t_2]$ is greater than or equal to the maximum possible inventory held, then there has to be at least one visit to this customer in the interval $[t_1, t_2]$:

$$\sum_{t'=t_1}^{t_2} y_i^{t'} \geq \left\lceil \frac{\sum_{t'=t_1}^{t_2} d_i^{t'}}{C_i} - 1 \right\rceil \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1. \tag{21}$$

Inequality (21) can be strengthened by considering that if the quantity needed to satisfy future demands is larger than the maximum inventory capacity, several visits are needed. Since the maximum delivery size is the minimum between the holding capacity and the vehicle capacity, one can round up the right-hand side of (22). Making the numerator tighter by considering the actual inventory hold instead of the maximum possible inventory held yields inequality (23), which cannot be rounded up or it becomes non-linear:

$$\sum_{t'=t_1}^{t_2} y_i^{t'} \geq \left\lceil \frac{\sum_{t'=t_1}^{t_2} d_i^{t'} - C_i}{\min\{Q, C_i\}} \right\rceil \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1 \tag{22}$$

$$\sum_{t'=t_1}^{t_2} y_i^{t'} \geq \frac{\sum_{t'=t_1}^{t_2} d_i^{t'} - I_i^{t_1}}{\min\{Q, C_i\}} \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1. \tag{23}$$

Even if these inequalities are redundant for our model, they are useful in helping CPLEX generate new cuts. A different version of the same inequality can be written as follows. It is related to whether the inventory held at each period is sufficient to fulfill future demands. In particular, if the inventory hold in period t_1 by customer i is sufficient to fulfill its demand for periods $[t_1, t_2]$, then no visit to customer i is required, i.e., if $I_i^{t_1} \geq \sum_{t'=t_1}^{t_2} d_i^{t'}$, then $\sum_{t'=t_1}^{t_2} y_i^{t'} \geq 0$. On the other hand, if the inventory is not sufficient to fulfill future demands, then a visit must take place. This can be enforced by the following set of valid inequalities:

$$\sum_{t'=t_1}^{t_2} y_i^{t'} \geq \frac{\sum_{t'=t_1}^{t_2} d_i^{t'} - I_i^{t_1}}{\sum_{t'=t_1}^{t_2} d_i^{t'}} \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1 \tag{24}$$

3.3 Extending the new valid inequalities for the multi-vehicle IRP

Inequalities (22), (23) and (24) can be easily adapted to the multi-vehicle case if the fleet is homogeneous:

$$\sum_{k \in \mathcal{K}} \sum_{t'=t_1}^{t_2} y_i^{kt'} \geq \left\lceil \frac{\sum\limits_{t'=t_1}^{t_2} d_i^{t'} - C_i}{\min\{Q, C_i\}} \right\rceil \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1 \tag{25}$$

$$\sum_{k \in \mathcal{K}} \sum_{t'=t_1}^{t_2} y_i^{kt'} \geq \frac{\sum\limits_{t'=t_1}^{t_2} d_i^{t'} - I_i^{t_1}}{\min\{Q, C_i\}} \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1 \tag{26}$$

$$\sum_{k \in \mathcal{K}} \sum_{t'=t_1}^{t_2} y_i^{kt'} \geq \frac{\sum\limits_{t'=t_1}^{t_2} d_i^{t'} - I_i^{t_1}}{\sum\limits_{t'=t_1}^{t_2} d_i^{t'}} \quad i \in \mathcal{V}' \quad t_1, t_2 \in \mathcal{T}, t_2 \geq t_1. \tag{27}$$

3.4 Input ordering

In this paper we apply to the IRP the concept of input ordering, an idea put forward by Jans and Desrosiers [16, 17] in the context of the binary clustering problem and of the job grouping problem. These authors have observed that the order in which the input data are loaded into the model can have a major effect on its solution, particularly on the value of the LP relaxation, on the number of nodes explored and ultimately on the solution time. To the best of our knowledge we are the first to apply input ordering to a routing problem.

The idea lies in the fact that assigning "critical" customers first can decrease the flexibility for the remaining customers, thus increasing the lower bound. If a customer with a relatively high demand is assigned first to a vehicle, then there will be little spare capacity in that vehicle, thus restricting the number of customers that can be partially assigned to it. We therefore try to order the customers in such a way that when an assignment decision is made for the first one, fewer options are available for the remaining customers. This procedure can help strengthen the effect of symmetry breaking constraints (19) and (20) in a branch-and-cut context. Since this is the first time that input ordering is tested in an

IRP framework, we propose three ways of initially sorting customers, besides the random order obtained directly from the instances when they are generated. One is based on the demand, while the other two are based on cost measures. The first order we propose is to rank the customers according to their total demand throughout the planning horizon. The idea is that customers with higher demand should be served more often, and should also fill the vehicle capacity more quickly than customers with lower demands. The second order gives higher priority to customers close to the supplier's location, while the third one is the opposite, putting first customers whose location is far from the supplier.

