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Multi-Period Vehicle Routing Problem with Due Dates

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Abstract. In this paper we study the Multi-period Vehicle Routing Problem with Due dates (MVRPD), where customers have to be served between a release and a due date. Customers with due dates exceeding the planning period may be postponed at a cost. A fleet of capacitated vehicles is available to perform the distribution in each day of the planning period. The objective of the problem is to find vehicle routes for each day such that the overall cost of the distribution, including transportation costs, inventory costs and penalty costs for postponed service, is minimized. We present alternative formulations for the MVRPD and enhance the formulations with valid inequalities. The formulations are solved with a branch-and-cut algorithm and computationally compared. Furthermore, we present a computational analysis aimed at highlighting managerial insights. We study the potential benefit that can be achieved by incorporating flexibility in the due dates and the number of vehicles. Finally, we highlight the effect of reducing vehicle capacity.

Keywords: Vehicle routing problems, city distribution, multiperiod transpiration planning.

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1. Introduction

In Vehicle Routing Problems (VRPs) the transportation planning period is a day and the service day of customers is assumed to be known. In a given day customers have to be assigned to vehicles and the order of visits of each vehicle has to be determined.

Several situations exist where some flexibility on the service time is possible but the quantities to be delivered are fixed. This is the case when customers make orders and the delivery is guaranteed within a certain number of days. This is in fact one of the most common situations. Often contracts are established between supplier and customers whose cost depend on the time-to-delivery. The shorter the time-to-delivery the more expensive the contract is. Similarly, in e-commerce, customers make orders and a due date is established at the time an order is made. The time of service is a decision variable while the quantities to be delivered are given.

The real problem that motivated this study arises in city logistics. City logistics aims to reduce the nuisances associated to freight transportation in urban areas. A study on ad-hoc freight transportation systems for congested urban areas was presented in [13] while in [14] different models are presented for the evaluation and planning of city logistic systems. For a recent reference on a heuristic algorithm for a vehicle routing problem arising in city logistics we refer to [20].

There are different settings in city logistic systems. The one we consider in this paper is composed by a central distribution center (CDC) which is used to consolidate distribution activities within an urban environment. Customers are private citizens, offices or shops. Customers have made orders and request the delivery to take place within a given due date. Trucks deliver goods to the CDC where they are consolidated in vehicles dedicated for conducting urban distribution activities. The problem is how to organize the distribution of goods to final customers. Goods have to be distributed from the CDC to the customers within the due dates in such a way that the distribution cost is minimized. We refer to this problem as the Multi-period Vehicle Routing Problem with Due dates (MVRPD), where a period corresponds to a day.

The MVRPD conceptually lies between the Periodic Vehicle Routing Problem (PVRP) and the Inventory Routing Problem (IRP). In the PVRP the planning period is made of a certain number of days. A customer may request to be served one or more times in the planning period. Alternative sequences of days of visit are pre-defined for each customer. Given a sequence of days of visit, the quantities to be delivered in each day of visit are known. For example the planning period may be made of 6 days. A customer may require two visits in a week and its possible alternative sequences may be: $(1,4),(2,5),(3,6)$. The problem becomes that of choosing for each customer one sequence of days and, for each day, assigning customers to vehicles and determining for each vehicle the order of visit. Therefore, the PVRPs model the situation of customers requesting a certain frequency of service with the flexibility of choosing the precise days of service. The customers determine the service frequency and the quantities to be delivered. The flexibility in the choice of the precise sequence of days of service creates saving opportunities, but makes the problem harder to be solved. A commonly used formulation is provided in [9]. A comprehensive survey on the PVRP and its extensions can be found in [19]. The exact algorithm proposed in [4] is currently the leading methodology for the exact solution of the PVRP. For a recent reference on a heuristic algorithm for the PVRP we refer to [22]. The issue of allowing more flexibility in PVRP is studied in [17] where the PVRP with Service Choice (PVRP-SC) is introduced, that is a PVRP in which service frequency is a decision of the model. The authors propose a mathematical formulation and an exact solution approach for the problem in [17] while in [16] a continuous approximation model for the same problem is proposed. In [18] the authors developed a tabu search method for the PVRP that can incorporate a range of operational flexibility options, like the possibility to increase the set of visit schedules, decide visit frequency, vary the driver who visits a customer and decide delivery amounts per visit. The authors analyze the trade-offs between the system performance improvements due to operational flexibility and the implementation, computational and modeling complexity. They introduce quantitative measures in order to evaluate the complexity increase and provide insights both from a managerial and a modeling perspective.

