CIRRELT

Centre interuniversitaire de recherche sur les réseaux d'entreprise, la logistique et le transport

Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation

Formulations for the Nonbifurcated Hop-Constrained Multicommodity Capacitated Fixed-Charge Network Design Problem

Babacar Thiongane
Jean-François Cordeau
Bernard Gendron

August 2013

CIRRELT-2013-53

Bureaux de Montréal :
Université de Montréal
C.P. 6128, succ. Centre-ville
Montréal (Québec)
Canada H3C 3J7
Téléphone : 514 343-7575
Télécopie : 514 343-7121

Bureaux de Québec :
Université Laval
2325, de la Terrasse, bureau 2642
Québec (Québec)
Canada G1V 0A6
Téléphone : 418 656-2073
Télécopie : 418 656-2624

www.cirrelt.ca

Formulations for the Nonbifurcated Hop-Constrained Multicommodity Capacitated Fixed-Charge Network Design Problem

Babacar Thiongane[1,2,*], Jean-François Cordeau[1,3], Bernard Gendron[1,4]

[1] Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation (CIRRELT)

[2] Kronos, 3535 Queen-Mary road, Suite 650, Montréal, Canada H3V 1H8

[3] Department of Logistics and Operations Management, HEC Montréal, 3000 Côte-Sainte-Catherine, Montréal, Canada H3T 2A7

[4] Université de Montréal, Pavillon André-Aisenstadt, Department of Computer Science and Operations Research, P.O. Box 6128, Station Centre-Ville, Montréal, Canada H3C 3J7

Abstract. This paper addresses the multicommodity capacitated fixed-charge network design problem with nonbifurcated flows and hop constraints. We present and compare mathematical programming formulations for this problem and we study different relaxations: linear programming relaxations, Lagrangean relaxations and partial relaxations of the integrality constraints. In particular, for the hop-indexed formulation, we show that the Lagrangean bound obtained by relaxing the flow conservation equations is tighter than the linear programming relaxation bound. We present computational results on a large set of randomly generated instances.

Keywords. Hop-constrained multicommodity network design problem, formulations, relaxations, Lagrangean relaxation.

Acknowledgements. This research was partly supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) for their financial support. This support is gratefully acknowledged.

---

* Corresponding author: Babacar.Thiongane@cirrelt.ca

Dépôt légal – Bibliothèque et Archives nationales du Québec
Bibliothèque et Archives Canada, 2013

1 Introduction

Let $G = (V, E)$ be a directed graph, where V is the set of nodes and E the set of arcs. Let also K be a set of commodities, where each commodity $k \in K$ is defined by an origin node s^k, a destination node t^k, and a demand d^k to be routed from s^k to t^k. Each arc $e \in E$ has a capacity u_e that satisfies $u_e \leq \sum_{k \in K} d^k$. For each unit of commodity k going through arc e, a nonnegative unit flow cost c_e^k has to be paid. Moreover, a nonnegative design cost f_e applies if there is a positive flow of any commodity on arc e. We consider the *multicommodity capacitated fixed-charge network design problem with nonbifurcated flows and hop constraints* (MCFDH) in which we want to minimize the sum of routing and design costs while satisfying the demands and the capacity constraints. In addition, each commodity k has to be routed on a single path (nonbifurcated or unsplittable flows) whose length must not exceed l^k. These hop constraints are useful in the context of reliability and quality of service in telecommunication and transportation networks, where limiting the number of arcs can reduce the probability of information loss or avoid unacceptable delays. When $f_e = 0$, $e \in E$, and $l^k = |E|$, $k \in K$, the MCFDH reduces to the *multicommodity integral flow problem*, which is NP-hard even if the number of commodities is two (Even, Itai and Shamir 1976). Thus, the MCFDH is itself NP-hard.

Network design problems with bifurcated (or splittable) flows have been much studied (see Frangioni and Gendron (2009), and the references therein). Problems in which demands cannot be split arise in several applications in the areas of telecommunication and transportation. Brockmüller, Günlük and Wolsey (1999) study a capacitated network design problem with non-linear costs arising in the design of private line networks; a similar problem is treated in Dahl, Martin and Stoer (1995). In Gavish and Altinkemer (1990) a non-linear network design problem is studied, while in Balakrishnan, Magnanti and Wong (1995) the authors present a decomposition algorithm for trees. Barhnart, Hane and Vance (2000) present a column generation model and a branch-and-price-and-cut algorithm for the integer multicommodity flow problem.

Problems involving hop constraints have been studied for minimum spanning tree problems by Gouveia (1995), Gouveia (1996), Gouveia and Requejo (2001), Dahl, Gouveia and Requejo (2003), Gouveia, Simonetti and Uchoa (2011) and for Steiner tree problems by Voß (1999) and Costa, Cordeau and Laporte (2009), who both model the design of centralized telecommunication networks with minimum cost, as well as by Balakrishnan and Altinkemer (1992) for more general telecommunication network design problems. Survivability in network design problems, which deals with the design of networks that can survive arc or node failures, is investigated in Wessäly (1996, 1997), Gouveia, Patrício and De Sousa (2006), Gouveia, Patrício and De Sousa (2008), Alevras, Grötschel and Botton et al. (2013). The effect of hop limits on the optimal cost is studied in Orlowski and Wessäly (2004) for a telecommunication network design problem. The convex hull of hop-constrained st-paths in a graph is studied in Dahl (1999) and Dahl and Gouveia (2004), who give a complete linear description when the number of hops is not larger than 3, and propose classes of facet-defining inequalities for the general case. To the best of our knowledge, the MCFDH has not been addressed before.

In this paper, we present four mathematical programming formulations for the MCFDH: the classical arc-based and path-based formulations, as well as the node-based and hop-indexed models. Different relaxations of these formulations are studied: linear programming (LP) relaxations, Lagrangean relaxations and partial relaxations of the integrality constraints. A theoretical comparison of the LP relaxations of the formulations is performed, showing that the path-based formulation and the hop-indexed formulation have the same LP relaxation value, which is not worse (and typically better) than the LP relaxation of the two other formulations. We then focus on the hop-indexed model and study two Lagrangean relaxations, one obtained by relaxing the capacity constraints and the other by relaxing the flow conservation constraints. We also compare these relaxations with those obtained by relaxing the integrality of either the design variables or the flow variables. The first Lagrangean relaxation can be decomposed into $|K|$ hop-constrained shortest path problems. Since the hop-indexed formulation for that problem has the integrality property, its associated Lagrangean dual has the same value as the LP relaxation value of the hop-indexed model. The second Lagrangean relaxation can be decomposed into $|E|$ 0-1 knapsack problems, which do not have the integrality property. Thus, the value of the Lagrangean dual associated with this relaxation is greater than, or equal to, the LP relaxation value (this is a major difference with the bifurcated case, where a similar Lagrangean relaxation provides the same bound as the LP relaxation).