4 Branch-and-Cut Algorithm

For very small instances sizes, the model presented in Section 2 can be fully described, and all variables and constraints can be explicitly generated. It can then be solved by feeding it directly into a powerful integer linear programming solver to be solved by branch-and-bound. However, for instances of realistic size, the number of subtour elimination constraints (9) is too large to allow full enumeration and these must be dynamically generated throughout the search process. The exact algorithm we present is a branch-and-cut scheme in which subtour eliminations constraints are only generated and added into the program whenever they are found to be violated. It works as follows. At a generic node of the search tree, a linear program containing the model with a subset of the subtour elimination constraints is solved, a search for violated inequalities is performed, and some of these are added to the current program which is then reoptimized. This process is reiterated until a feasible or dominated solution is reached, or until there are no more cuts to be added. At this point, branching on a fractional variable occurs. We provide a sketch of the branch-and-cut-and-bound scheme in Algorithm 1.

Algorithm 1 Branch-and-cut-and-bound algorithm

At the root node of the search tree, generate and insert all valid inequalities into the program.
Subproblem solution. Solve the LP relaxation of the current node.
Termination check:
if there are no more nodes to evaluate **then**
    Stop.
else
    Select one node from the branch-and-cut tree.
end if
while the solution of the current LP relaxation contains subtours **do**
    Identify connected components as in Padberg and Rinaldi [20].
    Determine whether the component containing the supplier is weakly connected as in Gendreau et al. [13].
    Add all violated subtour elimination constraints (9).
    Subproblem solution. Solve the LP relaxation of the current node.
end while
if the solution of the current LP relaxation is integer **then**
    Go to the termination check.
else
    Branching: branch on one of the fractional variables.
    Go to the termination check.
end if

5 Computational experiments

The algorithm just described was coded in C++ using the IBM Concert Technology and solved using CPLEX 12.5 as the solver running on six threads. All computations were executed on a grid of Intel Xeon™ processors running at 2.66 GHz with up to 48 GB of RAM installed per node, with the Scientific Linux 6.1 operating system.

We have used the benchmark instance set for the single vehicle case created by Archetti et al. [3], which is made up of instances with up to three time periods and 50 customers, and six time periods and 30 customers. In order to generate instances with several vehicles, we follow the same procedure of [1, 7, 10]. The overall capacity remains unchanged, but we divide the original vehicle capacity by the number of vehicles considered, varying from one to five. A time limit of two hours was imposed on the solution of each instance of this small set, compared with the six hours allowed by the competition.

In what follows, we detail our findings for each type of input ordering on these instances. We have compared our algorithm with respect to the best known solutions for the two benchmark instances available, as reported by Coelho and Laporte [7]. We summarize the performance of our method on this dataset in Table 1. For each of the 160 instances we have executed our algorithm with $K = 1, 2, 3, 4$, and 5 vehicles, totaling 800 different instances. We have solved each one with four different orders for the input data: random, higher demand first, proximity to the supplier and remoteness to the supplier, yielding a total of 3200 runs of the algorithm. Out of the 249 open instances we now provide values for 98 improved lower bounds, 96 new best known solutions, and we have proved optimality for 11 new instances. Detailed results can be downloaded from the website `http://www.leandro-coelho.com`.

We note that the valid inequalities alone, corresponding to a random input order of customers, have a very positive impact on solving these instances. In particular, the running time taken to prove optimality for all the instances with one vehicle is reduced by half. Moreover, two input orderings appear to yield the largest impacts. Both are based on ordering the data according the a greedy criterion: highest demand or highest travel cost first.

Table 1: Summary of the computational results for the IRP on the small instance set of Archetti et al. [3]

Input order		$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$	Total
	Open instances	0	8	48	84	109	249
Random	New LB	–	1	9	11	14	35
	New UB	–	4	3	6	8	21
	New optima	–	1	0	0	0	1
Demand	New LB	–	4	11	20	19	54
	New UB	–	3	9	21	25	58
	New optima	–	0	2	2	0	4
Proximity	New LB	–	1	9	14	14	38
	New UB	–	2	1	4	8	15
	New optima	–	0	0	0	0	0
Remoteness	New LB	–	3	13	18	25	59
	New UB	–	5	13	29	27	74
	New optima	–	2	1	4	2	9
Total without repetition	New LB	–	4	23	31	40	98
	New UB	–	7	9	34	36	86
	New optima	–	3	2	4	2	11

We also report on Table 2 the average improvements on the lower and upper bounds. These statistics are computed only over those instances for which improvements were obtained. It shows the average unit and percentage improvements. Note that for the smaller instances, i.e., for those with two and three vehicles, the improvements are small because the original bounds were already very tight. On the larger instances, our algorithms are able to obtain average solution values which can be up to 50% better. In particular, for each input ordering we show the average unit improvement on the bounds, followed by the percentage improvement in parenthesis.