In the IRPs, the planning period is made of a certain number of days, as in the PVRPs. However, the IRP includes more flexibility with respect to the PVRP. A customer may be visited any number of times and the quantities to be delivered have to be determined. The customer consumption is known, day by day, but, contrary to the VRPs and the PVRPs, the quantities to be delivered are not. The days of service and the quantities to be delivered are to be determined in such a way that a stock-out never occurs at any customer. In addition, as in VRPs and in PVRPs, for each day customers have to be assigned to vehicles and the order of visit has to be determined. Additional savings can be achieved with respect to VRPs and also with respect to the PVRPs. IRPs are interesting and challenging problems even when there is only one destination, i.e., when the routing side of the problem is trivially solved. An introduction to IRPs with a focus on the case of one origin and one destination can be found in [7], while a tutorial for the case of multiple destinations has been published shortly after in [8]. Surveys are also available, the most recent ones being [10] and [6].

The IRPs model different practical situations where the decision space is very broad. In particular, they model a management practice which is known as Vendor Managed Inventory (VMI). In VMI the supplier has regular information on the status of the inventory levels of its customers and of their consumptions and has the freedom to organize the distribution, provided that it guarantees no stock-out occurs at the customers. In the most basic IRP, customers are to be supplied over a certain number of days by a fleet of capacitated vehicles, based at a depot. Their consumption is known, day by day. Each customer has a maximum inventory capacity. Different replenishment policies may be adopted. The quantity delivered to a customer may be such that the inventory capacity is reached (Order-Up to level policy) or such that the inventory capacity is not exceeded (Maximum Level policy). The decisions include when to serve each customer (how many times and the precise days), how much to deliver when a customer is served and the routes followed by the vehicles. This problem was introduced in [5]. The first exact method for the solution of this problem was proposed in [2] for the case of one vehicle. Exact algorithms for the multi-vehicle extension were recently presented in [11], [12] and [15], while alternative formulations are compared in [3].

The decision space of the MVRPD is broader than that of the VRP, as the days of service have to be chosen, and more restricted than in IRPs, as the quantities are given. The MVRPD are close to the PVRPs but typically there is no periodicity in the service. Furthermore, we mention that in [24] and [1] the authors study the dynamic multi-period vehicle routing problem, where customers requests arrive dynamically over time and must be satisfied within a time window. The latter comprises several time periods of the planning horizon and thus resembles the due date in the MVRPD. In [24] the objective function comprises travelling cost, waiting time and balancing daily workload. In [1] the objective is to minimize travelling cost in a stochastic setting.

The contributions of this paper are fourfold. We first introduce the MVRPD. We investigate three alternative formulations and propose a set of valid inequalities for each one that exploit the problem structure. Each formulation is solved with a branch-and-cut algorithm and we identify the best one through computational experiments. Finally, we evaluate the impact of altering due dates, number of vehicles and vehicle capacity. Our analyses provide valuable managerial insights.

The rest of the paper is organized as follows. In Section 2 we develop three formulations together with valid inequalities. In Section 3 we present our computational experiments. Finally, we provide concluding remarks in Section 4.

2. Problem description and formulations

We consider a planning horizon, composed of a certain number of days. A set of customers have to be served. Each customer has placed an order that has to be satisfied within a certain due date. Multiple orders of the same customers may be modelled through different co-located customers. In the following, we will use the terms 'order' and 'customer' with the same meaning. A fleet of capacitated vehicles, based at a depot, are available to serve the customers. The goods requested by a customer may not be available at the beginning of the planning horizon but are known to become available at a later time. If the due date of a customer exceeds the planning horizon, its service may be postponed. In this case, a penalty will be charged. The latter cost is assumed to encompass the inventory holding cost of customers beyond the planning horizon. The problem is to design daily distribution routes for the given planning horizon. We refer to this problem as the Multi-period Vehicle Routing Problem with Due dates (MVRPD).