The paper is organized as follows. Mathematical programming formulations of the problem are presented Section 2. In Section 3, we compare the LP relaxation values of the formulations, while in Section 4, we present the Lagrangean relaxations and compare them to the partial relaxations of the integrality constraints. Computational results are presented and analyzed in Section 5. Section 6 concludes the paper.

2 Problem Formulations

This section presents four mathematical programming formulations for the MCFDH, namely, the classical arc-based and path-based models, as well as the node-based and hop-indexed formulations.

2.1 Classical arc-based formulation

This formulation is obtained by adding the hop constraints to the classical arc-based formulation of the multicommodity capacitated fixed-charge network design problem. It uses binary variables x_e^k taking value 1 if the path of commodity k goes through arc e, and 0 otherwise, as well as binary variables y_e taking value 1 if arc e carries flow for at least one commodity, and 0, otherwise. Given $v \in V$, we denote by $\omega^+(v)$ the set of outgoing arcs from v and by $\omega^-(v)$ the set of incoming arcs to v.

$$\text{(C)} \qquad \min \sum_{k \in K} \sum_{e \in E} d^k c_e^k x_e^k + \sum_{e \in E} f_e y_e$$

$$\sum_{e \in \omega^+(v)} x_e^k - \sum_{e \in \omega^-(v)} x_e^k = \begin{cases} 1 & v = s^k \\ -1 & v = t^k \\ 0 & v \in V \setminus \{s^k, t^k\} \end{cases} \qquad k \in K \qquad (1)$$

$$\sum_{k \in K} d^k x_e^k \leq u_e y_e \qquad e \in E \qquad (2)$$

$$x_e^k \leq y_e \qquad e \in E, k \in K \qquad (3)$$

$$\sum_{e \in E} x_e^k \leq l^k \qquad k \in K \qquad (4)$$

$$x_e^k \in \{0,1\} \qquad e \in E, k \in K \qquad (5)$$

$$y_e \in \{0,1\} \qquad e \in E. \qquad (6)$$

Constraints (1) are the flow conservation constraints, while (2) are the capacity constraints. Constraints (3) are redundant strong linking inequalities which significantly improve the LP relaxation of the model. Inequalities (4) represent the hop constraints, which are valid because the flows are nonbifurcated.

2.2 Path-based formulation

For every $k \in K$, let P^k be the set of paths from s^k to t^k whose length is less than or equal to l^k. The formulation uses binary variables y_e as in the classical arc-based model, as well as binary variables x_p taking value 1 if $p \in P^k$ is used to satisfy the demand for commodity k, and 0 otherwise. Given a path p, we define $a_{ep} = 1$ if arc e belongs to path p, and 0 otherwise. The cost per unit of flow of a path $p \in P^k$ is then $c_p = \sum_{e\in E} a_{ep}c_e^k$.

$$(\mathrm{P}) \qquad \min \sum_{k\in K}\sum_{p\in P^k} d^k c_p x_p + \sum_{e\in E} f_e y_e$$

$$\sum_{p\in P^k} x_p = 1 \qquad k \in K \tag{7}$$

$$\sum_{k\in K}\sum_{p\in P^k} a_{ep} d^k x_p \le u_e y_e \qquad e \in E \tag{8}$$

$$\sum_{p\in P^k} a_{ep} x_p \le y_e \qquad e \in E, k \in K \tag{9}$$

$$x_p \in \{0,1\} \qquad p \in P^k, k \in K \tag{10}$$

$$y_e \in \{0,1\} \qquad e \in E. \tag{11}$$

Constraints (7) ensure that a single path is selected for each commodity. Capacity and strong linking constraints are represented by (8) and (9), respectively. Finally, as a feasible path $p \in P^k$ has a length smaller than or equal to l^k, the hop constraints are satisfied by any solution to this formulation.

2.3 Node-based formulation

This formulation is derived from the classical arc-based model by using a variant of the subtour elimination constraints of Miller, Tucker and Zemlin (1960) (see also Desrochers and Laporte 1991). It has been used in Gouveia (1995), Voß (1999) and Costa, Cordeau and Laporte (2009). For each commodity $k \in K$ and node $v \in V$, we introduce the node variable $\pi_v^k \ge 0$, which represents the distance from s^k to v in terms of the number of arcs.

$$(\mathrm{N}) \qquad \min \sum_{k\in K}\sum_{e\in E} d^k c_e^k x_e^k + \sum_{e\in E} f_e y_e$$

$$s.t.\ (1)-(3),(5)-(6) \tag{12}$$

$$\pi_v^k - \pi_{v'}^k + (l_k+1)x_e^k \le l^k \qquad k \in K, e=(v,v') \in E \tag{13}$$

$$\sum_{e\in\omega^-(v)} x_e^k \le \pi_v^k \le l^k \sum_{e\in\omega^-(v)} x_e^k \qquad k \in K, v \in V \setminus \{s^k\} \tag{14}$$

$$\pi_{s^k}^k = 0 \qquad k \in K. \tag{15}$$

Constraints (1)-(3) and (5)-(6) are similar to the classical arc-based formulation. The difference is in the modeling of the hop constraints. From (13), if $e = (v, v')$ is an arc of the path for commodity k selected in an optimal solution, i.e., $x_e^k = 1$, then we have $\pi_{v'}^k \ge \pi_v^k + 1$. Thus, as $\pi_{s^k}^k = 0$ from (15) and $1 \le \pi_v^k \le l_k$, $v \ne s^k$, from (14), the node variable π_v^k represents the number of arcs from s^k to v in the optimal path for commodity k. Since (14) impose that $\pi_v^k \le l_k$, no path from s^k to t^k can have a length greater than l_k.

2.4 Hop-indexed formulation

For every commodity k, every arc e and every possible position q with $1 \leq q \leq l^k$, we define variable x^k_{eq} equal to 1 if arc e appears in position q in the path from s^k to t^k and 0, otherwise.

$$\text{(I)} \qquad \min \sum_{k \in K} \sum_{e \in E} \sum_{q=1}^{l^k} d^k c^k_e x^k_{eq} + \sum_{e \in E} f_e y_e$$

$$\sum_{e \in \omega^+(v)} \sum_{q=1}^{l^k} x^k_{eq} - \sum_{e \in \omega^-(v)} \sum_{q=1}^{l^k} x^k_{eq} = \begin{cases} 1 & v = s^k \\ -1 & v = t^k \end{cases} \qquad k \in K \tag{16}$$

$$\sum_{e \in \omega^+(v)} x^k_{eq} - \sum_{e \in \omega^-(v)} x^k_{eq-1} = 0 \qquad k \in K, v \in V \setminus \{s^k, t^k\}, q = 2, \ldots, l^k \tag{17}$$

$$\sum_{k \in K} \sum_{q=1}^{l^k} d^k x^k_{eq} \leq u_e y_e \qquad e \in E \tag{18}$$

$$\sum_{q=1}^{l_k} x^k_{eq} \leq y_e \qquad e \in E, k \in K \tag{19}$$

$$x^k_{eq} \in \{0,1\} \qquad k \in K, e \in E, q = 1, \ldots, l^k \tag{20}$$

$$y_e \in \{0,1\} \qquad e \in E. \tag{21}$$

Constraints (18) and (19) are capacity and strong linking constraints, respectively. Constraints (16) and (17) are the flow conservation constraints at the origin/destination nodes and at intermediate nodes, respectively. Moreover, (17) guarantee the increasing positions of the arcs on any path. Similar hop-indexed formulations, which capture the hop constraints inside the definition of the variables, have already been used in Gouveia (1998), Voß (1999), Gouveia, Patrício and De Sousa (2006, 2007) and Costa, Cordeau and Laporte (2009).