Table 2: Average improvements on the bounds for the open instances of the IRP on the small instance set of Archetti et al. [3]

Input order	in units (in %)	$K = 2$	$K = 3$	$K = 4$	$K = 5$	Average
Random	Avg LB increase	74.83 (0.72)	56.83 (0.77)	173.83 (1.12)	108.72 (1.27)	103.55 (0.97)
	Avg UB decrease	48.84 (0.58)	212.25 (2.49)	9120.72 (53.29)	4668.20 (28.01)	3512.50 (21.09)
Demand	Avg LB increase	100.88 (0.90)	80.97 (0.81)	226.91 (2.12)	235.11 (2.62)	160.96 (1.61)
	Avg UB decrease	14.34 (0.16)	163.31 (1.92)	5241.54 (34.41)	3443.33 (20.00)	2215.65 (14.12)
Proximity	Avg LB increase	156.68 (1.44)	134.00 (0.69)	261.14 (1.78)	127.85 (0.98)	169.91 (1.22)
	Avg UB decrease	1.90 (0.02)	159.43 (1.60)	7300.06 (44.53)	4636.07 (27.65)	3024.36 (18.45)
Remoteness	Avg LB increase	90.92 (0.81)	147.24 (1.15)	200.19 (1.91)	219.83 (2.18)	164.54 (1.51)
	Avg UB decrease	10.99 (0.11)	89.25 (0.91)	5819.62 (37.83)	2653.50 (15.31)	2143.34 (13.54)
Total	Avg LB increase	105.82 (0.96)	104.76 (0.85)	215.51 (1.73)	172.87 (1.76)	149.74 (1.32)
	Avg UB decrease	19.01 (0.21)	156.06 (1.73)	6870.48 (42.51)	3850.27 (22.74)	2723.96 (16.80)

We have also applied our algorithm to the newer and larger testbed proposed in [4], which contains 60 instances with six time periods and up to 200 customers. There are 20 instances with 50 customers, 20 instances with 100 customers, and 20 instances with 200 customers. A time limit of four hours was imposed on the solution of each instance, only one sixth of the time allowed by Coelho and Laporte [7]. We have succeeded in solving all instances optimally for the single vehicle case. For all the 20 instances containing 50 customers and with two vehicles we have obtained significant improvements. We do not report solutions for larger instances because we have observed that the branch-and-cut algorithm alone is rarely capable of finding a feasible solution within the allotted time. Detailed results on all instances are available for download from `http://www.leandro-coelho.com/`.

Finally we report on Table 4 the average unit and percentage improvements of the lower and upper bounds for the instances where improvements were observed. Note again that

Table 3: Summary of the computational results for the IRP on the large instance set of Archetti et al. [4]

Input order		$K = 1$	$K = 2$	Total
	Open instances	43	20	63
Random	New LB	14	10	24
	New UB	10	2	12
	New optima	3	0	3
Demand	New LB	12	9	21
	New UB	11	2	13
	New optima	3	0	3
Proximity	New LB	14	7	21
	New UB	9	2	11
	New optima	3	0	3
Remoteness	New LB	10	9	19
	New UB	8	4	12
	New optima	3	0	3
Total without repetition	New LB	18	14	32
	New UB	14	6	20
	New optima	3	0	3

the best improvements in the lower bounds were obtained by ordering the input from high to low with respect to a given criterion, in this case the demand and the travel cost to the supplier.

Table 4: Average improvements on the bounds for the open instances of the IRP on the large instance set of Archetti et al. [3]

Input order	in units (in %)	$K = 1$	$K = 2$	Average
Random	Avg LB increase	46.08 (0.35)	59.80 (0.18)	52.94 (0.26)
	Avg UB decrease	310.59 (1.63)	124.23 (0.75)	217.41 (1.19)
Demand	Avg LB increase	43.58 (0.27)	84.95 (0.28)	64.26 (0.27)
	Avg UB decrease	437.01 (2.04)	191.46 (1.25)	314.23 (1.64)
Proximity	Avg LB increase	46.82 (0.30)	60.65 (0.20)	53.73 (0.25)
	Avg UB decrease	322.27 (1.46)	212.50 (1.24)	267.38 (1.35)
Remoteness	Avg LB increase	46.92 (0.35)	76.02 (0.30)	61.47 (0.32)
	Avg UB decrease	380.92 (2.01)	150.30 (0.80)	265.61 (1.40)
Total	Avg LB increase	45.85 (0.31)	70.35 (0.24)	58.10 (0.27)
	Avg UB decrease	362.69 (1.78)	169.62 (1.01)	266.15 (1.39)

6 Conclusions

We have developed new valid inequalities which hold for several classes of IRPs, and we have tested the effect of changing the order of the input data on the quality of the bounds obtained and on the running time. We have generated new best known solutions for several large instances of the multi-vehicle IRP. We have also obtained improved lower bounds for several instances when compared to previous best known solutions, besides identifying new optimal solutions. We have increased the size of instances which we are now capable of solving exactly within short computational times.

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