A planning horizon $T = \{1, \ldots, H\}$ is given. The MVRPD is defined on a complete directed graph $G = (V, A)$, where $V = \{1, \ldots, n\}$ is the vertex set and $A = \{(i, j) : i, j \in V\}$ is the arc set. Vertex 1 is the depot at which m identical vehicles of capacity Q are based, whereas the remaining vertices represent customers. An order quantity q_i is associated with customer i, together with a release date r_i, $1 \leq r_i \leq H$ and a due date d_i, $d_i \geq r_i$. The due date may exceed the planning horizon. If it does, the customer may be served within the planning horizon or its service may be postponed. A penalty cost p_i is charged if customer i is postponed. A nonnegative cost c_{ij} is associated with each arc $(i, j) \in V$ and represents the transportation cost incurred by travelling directly from i to j. For all periods $t \in T$ routes are constructed such that each customer order is delivered at most once by one vehicle (exactly once if the due date does not exceed the planning horizon), all routes start and end at the depot and the total quantity on any route does not exceed the vehicle capacity Q. Furthermore, the routes must be such that each customer is not served before its release date. For each customer i, an inventory holding cost h_i is charged for each day that order i spends at the depot. We assume that the depot has sufficient capacity to hold the entire demand delivered to it. We call the period $[r_i, d_i]$ the window associated with customer i. Let $C \subset \{V \setminus \{1\}\}$ be the set of customers whose due date is greater than H. Each order $i \in C$, if not served within H, incurs a holding cost $h_i[H - r_i]$ as well as the penalty cost p_i.

The objective of the problem is to minimize the total cost, which is comprised of the following elements:

(i) Transportation cost for served orders: the cost of the distance travelled by the vehicles to distribute the orders from the depot;

(ii) Inventory holding cost of served and unserved orders: the cost of holding inventory between the release date and the actual delivery day (if the order is served) or H (if the order is unserved);

(iii) Unserved order cost: the penalty cost incurred for unserved orders.

In the following sections we present three formulations for the MVRPD, each of which is followed by a set of valid inequalities. In Section 2.1 we present a flow based formulation. This is extended in Section 2.2 to include assignment variables. Finally, in Section 2.3 we present a load based formulation.

2.1 A flow based formulation

The flow based formulation makes use of the following decision variables:

$$x_{ijk}^t = \begin{cases} 1 & \text{if arc } (i,j) \text{ is traversed by vehicle } k \text{ on day } t, \\ 0 & \text{otherwise,} \end{cases}$$

and is as follows:

$$\text{(MVRPD-F1)} \quad \text{Min} \sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=r_i}^{H} c_{ij}x_{ijk}^t + \sum_{i=2}^{n} h_i \sum_{t=r_i+1}^{\min\{d_i,H\}} (t-r_i) \sum_{k=1}^{m}\sum_{j=1}^{n} x_{ijk}^t + \\ + \sum_{i\in C} \big(h_i(H-r_i)+p_i\big)\Big(1-\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=r_i}^{H} x_{ijk}^t\Big) \tag{1}$$

subject to

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=r_i}^{d_i} x_{ijk}^t = 1 \qquad \forall i \in \{V \setminus \{C,1\}\}, \tag{2}$$