3 Linear Programming Relaxations

Let $(\overline{X})$ denote the LP relaxation of problem (X), and $v(\overline{X})$ its value. The following proposition establishes a comparison between the LP relaxations of the four formulations.

Proposition 1
$v(\overline{N}) \leq v(\overline{C}) \leq v(\overline{I}) = v(\overline{P})$ and there exist instances for which the inequalities are strict.

Proof.
$v(\overline{N}) \leq v(\overline{C})$
Let $(\overline{x}, \overline{y})$ be an optimal solution to $(\overline{C})$. The result follows if we can show that, for each k, there exists $\pi^k = (\pi^k_v)_{v \in V}$ such that (13)-(15) are satisfied for $x = \overline{x}$. We assume that this is not true and we will derive a contradiction, therefore showing the result. The hypothesis implies that for some commodity k, the following LP is infeasible:

$$\min \sum_{v \in V} 0 \pi^k_v$$

$$\pi^k_v - \pi^k_{v'} \leq l^k - (l^k + 1)\overline{x}^k_e, \; e = (v, v') \in E$$

$$\sum_{e \in \omega^-(v)} \overline{x}^k_e \leq \pi^k_v \leq l^k \sum_{e \in \omega^-(v)} \overline{x}^k_e, \; v \in V \setminus \{s^k\}$$

$$\pi^k_{s^k} = 0.$$

By strong duality, this implies that the dual of this LP is either infeasible or unbounded:

$$\max \sum_{e\in E} \Big(l^k - (l^k+1)\overline{x}^k_e\Big) y^k_e + \sum_{v\in V\setminus\{s^k\}} \Big[\Big(l^k\Big(\sum_{e\in\omega^-(v)} \overline{x}^k_e\Big)\Big) w^k_v - \Big(\sum_{e\in\omega^-(v)} \overline{x}^k_e\Big) u^k_v\Big]$$

$$\sum_{e\in\omega^+(v)} y^k_e - \sum_{e\in\omega^-(v)} y^k_e + w^k_v - u^k_v = 0,\ v \in V\setminus\{s^k\}$$

$$y^k_e \ge 0,\ e\in E,\ w^k_v \ge 0,\ u^k_v \ge 0,\ v\in V\setminus\{s^k\}.$$

Since 0 is a solution to this problem, it must be unbounded. It is well-known that this flow circulation problem is unbounded iff there exists an elementary circuit W in G (not containing s^k) with negative cost:

$$\sum_{e\in W} \Big(l^k - (l^k+1)\overline{x}^k_e\Big) + \sum_{v\in W} \Big[(l^k-1)\Big(\sum_{e\in\omega^-(v)} \overline{x}^k_e\Big)\Big] < 0.$$

Since $\sum\limits_{v\in W}\sum\limits_{e\in\omega^-(v)} \overline{x}^k_e \ge \sum_{e\in W} \overline{x}^k_e$*, we obtain:*

$$\sum_{e\in W} \Big(l^k - (l^k+1)\overline{x}^k_e\Big) + (l^k-1)\Big(\sum_{e\in W} \overline{x}^k_e\Big) < 0.$$

After simplifications, this gives the following inequality: $2\sum_{e\in W} \overline{x}^k_e > |W|\, l^k$*. But then, since* $\overline{x}$ *satisfies the hop constraints in* $\overline{C}$*, we have:* $2\,l^k \ge 2\sum_{e\in E} \overline{x}^k_e \ge 2\sum_{e\in W} \overline{x}^k_e > |W|\, l^k$*, a contradiction.*

$v(\overline{C}) \le v(\overline{I})$

Let $(\overline{x},\overline{y})$ *be an optimal solution to* $(\overline{I})$*. Let us show that* $(\tilde{x},\tilde{y})$ *is feasible for* $(\overline{C})$*, where* $\tilde{y} = \overline{y}$ *and* $\tilde{x}^k_e = \sum\limits_{q=1}^{l^k} \overline{x}^k_{eq}, e\in E, k\in K$*. Constraints (2) and (3) are clearly satisfied. Constraints (1) follow from (16), (17). As remarked in Section 2.4, we have* $\tilde{x}^k_e = \sum\limits_{q=1}^{l_k} \overline{x}^k_{eq} \in [0,1]$*. Also, since* $(\overline{x},\overline{y})$ *is an optimal solution to* $(\overline{I})$*, the outgoing flow from* s^k *is 1. Thus, with constraint (17), the quantity* $\sum\limits_{e\in E}\sum\limits_{q=1}^{l_k} \overline{x}^k_{eq}$ *cannot exceed* l^k*. Hence,* $\sum\limits_{e\in E} \tilde{x}^k_e \le l_k$*. Finally, we have:*

$$v(\overline{I}) = \sum_{k\in K}\sum_{e\in E} c^k_e\Big(\sum_{q=1}^{l^k} \overline{x}^k_{eq}\Big) + \sum_{e\in E} f_e \overline{y}_e \sum_{k\in K} = \sum_{e\in E} c^k_e \tilde{x}^k_e + \sum_{e\in E} f_e \tilde{y}_e \ge v(\overline{C}).$$

$v(\overline{I}) = v(\overline{P})$

Let $(\overline{x},\overline{y})$ *be an optimal solution to* $(\overline{P})$*. Let* $(\tilde{x},\tilde{y})$ *defined by* $\tilde{y} = \overline{y}$ *and* $\tilde{x}^k_{eq} = \sum\limits_{p\in P_k} a^q_{ep}\overline{x}_p, e \in E, k\in K, q = 1,\ldots,l^k$*, where* $a^q_{ep} = 1$ *if* e *is the* q*-th arc in path* p *and* $a^q_{ep} = 0$ *otherwise. All constraints of* $(\overline{I})$ *are clearly satisfied. Since* $\sum\limits_{q=1}^{l^k} a^q_{ep} = a_{ep}$*, the value of* $(\tilde{x},\tilde{y})$ *is* $v(\overline{P})$ *and* $v(\overline{P}) \ge v(\overline{I})$*.*
Conversely, an optimal solution to $(\overline{I})$ *gives a set of paths from* s^k *to* d^k *with a length not exceeding* l^k*. Therefore, by expressing this solution in terms of the path-based variables, we*