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=r_i}^{H} x_{ijk}^t \le 1 \qquad \forall i \in \{C\}, \tag{3}$$

$$\sum_{i=2}^{n} q_i \sum_{j=1}^{n} x_{ijk}^t \le Q \qquad (k=1,\ldots,m),(t=1,\ldots,H), \tag{4}$$

$$\sum_{j=1}^{n} x_{ijk}^{t} = \sum_{j=1}^{n} x_{jik}^{t} \qquad (i = 1, \ldots, n), (k = 1, \ldots, m), (t = 1, \ldots, H), \tag{5}$$

$$\sum_{i,j \in S} x_{ijk}^{t} \leq |S| - 1 \qquad (S \subset V \setminus \{1\}, 2 \leq |S|), (k = 1, \ldots, m), (t = 1, \ldots, H), \tag{6}$$

$$\sum_{j=1}^{n} x_{1jk}^{t} \leq 1 \qquad (k = 1, \ldots, m), (t = 1, \ldots, H), \tag{7}$$

$$\sum_{j=1}^{n} x_{j1k}^{t} \leq 1 \qquad (k = 1, \ldots, m), (t = 1, \ldots, H), \tag{8}$$

$$x_{ijk}^{t} \in \{0, 1\} \qquad (i, j = 1, \ldots, n), (k = 1, \ldots, m), (t = 1, \ldots, H). \tag{9}$$

The objective function (1) includes the transportation cost. The term

$$\sum_{i=2}^{n} h_i \sum_{t=r_i+1}^{\min\{d_i, H\}} (t - r_i) \sum_{k=1}^{m} \sum_{j=1}^{n} x_{ijk}^{t}$$

accounts for the inventory holding cost for the served orders until H. The term

$$\sum_{i \in C} h_i (H - r_i) \Big[1 - \sum_{k=1}^{m} \sum_{j=1}^{n} \sum_{t=r_i}^{H} x_{ijk}^{t} \Big]$$

accounts for the inventory holding cost for unserved customers within the interval $[1, H]$. Finally, the term

$$\sum_{i \in C} p_i (1 - \sum_{k=1}^{m} \sum_{j=1}^{n} \sum_{t=r_i}^{H} x_{ijk}^{t})$$

expresses the penalty of unserved customers until period H. Constraints (2) ensure that each customer order, which must be delivered within H, is served within its window by a single vehicle on a single day. Constraints (3) ensure that each customer order, which could be delivered after H, is delivered at most once within its release date and H. Constraints (4) guarantee that the capacity of each vehicle is respected on each day. Constraints (5) ensure the flow conservation. Constraints (6) are the subtour elimination constraints. Constraints (7) and (8) ensure that at most one tour is performed by each vehicle per day.

Valid inequalities

In what follows we present a set of valid inequalities for the flow based formulation.

Inequality 2.1. *The inequalities*

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ijk}^t \leq mQt' \qquad (t'=1,\ldots,H),$$

are valid, since the total delivered demand until day t' should not exceed the total available vehicle capacity over the t' days.

Inequality 2.2. *The inequalities*

$$\sum_{i=1}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ijk}^t \leq Qt' \qquad (t'=1,\ldots,H),(k=1,\ldots,m)$$

are valid, since the total satisfied demand until day t' by vehicle k should not exceed the total available capacity of the vehicle over the t' days.

We note that inequalities (2.1) are obtained by aggregating (2.2) over all vehicles.

Inequality 2.3. *The inequalities*

$$\sum_{i=2}^{n} q_i - \sum_{k=1}^{m}\sum_{i=2}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ijk}^t - \sum_{i\in C}\Big(q_i - \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{H} q_i x_{ijk}^t\Big) \leq mQ(H-t') \qquad (t'=1,\ldots,H),$$

are valid.

The customer demand that is not satisfied until H is

$$\sum_{i\in C}\Big(q_i - \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{H} q_i x_{ijk}^t\Big).$$

Therefore, the unsatisfied demand until day t' which is satisfied by H is

$$\sum_{i=2}^{n} q_i - \sum_{k=1}^{m}\sum_{i=2}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ijk}^t - \sum_{i\in C}\Big(q_i - \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{H} q_i x_{ijk}^t\Big).$$

This should not exceed the residual capacity within the interval $[t'+1,H]$, which is $mQ(H-t')$.

Inequalities (2.3) can be strengthened in the following way.

Inequality 2.4. *The inequalities*

$$\sum_{i=2}^{n} q_i - \sum_{k=1}^{m}\sum_{i=2}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ijk}^t - \sum_{i\in C}\left(q_i - \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{H} q_i x_{ijk}^t\right) \leq mQ(H-t') - \sum_{k=1}^{m}\sum_{t=t'+1}^{H} Q \sum_{j=2}^{n}(1-x_{1jk}^t)\,(t'=1,\ldots,H-1)$$

are valid. The summation

$$\sum_{k=1}^{m}\sum_{t=t'+1}^{H}\sum_{j=2}^{n} x_{1jk}^t$$

expresses the number of times the vehicles are utilized within the interval $[t'+1,H]$. *The term*

$$\sum_{k=1}^{m}\sum_{t=t'+1}^{H} Q \sum_{j=2}^{n}(1-x_{1jk}^t)$$

expresses the unutilized capacity during $[t'+1,H]$. *Therefore, inequalities (2.4) follow from (2.3), while the former account for unutilized vehicle capacity within the interval* $[t'+1,H]$.

The three following inequalities are based on the required demand per interval. For each combination of periods t_1 and t_2, such that $t_1 < t_2$, let q_{t_1,t_2} denote the demand that should be delivered within $[t_1,t_2]$. This is expressed as

$$q_{t_1,t_2} = \sum_{\substack{i=1 \\ r_i\geq t_1, d_i\leq t_2,}}^{n} q_i.$$

Furthermore, we define S_{t_1,t_2} as the set of customers whose release date is greater than or equal to t_1, and whose due dates are less than or equal to t_2.

Inequality 2.5. *The inequalities*

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=t_1}^{t_2} x_{1jk}^t \geq \left\lceil \frac{q_{t_1,t_2}}{Q} \right\rceil \quad (t_1=1,\ldots,H),(t_2=t_1,\ldots,H)$$

are valid. This follows from the fact that the number of vehicles required for serving S_{t_1,t_2} *is bounded by* $\left\lceil \frac{q_{t_1,t_2}}{Q} \right\rceil$.

Inequality 2.6. *The inequalities*

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{\substack{j\in S_{t_1,t_2}\\ j\neq i}}^{n}\sum_{t=t_1}^{t_2} q_j x_{ijk}^t = q_{t_1,t_2} \quad (t_1 = 1,\dots,H),(t_2 = t_1,\dots,H)$$

are valid since all customers S_{t_1,t_2} must be served within $[t_1, t_2]$.

Inequality 2.7. *The inequalities*

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{\substack{j\in S_{t_1,t_2}\\ j\neq i}}^{n}\sum_{t=t_1}^{t_2} q_j x_{ijk}^t \leq mQ(t_2 - t_1 + 1) \quad (t_1 = 1,\dots,H),(t_2 = t_1,\dots,H)$$

are valid, since the total customer demand that must be served within $[t_1, t_2]$ should not exceed the available vehicle capacity during the interval $[t_1, t_2]$.

Furthermore, for $S \subset V$, let $\delta(S)$ denote the set of edges with one endpoint in S and the other in $\{V \setminus S\}$. We add the following rounded capacity constraints

$$\sum_{k=1}^{m}\sum_{(i,j)\in\delta(S)} x_{ijk}^t \geq 2\left\lceil \frac{\sum_{i\in S} q_i}{Q}\right\rceil \quad (t = 1,\dots,H),(S \subset V \setminus \{1\}, 2 \leq |S|).$$

2.2 A flow based formulation with assignment variables

We define the following additional assignment variables:

$$z_{ik}^t = \begin{cases} 1 & \text{if customer } i \text{ is served by vehicle } k \text{ on day } t,\\ 0 & \text{otherwise;}\end{cases}$$

and extend the flow based formulation as follows:

$$\text{(MVRPD-F2)}\quad \text{Min} \sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=r_i}^{H} c_{ij}x_{ijk}^t + \sum_{i=2}^{n} h_i \sum_{t=r_i+1}^{\min\{d_i,H\}} (t - r_i)\sum_{k=1}^{m} z_{ik}^t +$$
$$+\sum_{i\in C}\big(h_i(H - r_i) + p_i\big)\Big(1 - \sum_{k=1}^{m}\sum_{t=r_i}^{H} z_{ik}^t\Big) \qquad (10)$$

subject to

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=r_i}^{d_i} x_{ijk}^t = 1 \qquad \forall i \in \{V \setminus \{C,1\}\}, \tag{11}$$