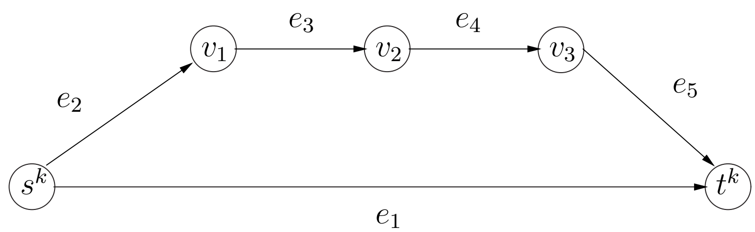

Figure 1: Instance showing that $v(\overline{N}) < v(\overline{C}) < v(\overline{I})$.

obtain a solution that satisfies all the constraints of $(\overline{P})$ *and the relation* $v(\overline{P}) \leq v(\overline{I})$ *also holds.*

Now, consider the graph defined in Figure 1 with $K = \{k\}$, *flow costs* $c_{e_1} = c > 0$, $c_{e_i} = c/4$ *for* $i = 2,..,5$, *and design costs* $f_{e_1} = f > 0$, $f_{e_i} = f/4 - \epsilon$ *where* $\epsilon > 0$. *Suppose that* $u_{e_i} = d^k = 1$ *and* $l^k = 3$. *Thus, an optimal solution to* $(\overline{I})$ *is* $x^k_{e_1 1} = 1$ *and*

$$v(\overline{I}) = c + f.$$

However, an optimal solution to $(\overline{C})$ *is* $x^k_{e_1} = y_{e_1} = 1/3$, $x^k_{e_i} = y_{e_i} = 2/3$, $i = 2,..,5$ *and its optimal value is*

$$v(\overline{C}) = c + f - 8\epsilon/3 = v(\overline{I}) - 8\epsilon/3.$$

Moreover, an optimal solution to $(\overline{N})$ *is given by* $x^k_{e_1} = y_{e_1} = 1/4$, $x^k_{e_i} = y_{e_i} = 3/4$, $i = 2, \ldots, 5$, $\pi^k_{v_i} = 3/4$, $i = 1, 2, 3$, $\pi^k_{s^k} = 0$, $\pi^k_{t^k} = 1$ *and* $v(\overline{N}) = c + f - 3\epsilon$. *Thus, for this instance* $v(\overline{N}) < v(\overline{C}) < v(\overline{I})$. ■

Note that the dominance of the hop-indexed formulation over the node-based formulation has been shown in Costa, Cordeau and Laporte (2009) for the Steiner tree problem with revenues, budget and hop constraints. The equivalence between the LP relaxation of the path-based and hop-indexed formulations has been proved for the hop-constrained minimum spanning tree problem in Dahl, Gouveia and Requejo (2003), where the authors mention that the arguments of the proof can be used to prove the same result for more general problems.

The next section focuses on improving the lower bound $v(\overline{I})$ by studying various relaxations of the hop-indexed formulations, namely Lagrangean relaxations, as well as partial relaxations of the integrality constraints.

4 Hop-Indexed Relaxations

We consider two Lagrangean relaxations for the hop-indexed formulation: the relaxation of the capacity and strong linking constraints, which we call the hop-constrained shortest path relaxation, and the relaxation of the flow conservation constraints, which we call the 0-1 knapsack relaxation. In addition, we study the relationships between these Lagrangean relaxations and the partial relaxations of the integrality constraints either on the design variables or on the flow variables.

4.1 Hop-constrained shortest path relaxation

Let us associate multipliers $\gamma \in \mathbb{R}^{m_1}_+$, $m_1 = |E|$ and $\delta \in \mathbb{R}^{m_2}_+$, $m_2 = |E||K|$, respectively, to the capacity constraints (18) and to the strong linking constraints (19). When relaxing these constraints, the Lagrangean subproblem is defined as:

$$(LR^{HSP}(\gamma, \delta)) \quad \min \sum_{k \in K} \sum_{e \in E} \sum_{q=1}^{l^k} (c^k_e + \gamma_e + \delta^k_e) d^k x^k_{eq} + \sum_{e \in E} (f_e - \gamma_e u_e - \sum_{k \in K} \sum_{q=1}^{l^k} b^k_e \delta^k_e) y_e$$

subject to (16)-(17) and (20)-(21). Problem $(LR^{HSP}(\gamma,\delta))$ can be decomposed into two parts: a subproblem in the x variables that separates into $|K|$ hop-constrained shortest path problems and a subproblem in y variables that can be solved by looking at the sign of the cost of each variable. Hop-constrained shortest path problems are known to be NP-hard in general, but can be solved in polynomial time when the costs are nonnegative (Dahl and Gouveia 2004), which is the case here.

By using the same arguments as in the proof of the equality $v(\overline{I}) = v(\overline{P})$ in Proposition 1, one can see that the LP relaxation of the hop-indexed formulation of the hop-constrained shortest path problems in $(LR^{HSP}(\gamma,\delta))$ gives, for each commodity k, a set of paths from s^k to t^k with a length not exceeding l^k; therefore, we can relax the integrality of the x variables without changing the optimal value. Also, it is trivial to see that the subproblem in y variables can be solved by relaxing the integrality constraints. Therefore, the Lagrangean subproblem can be solved by relaxing the integrality constraints on all variables, i.e., $(LR^{HSP}(\gamma,\delta))$ has the integrality property.

The Lagrangean dual associated with this relaxation is defined as:

$$(LD^{HSP}) \ \max\{v(LR^{HSP}(\gamma,\delta)) \mid (\gamma,\delta) \in \mathbb{R}_+^{m_1+m_2})\}.$$

Since $(LR^{HSP}(\gamma,\delta))$ has the integrality property, by standard Lagrangean duality, we have:

Proposition 2 $v(LD^{HSP}) = v(\overline{I})$.