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=r_i}^{H} x_{ijk}^t \le 1 \qquad \forall i \in \{C\}, \tag{12}$$

$$\sum_{i=2}^{n} q_i \sum_{j=1}^{n} x_{ijk}^t \le Q \qquad (k = 1,\dots,m),(t = 1,\dots,H), \tag{13}$$

$$\sum_{j=1}^{n} x_{ijk}^t = \sum_{j=1}^{n} x_{jik}^t \qquad (i = 1,\dots,n),(k = 1,\dots,m),(t = 1,\dots,H), \tag{14}$$

$$z_{ik}^t = \sum_{j=1}^{n} x_{jik}^t \qquad (i = 1,\dots,n),(k = 1,\dots,m),(t = 1,\dots,H), \tag{15}$$

$$\sum_{i,j\in S} x_{ijk}^t \le \sum_{i\in S} z_{ik}^t - z_{jk}^t \qquad (S \subset V \setminus \{1\}),(k = 1,\dots,m),(t = 1,\dots,H),\forall j \in S, \tag{16}$$

$$\sum_{j=1}^{n} x_{1jk}^t \le 1 \qquad (k = 1,\dots,m),(t = 1,\dots,H), \tag{17}$$

$$\sum_{j=1}^{n} x_{j1k}^t \le 1 \qquad (k = 1,\dots,m),(t = 1,\dots,H), \tag{18}$$

$$x_{ijk}^t \in \{0,1\} \qquad (i,j = 1,\dots,n),(k = 1,\dots,m),(t = 1,\dots,H), \tag{19}$$

$$z_{ik}^t \in \{0,1\} \qquad (i = 1,\dots,n),(k = 1,\dots,m),(t = 1,\dots,H). \tag{20}$$

The objective function (10) is adapted from the flow based formulation presented in Section 2.1 to include the assignment variables. Constraints (11)–(14) and constraints (17)–(18) directly follow the flow based formulation presented in Section 2.1. Constraints (15) link the assignment variables with the flow variables. Constraints (16) are the subtour elimination constraints formulated in terms of the z_{ik}^t variables. Note that they are more binding than (6).

Valid inequalities

In the following we present the formulation of the valid inequalities presented in Section 2.1 adapted to the flow based formulation with assignment variables.

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{t=r_i}^{\min\{t',d_i\}} q_i z_{ik}^t \leq mQt' \qquad (t' = 1,\ldots,H), \tag{21}$$

$$\sum_{i=1}^{n}\sum_{t=r_i}^{\min\{t',d_i\}} q_i z_{ik}^t \leq Qt' \qquad (t' = 1,\ldots,H), (k = 1,\ldots,m), \tag{22}$$

$$\sum_{i=2}^{n} q_i - \sum_{k=1}^{m}\sum_{i=2}^{n}\sum_{t=r_i}^{\min\{t',d_i\}} q_i z_{ik}^t - \sum_{i\in C}\Big(q_i - \sum_{k=1}^{m}\sum_{t=1}^{H} q_i z_{ik}^t\Big) \leq mQ(H-t') \qquad (t' = 1,\ldots,H), \tag{23}$$

$$\sum_{i=2}^{n} q_i - \sum_{k=1}^{m}\sum_{i=2}^{n}\sum_{t=r_i}^{\min\{t',d_i\}} q_i z_{ik}^t - \sum_{i\in C}\Big(q_i - \sum_{k=1}^{m}\sum_{t=1}^{H} q_i z_{ik}^t\Big) \leq mQ(H-t') - \sum_{k=1}^{m}\sum_{t=t'+1}^{H} Q \sum_{j=2}^{n}(1 - x_{1jk}^t)\, (t' = 1,\ldots,H-1), \tag{24}$$