4.2 0-1 knapsack relaxation

Let us associate multipliers $\alpha \in \mathbb{R}^{n_1}$, $n_1 = 2|K|$, to (16), and $\beta \in \mathbb{R}^{n_2}$, $n_2 = (|V|-2)\sum_{k\in K}(l_k - 1)$, to (17). When relaxing these constraints, the Lagrangean subproblem is defined as:

$$(LR^{01K}(\alpha,\beta)) \quad \min \sum_{k\in K}\sum_{e\in E}\sum_{q=1}^{l_k} \tilde{c}_{eq}^k(\alpha,\beta)x_{eq}^k + \sum_{e\in E} f_e y_e + \sum_{k\in K}(-\alpha_{s^k}^k + \alpha_{t^k}^k)$$

subject to (18)-(21), where $\tilde{c}_{eq}^k(\alpha,\beta)$ is defined by:

$$\tilde{c}_{eq}^k(\alpha,\beta) = \begin{cases} d^k c_{eq}^k + \zeta_u^{s^k}\alpha_{s^k}^k - \zeta_v^{t^k}\alpha_{t^k}^k - (1-\zeta_v^{t^k})\beta_{vq+1}^k & q = 1 \\ d^k c_{eq}^k + (1-\zeta_u^{s^k})\beta_{uq}^k - \zeta_v^{t^k}\alpha_{t^k}^k - (1-\zeta_v^{t^k})\beta_{vq+1}^k & q = 2,\dots,l^k-1 \\ d^k c_{eq}^k + (1-\zeta_u^{s^k})\beta_{uq}^k - \zeta_v^{t^k}\alpha_{t^k}^k & q = l^k \end{cases}$$

with $e = (u,v)$, and $\zeta_u^w = 1$ if $u = w$ and 0 otherwise. $(LR^{01K}(\alpha,\beta))$ decomposes by arc. For each arc e, we consider two cases:
$y_e = 0$: We then have $x_{eq}^k = 0$ for all $k \in K$, $q = 1,\dots,l^k$ with an objective function value of 0.
$y_e = 1$: In this case, the objective function value and the values of the x variables can be obtained as follows. First, for each $k \in K$, we determine $q^k = \arg\min_{q=1\dots l^k}\{\tilde{c}_{eq}^k(\alpha,\beta)\}$ to ensure that constraint $\sum_{q=1}^{l^k} x_{eq}^k \le 1$ is satisfied. Then, we solve the following a 0-1 knapsack problem:

$$(\mathcal{Q}_e) \quad \min\{\sum_{k\in K}\tilde{c}_{eq^k}^k(\alpha,\beta)x_{eq^k}^k \quad \text{s.t.} \sum_{k\in K} d^k x_{eq^k}^k \le u_e,\ x_{eq^k}^k \in \{0,1\},\ k \in K\}.$$

Thus, for each arc e, we pick the cheapest of the two alternatives, $y_e = 0$ or $y_e = 1$, and derive an optimal solution out of it.

Note that if we relax the integrality of the y variables, the Lagrangean subproblem would be solved in the same way, since it is easy to show then that the optimal solution for each arc

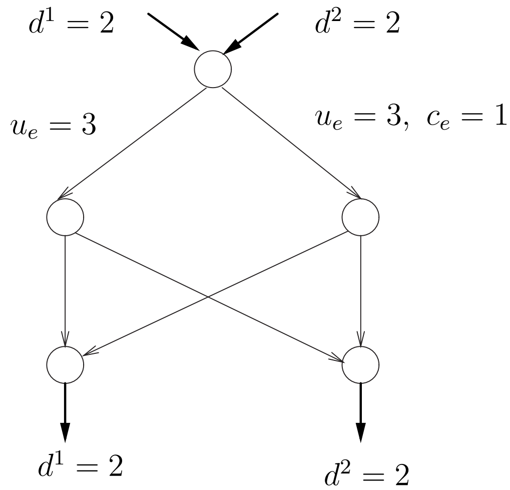

Figure 2: Instance showing that $v(\overline{I}) = 1 < 2 = v(LD^{01K})$.

e must satisfy $y_e = 0$ or $y_e = 1$. However, because the Lagrangean subproblem decomposes into $|E|$ 0-1 knapsack problems, $(LR^{01K}(\alpha, \beta))$ does not have the integrality property. If bifurcated flows were allowed instead of nonbifurcated flows, the Lagrangean subproblem would be solved in the same way by considering for each arc e the two alternatives $y_e = 0$ and $y_e = 1$ (whether or not the integrality of the y variables is relaxed). The subproblem solved for the case $y_e = 1$ would then be a continuous knapsack problem and the Lagrangean subproblem would have the integrality property (Crainic, Frangioni, Gendron 2001).

The Lagrangean dual associated with this relaxation is defined as:

$$(LD^{01K})\ \max\{v(LR^{01K}(\alpha, \beta)) \mid (\alpha, \beta) \in \mathbb{R}^{n_1+n_2})\}.$$

Since $(LR^{01K}(\alpha, \beta))$ does not have the integrality property, by standard Lagrangean duality, we have:

Proposition 3 *$v(LD^{01K}) \geq v(\overline{I})$ and there exist instances for which the inequality is strict.*

Proof. *Figure 2 shows a capacitated instance with no design costs and no hop constraints. There are two commodities, each with a demand of 2, that share the same origin, but have different destinations. Arcs leaving the origin of each commodity have a capacity of 3. One of these two arcs has a flow cost of 1, while all other arcs have no flow costs. The optimal solution to $(\overline{I})$ sends 3 units of flow on the arc with no cost leaving the origin and 1 unit of flow on the arc with flow cost equal to 1. This means that one of the two commodities has its demand split between two paths, which implies values of 1/2 for the corresponding x variables. The optimal value is $v(\overline{I}) = 1$. By standard Lagrangean duality, $v(LD^{01K})$ can be computed by optimizing over the polytope defined by the intersection of the flow conservation equations and the convex hulls of the 0-1 knapsack set for each arc. Clearly, the optimal solution to $(\overline{I})$ is not feasible for this polytope. The optimal solution is in fact obtained by sending two units of flow of each commodity on each of the arcs from the origin, which is also the optimal solution to I. The optimal value is $v(LD^{01K}) = 2$. Thus, in this case, $v(\overline{I}) = 1 < 2 = v(LD^{01K})$.* ■

4.3 Partial relaxations of integrality constraints

In the LP relaxation of (I), both the x and y variables can be fractional, i.e., $(x, y) \in [0,1]^M \times [0,1]^{|E|}$, where $M = |E| \sum\limits_{k \in K} l^k$. When x is fractional, the flows can be bifurcated, i.e., several paths can be used to satisfy the demand for any commodity. When y is fractional, it is possible that only a fraction of the design cost associated with any arc is taken into account in an optimal solution in which there is flow on that arc, but there is also a residual capacity. One possibility to improve the lower bound $v(\overline{I})$ is to relax the integrality constraints only on a subset of the variables, either the x variables or the y variables, but not on both, as in $(\overline{I})$.