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{t=t_1}^{t_2} x_{1jk}^t \geq \left\lceil \frac{q_{t_1,t_2}}{Q} \right\rceil \quad (t_1 = 1,\ldots,H), (t_2 = t_1,\ldots,H), \tag{25}$$

$$\sum_{k=1}^{m}\sum_{j\in S_{t_1,t_2}}^{n}\sum_{t=t_1}^{t_2} q_j z_{jk}^t = q_{t_1,t_2} \quad (t_1 = 1,\ldots,H), (t_2 = t_1,\ldots,H), \tag{26}$$

$$\sum_{k=1}^{m}\sum_{j\in S_{t_1,t_2}}^{n}\sum_{t=t_1}^{t_2} q_j z_{jk}^t \leq mQ[t_2 - t_1 + 1] \quad (t_1 = 1,\ldots,H), (t_2 = t_1,\ldots,H), \tag{27}$$

$$\sum_{k=1}^{m}\sum_{(i,j)\in\delta(S)} x_{ijk}^t \geq 2\left\lceil \frac{\sum_{i\in S} q_i}{Q} \right\rceil \quad (t = 1,\ldots,H), (S \subset V \setminus \{1\}, 2 \leq |S|). \tag{28}$$

2.3 A load based formulation

We present an aggregated load formulation where arc variables are aggregated over all vehicles. The formulation is based on the following decision variables:

$$x_{ij}^t = \begin{cases} 1 & \text{if arc } (i,j) \text{ is traversed on day } t, \\ 0 & \text{otherwise;} \end{cases}$$

$$l_{ij}^t = \Big\{ \quad \text{the load of the vehicles upon traversing arc } (i,j) \text{ on day } t,$$

and is expressed as follows:

$$\text{(MVRPD-F3)} \qquad \text{Minimize} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=r_i}^{H} c_{ij}x_{ij}^t + \sum_{i=1}^{n} h_i \sum_{t=r_i}^{\min\{d_i,H\}} (t-r_i)\sum_{j=1}^{n} x_{ij}^t$$

$$+\sum_{i\in C}\big(h_i(H-r_i)+p_i\big)\Big(1-\sum_{j=1}^{n}\sum_{t=r_i}^{H} x_{ij}^t\Big) \tag{29}$$

subject to

$$\sum_{j=1}^{n}\sum_{t=r_i}^{d_i} x_{ij}^t = 1 \qquad \forall i \in \{V \setminus \{C,1\}\}, \tag{30}$$

$$\sum_{j=1}^{n}\sum_{t=r_i}^{H} x_{ij}^t \leq 1 \qquad \forall i \in \{C\}, \tag{31}$$

$$\sum_{j=1}^{n} x_{ij}^t = \sum_{j=1}^{n} x_{ji}^t \qquad (i=1,\ldots,n),(t=1,\ldots,H), \tag{32}$$

$$\sum_{j=1}^{n} x_{1j}^t \leq m \qquad (t=1,\ldots,H), \tag{33}$$

$$\sum_{j=1}^{n} l_{ji}^t - \sum_{j=1}^{n} l_{ij}^t = q_i\sum_{j=1}^{n} x_{ij}^t \qquad (i=2,\ldots,n),(t=1,\ldots,H), \tag{34}$$

$$\sum_{j=1}^{n} l_{1j}^t - \sum_{j=1}^{n} l_{j1}^t = \sum_{i=1}^{n}\sum_{j=1}^{n} q_i x_{ij}^t \qquad (t=1,\ldots,H), \tag{35}$$

$$l_{ij}^t \leq Q x_{ij}^t \qquad (i,j=1,\ldots,n),(t=1,\ldots,H), \tag{36}$$

$$x_{ij}^t \in \{0,1\} \qquad (i,j = 1,\dots,n),(t = 1,\dots,H), \tag{37}$$

$$l_{ij}^t \geq 0 \qquad (i = 1,\dots,n),(t = 1,\dots,H). \tag{38}$$

The objective function (29) and constraints (30) – (32) are adapted from the flow based formulation presented in Section 2.1. Constraints (33) ensure that at most m vehicles are used each day. Constraints (34) link the load variables with the flow variables, while constraints (35) entail that for each day the difference between the total load leaving the depot and the total load entering the depot equals the total demand served on that day. Constraints (36) guarantee that the load on each arc does not exceed the vehicle capacity.