First, we consider the relaxation of the integrality of the y variables, denoted (I_y). We then have the following result:

Proposition 4 *$v(I_y) = v(I)$.*

Proof. *Since the design costs are nonnegative and the y variables appear only in the capacity and strong linking constraints, there is an optimal solution to (I_y) that satisfies, for each arc e:*

$$y_e = \max\Big\{ (\sum_{k\in K} \sum_{q=1}^{l^k} d^k x^k_{eq})/u_e, \max_{k\in K}\Big\{ \sum_{q=1}^{l_k} x^k_{eq} \Big\}\Big\}.$$

Now, since the x variables are binary and $\sum\limits_{q=1}^{l_k} x^k_{eq} \leq 1$, we must have $\sum\limits_{q=1}^{l_k} x^k_{eq} \in \{0,1\}$. Moreover, since $(\sum\limits_{k\in K} \sum\limits_{q=1}^{l^k} d^k x^k_{eq})/u_e \leq 1$ and $(\sum\limits_{k\in K} \sum\limits_{q=1}^{l^k} d^k x^k_{eq})/u_e = 0$ iff $\max_{k\in K}\{\sum\limits_{q=1}^{l_k} x^k_{eq}\} = 0$, it follows that $y_e \in \{0,1\}$, i.e., there is an optimal solution to (I_y) that is also an optimal solution to I. ■

Thus, when relaxing the integrality of the y variables, we obtain, in fact, a model that solves the MCFDH. It is easy to see that the same result applies not only to the hop-indexed formulation (I), but also to the other models (C), (P) and (N).

Second, we consider the relaxation of the integrality of the x variables, denoted (I_x). We then have the following result:

Proposition 5 *$v(I_x) = v(I)$ for any uncapacitated instance (i.e., $u_e = \sum_{k\in K} d^k$, $e \in E$).*

Proof. *Let $\overline{y}$ be the values assigned to the y variables in an optimal solution to (I_x). We can obtain the optimal values $\overline{x}$ to the x variables by solving the LP relaxation of the hop-indexed formulation of $|K|$ hop-constrained shortest path problems. As already remarked above, this LP relaxation provides, for each commodity k, a set of paths from s^k to t^k with a length not exceeding l^k. Hence, $(\overline{x}, \overline{y})$ is also an optimal solution to (I).* ■

When there are capacities, we have in general $v(\overline{I}) \leq v(I_x) \leq v(I)$. It is interesting to know how the Lagrangean bound $v(LD^{01K})$ compares with $v(I_x)$, since both are never worse that $v(\overline{I})$.

Proposition 6 *There is no dominance between $v(I_x)$ and $v(LD^{01K})$.*

Proof. *Figure 3 shows an uncapacitated instance with no hop constraints and no flow costs. There are three commodities, each with a demand of 1, that share the same origin, but have different destinations. Arcs leaving the origin of each commodity have a design cost equal to 1, while all other arcs have no design costs. Since the instance is uncapacitated, (LR^{01K}) has the integrality property and $v(LD^{01K}) = v(\overline{I})$. In addition, by Proposition 5, $v(I_x) = v(I)$. The optimal solution to $(\overline{I})$ assigns the value $1/2$ to all (possibly non-zero) variables with an optimal value $v(\overline{I}) = 3/2$. The optimal solution to (I) consists in choosing two of the three arcs leaving the origin of each commodity and to send three units of flow on these arcs, one for each commodity. The optimal value is $v(I) = 2$. Thus, for this instance,*

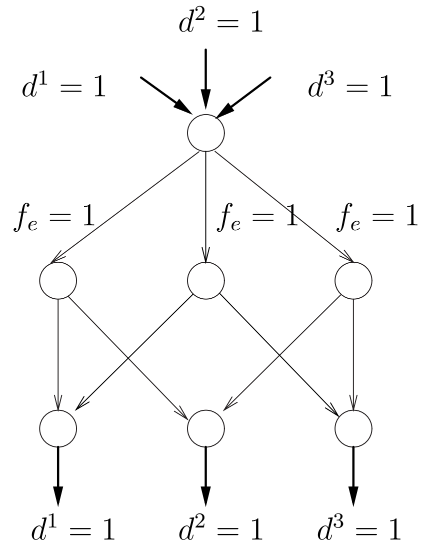

Figure 3: Instance showing that $v(LD^{01K}) = v(\overline{I}) = 3/2 < 2 = v(I) = v(I_x)$.

we have $v(LD^{01K}) = v(\overline{I}) = 3/2 < 2 = v(I) = v(I_x)$. *This example is an adaptation of Gomory's example (reported in Krarup and Pruzan 1983) of a fractional instance to the so-called strong formulation of the uncapacitated facility location problem.*
Consider now the instance illustrated in Figure 2 (see proof of Proposition 3). Since there are no design costs, (I_x) *is the same as* $(\overline{I})$ *for this instance. Hence, we have* $v(I_x) = v(\overline{I}) = 1 < 2 = v(LD^{01K})$. ■

4.4 Summary of bound relationships

The following proposition summarizes the relationships between the different lower bounds derived so far, including the LP relaxation bounds of the different formulations of the MCFDH, but with an emphasis on the stronger lower bounds derived from the hop-indexed model.

Proposition 7

$$\begin{aligned} v(I) &= v(I_y) \\ &\geq \max\{v(I_x), v(LD^{01K})\} \\ &\geq \min\{v(I_x), v(LD^{01K})\} \\ &\geq v(LD^{HSP}) \\ &= v(\overline{I}) \\ &= v(\overline{P}) \\ &\geq v(\overline{C}) \\ &\geq v(\overline{N}). \end{aligned}$$

5 Computational Results

The purpose of our computational experiments is to compare in practice 1) the LP relaxation values of the hop-indexed and classical arc-based formulations, $v(\overline{I})$ and $v(\overline{C})$; 2) the LP relaxation and the Lagrangean dual bounds, $v(\overline{I})$ and $v(LD^{01K})$; 3) the Lagrangean dual bound and the value of the partial relaxation of the integrality constraints on the flow variables, $v(LD^{01K})$ and $v(I_x)$. We have performed a series of experiments on 137 instances among those used by Crainic, Frangioni and Gendron (2001). The hop constraint parameter l^k of a commodity k has been generated according to $\kappa^k + 1 + r^k$, where r^k is uniformly distributed over the interval $\left[0, \frac{|V|}{4}\right]$ and κ^k denotes the length of a shortest path from s^k to t^k. The characteristics of the instances are described in Table 1.

Problem	#Instances	#Nodes	#Arcs	#Commodities
R1	6	10	35	10
R2	2	10	35	25
R3	6	10	60	10
R4	3	10	60	25
R5	7	10	60	50
R6	6	10	82	10
R7	6	10	83	25
R8	6	10	83	50
R9	5	20	120	40
R10	6	20	120	100
R11	5	20	120	200
R12	6	20	220	40
R13	9	20	220	100
R14	6	20	220	200
R15	9	20	314	40
R16	9	20	318	100
R17	9	20	315	200
C1	3	20	230	40
C2	4	20	230	200
C3	4	20	300	40
C4	4	20	300	200
C5	4	30	520	100
C6	4	30	520	400
C7	4	30	700	100
C8	4	30	700	400

Table 1: Characteristics of the 137 instances used

All bounds are computed with CPLEX 12.3, except $v(LD^{01K})$, which is obtained with the following subgradient method. Starting from an initial multiplier u^0, the method generates a sequence of multipliers using the formula:

$$u^{l+1} = u^l - \lambda^l \frac{d^l}{||d^l||}.$$

Here, λ^l is the step size and d^l is the step direction. We consider the modified Camerini-Fratta-Maffioli rule (Camerini, Fratta and Maffioli 1975):

$$d^l = \sigma^l + \mu^l d^{l-1},$$

where $\mu^l = \frac{||\sigma^l||}{||d^{l-1}||}$ if the scalar product $< \sigma^l, d^{l-1} >$ is negative, and $\mu^l = 0$ otherwise. The step size is computed as:

$$\lambda_l = \tau^l \frac{v^l - v(LR^{01K}(u^l))}{||d^l||},$$

where $0 < \tau^l < 2$, v^l is an estimation of the optimal value of the MCFDH and $v(LR^{01K}(u^l))$ denotes the value of the Lagrangean relaxation problem with the current Lagrange multipliers

u^l. In our experimentation, v^l is set to $2\,v^*$, where v^* is the best Lagrangean bound obtained so far. The value of τ^l is adjusted as follows: it is multiplied by a factor $\rho < 1$ if, after 10 iterations, there is no improvement, but its minimum value is set to 10^{-3}. The algorithm stops after 1000 iterations or when the relative distance between u^{l+1} and u^l is lower than 10^{-7}. The 0-1 knapsack problem arising in the Lagrangean subproblem is solved using the hybrid code of Bourgeois and Plateau (1992), which combines dynamic programming and enumeration, shown to be efficient for difficult instances. The maximum size of the subproblem being solved by dynamic programming has been set to 50. The code has been implemented in C++, using the g++ compiler and all experiments have been performed on a Linux machine, operating at 3.07 GHz.

The computational results are presented in Table 2, which displays:

- T(X), the average computing time in seconds for model X, where X represents one of the five formulations: $(\overline{C})$, $(\overline{I})$, (LD^{01K}), (I_x) and (I); note that formulation (I) has been used instead of (I_y), as it gave slightly faster computing times
- G(X), the average relative gap (in percentage) between model X (X different from $\overline{C}$) and formulation $\overline{C}$; a positive value means that (X) is better than $(\overline{C})$.

Problem	T($\overline{C}$)	G($\overline{I}$)	T($\overline{I}$)	G(LD^{01K})	T(LD^{01K})	G(I_x)	T(I_x)	G(I)	T(I)
R1	0.00	0.05	0.00	13.60	0.47	3.47	0.04*	14.21	0.03*
R2	0.00	0.03	0.00	9.53	1.30	1.00	0.06*	9.61	0.06*
R3	0.00	1.28	0.00	10.44	0.50	3.64	0.04*	10.50	0.04*
R4	0.01	0.73	0.01	2.95	2.00	2.22	0.15*	3.42	0.13*
R5	0.07	0.76	0.08	6.05	4.25	3.57	8.75*	8.27	7.58*
R6	0.00	0.19	0.01	10.10	0.71	4.11	0.06*	10.22	0.06*
R7	0.01	0.67	0.02	3.60	2.05	2.28	0.33*	3.85	0.18*
R8	0.12	1.74	0.12	8.19	5.23	4.38	8.65*	10.35	25.93*
R9	0.17	1.49	0.21	8.77	10.75	5.58	56.05*	10.94	19.54*
R10	2.04	1.51	3.32	5.32	27.40	4.65	2096.03*	9.30	5065.66*
R11	6.65	0.38	14.27	3.42	57.90	2.84	2992.89*	9.66	7178.70*
R12	0.57	2.01	0.83	3.77	17.66	5.64	935.11*	6.27	395.69*
R13	6.58	1.27	9.41	10.87	42.63	5.42	5027.53*	14.89	4162.07*
R14	46.36	1.77	99.12	2.58	90.29	7.21	5018.21*	13.70	6218.10*
R15	1.13	0.64	1.69	9.55	22.38	4.68	2637.46*	11.16	67.90*
R16	14.41	1.68	22.67	6.84	58.39	6.36	5302.25*	13.69	5649.41*
R17	99.72	1.34	203.31	5.59	123.60	11.35	6477.25*	15.19	7179.23*
C1	0.04	0.25	0.10	5.49	18.77	1.12	6.15*	5.53	2.13*
C2	36.84	2.26	53.23	2.28	100.85	6.63	7179.99*	9.07	7178.90*
C3	10.61	0.54	16.77	3.51	41.00	2.22	1801.73*	6.27	1797.15*
C4	23.18	0.55	49.68	2.73	94.79	3.39	5385.87*	7.41	5385.17*
C5	16.84	0.93	42.64	2.63	124.21	3.56	5562.18*	6.36	4150.08*
C6	187.85	0.45	582.68	1.38	443.71	2.63	34105.48**	5.35	34099.42**
C7	90.36	0.46	252.05	2.20	253.19	2.15	32398.15**	4.50	22458.45**
C8	486.68	0.39	1245.42	1.44	550.73	4.07	42723.62**	6.58	40886.88**

* : computing time limited to 2 hours
**: computing time limited to 12 hours

Table 2: Comparison between the different lower bounds over 137 instances

From the results shown in Table 2, we can draw the following observations:

- When comparing $(\overline{I})$ and $(\overline{C})$, we observe that the average gap G($\overline{I}$) obtained, although always positive, is never greater than 2.26% (problem C2), and that the computing time for $(\overline{C})$ is up to three times better.

- The Lagrangean bound $v(LD^{01K})$ is consistently better than $v(\overline{I})$, the lower bound obtained from the hop-indexed LP relaxation. In particular, we observe that G($\overline{I}$) is lower than or equal to 2.26%, while G(LD^{01K}) is always greater than this percentage, except for the last three sets of instances, C6, C7 and C8. The average gap difference between (LD^{01K}) and ($\overline{I}$) reaches up to 13% (for R1 instances), although the average gap difference is less than or equal to 10%.
- The average computing time for the hop-indexed LP relaxation ($\overline{I}$) is most of the time smaller than that for the subgradient method to compute $v(LD^{01K})$. Indeed, it takes less than one second for ($\overline{I}$) to solve 11 of the sets of problems, while the subgradient method took up to 18 seconds for the same sets. For larger problems such as R17 and C8, however, the Lagrangean dual solution time dominates that of ($\overline{I}$), performing better, with relative improvements of 39% and 55%, respectively.
- For 17 problems out of the 25, (LD^{01K}) generated lower bound values which are greater than or equal to the bound values obtained by solving the mixed-integer relaxation (I_x).
- Solving the mixed-integer formulations takes much more time than solving the Lagrangean dual formulation, except for small instances. The subgradient method never took more than 9 minutes to solve any of the sets of problems, while the computing times have been restricted to 2 hours, and sometimes to 12 hours, for both (I_x) and (I). Note that these time limits have been reached for both formulations in many instances. We also observe that no formulation between (I_x) and (I) is clearly faster than the other.

6 Conclusion

In this paper, we have studied the MCFDH which, to the best of our knowledge, has not been addressed before. Several formulations have been presented and we have given relationships between different lower bounds. The mixed-integer formulation obtained by relaxing the integrality of the design variables is equivalent to solving the MCFDH. There is no dominance between the mixed-integer relaxation derived from the problem with bifurcated flows and the Lagrangean relaxation of the flow constraints of the hop-indexed formulation. However, our computational experiments have shown that the Lagrangean bound is better for most instances, while also being faster to compute. The Lagrangean bound is also better in theory and in practice than the hop-indexed LP relaxation bound, although it generally takes more time to compute. Overall, the Lagrangean relaxation method seems to offer a good tradeoff between computing time and bound quality. Further investigations involving combinations of this Lagrangean relaxation with primal heuristics would be of interest.

References

Alevras D., Grötschel M., Wessäly R. “A network dimensioning tool”, *Technical Report SC 96-49, Konrad-Zuse-Zentrum fur Informationstechnik, Germany*, 1996.

Alevras D., Grötschel M., Wessäly R. “Capacity and survivability models for telecommunication netoworks”, *Technical Report SC 97-24, Konrad-Zuse-Zentrum fur Informationstechnik, Germany*, 1997.

Balakrishnan A., Altinkemer K. “Using a hop-constrained model to generate alternative communication network design”, *INFORMS Journal on Computing* 4 (1992) 195-205.

Balakrishnan A., Magnanti T.L., Wong R.T. “A decomposition algorithm for local access telecommunications network expansion planning", *Operations Research* 43 (1995) 58-76.

Barnhart C., Hane C.A., Vance P. "Using branch-and-price-and-cut to solve origin-destination integer multicommodity flow problems", *Operations Research* 48 (2000) 318-326.

Botton Q., Fortz B., Gouveia L., Poss M. "Benders decomposition for the hop-constrained survivable network design problem", *INFORMS Journal on Computing* 25 (2013) 13-26.

Brockmüller B., Günlük O., Wolsey L.A. "Designing private line networks - Polyhedral analysis and computation", *CORE DP9647, Université Catholique de Louvain*, 1999.

Camerini P.M., Fratta L., Maffioli F. "On improving relaxation methods by modified gradient techniques", *Mathematical Programming Study* 3 (1975) 26-34.

Costa A., Cordeau J.-F., Laporte G. "Models and branch-and-cut algorithms for the Steiner tree problem with revenues, budget and hop constraints", *Networks* 53 (2009) 141-159.

Crainic T.G, Frangioni A., Gendron B. "Bundle-based relaxation methods for multicommodity capacitated fixed-charge network design problems", *Discrete Applied Mathematics* 112 (2001) 73-99.

Dahl G. "Notes on polyhedra associated with hop-constrained paths", *Operations Research Letters* 25 (1999) 97-100.

Dahl G., Gouveia L. "On the directed hop-constrained shortest path problem", *Operations Research Letters* 32 (2004) 15-22.

Dahl G., Gouveia L., Requejo C. "On formulations and methods for the hop-constrained minimum spanning tree problem", *In: Handbook of Optimization in Telecommunications, ed. M.G.C. Resende and P.M. Pardalos, Springer Science + Business Media*, 2006.

Dahl G., Martin A., Stoer M. "Routing through virtual paths in layered telecommunication networks", *Operations Research* 47 (1999) 693-702.

Desrochers M., Laporte G. "Improvements and extensions to the Miller-Tucker-Zemlin subtour elimination constraints", *Operations Research Letters* 10 (1991) 27-36.

Even S., Itai A., Shamir A. "On the complexity of timetable and multicommodity flow problems", *SIAM Journal of Computing* 5 (1976) 184-183.

Frangioni A., Gendron B. "0-1 reformulations of the multicommodity capacitated network design problem", *Discrete Applied Mathematics*, 157 (2009) 1229-1241.

Gavish B., Altinkemer K. "Backbone network design tools with economic trade-offs", *ORSA Journal on Computing* 2 (1990) 58-76.

Gouveia L. "Using the Miller-Tucker-Zemlin constraints to formulate a minimal spanning tree problem with hop constraints", *Computers and Operations Research* 22

(1995) 959-970.

Gouveia L. "Multicommodity flow models for spanning trees with hop constraints", *European Journal of Operational Research* 95 (1996) 178-190.

Gouveia L. "Using variable redefinition for computing lower bounds for minimum spanning and Steiner trees with hop constraints" *INFORMS Journal on Computing* 10 (1998) 180-188.

Gouveia L., Patrício P.F., De Sousa A.F. "Compact models for hop-constrained node survivable network design", *In: Telecommunications Network Planning: Innovations in Pricing, Network Design and Management, ed. G. Anandaligam and S. Raghavan, Springer,* (2006) 167-180.

Gouveia L., Patrício P.F., De Sousa A.F. "Hop-constrained node survivable network design: an application to MPLS over WDM", *Networks and Spatial Economics* 8 (2008) 3-21.

Gouveia L., Simonetti L., Uchoa E. "Modeling hop-constrained and diameter-constrained minimum spanning tree problems as Steiner tree problems over layered graphs", *Mathematical Programming* 128 (2011) 123-148.

Gouveia L., Requejo C. "A new Lagrangean relaxation approach for the hop-constrained minimum spanning tree problem", *European Journal of Operational Research* 132 (2001) 539-552.

Krarup J., Pruzan P.M. "The simple plant location problem: survey and synthesis", *European Journal of Operational Research* 12 (1983) 36-81.

Magnanti T.L., Wong R.T. "Network Design and Transportation Planning: Models and Algorithms", *Transportation Science* 18 (1980) 1-55.

Miller C.E., Tucker A.W., Zemlin R.A. "Integer programming formulations and traveling salesman problems", *Journal of the ACM* 7 (1960) 326-329.

Orlowsky S., Wessäly R. "The effect of hop limits on optimal cost in survivable network design", *In: Telecommunications Network Planning: Innovations in Pricing, Network Design and Management, ed. G. Anandaligam and S. Raghavan, Springer* (2006) 167-180.

Plateau G., Elkihel M. "A hybrid method for the 0-1 knapsack problem", *Methods of Operations Research* 49 (1985) 277-293.

Voß S. "The steiner tree problem with hop constraints", *Annals of Operations Research* 86 (1999) 321-345.