Valid inequalities

In the following we present the formulation of the valid inequalities presented in Section 2.1 adapted to the load based formulation.

$$\sum_{i=1}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ij}^t \leq mQt' \qquad (t' = 1,\dots,H), \tag{39}$$

$$\sum_{i=2}^{n} q_i - \sum_{i=2}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ij}^t - \sum_{i\in C}\Big(q_i - \sum_{j=1}^{n}\sum_{t=1}^{H} q_i x_{ij}^t\Big) \leq mQ(H-t') \qquad (t' = 1,\dots,H), \tag{40}$$

$$\sum_{i=2}^{n} q_i - \sum_{i=2}^{n}\sum_{j\neq i}\sum_{t=r_i}^{\min\{t',d_i\}} q_i x_{ij}^t - \sum_{i\in C}\Big(q_i - \sum_{j=1}^{n}\sum_{t=1}^{H} q_i x_{ij}^t\Big) \leq mQ(H-t') - \sum_{t=t'+1}^{H}(Qm - Q\sum_{j=2}^{n} x_{1j}^t)\,(t' = 1,\dots,H-1), \tag{41}$$

$$\sum_{j=1}^{n}\sum_{t=t_1}^{t_2} x_{1j}^t \geq \left\lceil \frac{q_{t_1,t_2}}{Q} \right\rceil \quad (t_1 = 1,\dots,H),(t_2 = t_1,\dots,H), \tag{42}$$

$$\sum_{i=1}^{n} \sum_{\substack{j \in S_{t_1,t_2} \\ j \neq i}}^{n} \sum_{t=t_1}^{t_2} q_j x_{ij}^t \geq q_{t_1,t_2} \quad (t_1 = 1, \ldots, H), (t_2 = t_1, \ldots, H), \tag{43}$$

$$\sum_{i=1}^{n} \sum_{\substack{j \in S_{t_1,t_2} \\ j \neq i}}^{n} \sum_{t=t_1}^{t_2} q_j x_{ij}^t \leq mQ[t_2 - t_1 + 1] \quad (t_1 = 1, \ldots, H), (t_2 = t_1, \ldots, H). \tag{44}$$

3. Computational experiments

The three formulations presented in Section 2 were coded in a C++ environment with CPLEX 12.5. All experiments were conducted on an Intel(R) Xeon(R) CPU X5675 with 12-Core 3.07 GHz and 96 GB of RAM (by using a single thread). The maximum run time for each instance was set to three hours. The polynomial time exact separation algorithm presented in [21] is used to detect violated subtour elimination constraints. Capacity cuts are separated heuristically using the extended shrinking heuristic and the greedy shrinking heuristic presented in [23].

The instances were generated by adapting the benchmark IRP instances presented in [2]. The testbed consists of 180 instances, classified into the following four categories:

- Instances with high inventory cost and $H = 6$;
- Instances with low inventory cost and $H = 6$;
- Instances with high inventory cost and $H = 3$;
- Instances with low inventory cost and $H = 3$.

Each category contained instances with $10, 15, \ldots, 50$ customers, five instances were generated for each combination of category and number of customers. Note that the IRP instances presented in [2] comprises also instances with 5 customers. We decided to skip those instances as the dimension is too small to give significant insights. Data concerning customer locations, inventory costs and vehicle capacity has been kept equal to the data of the original IRP instances. Customer demands has been set equal to the daily customer demand of the IRP instance multiplied by H. Release dates are set as follows: starting with a value $t = 1$, customers are considered sequentially and their release date is set equal to t. The value of t

is increased by one every time a new customer is considered until it reaches the value of H, after which it is set again to one. The difference between the release date and the due date of each customer is set to α, i.e., given the release date r_i for customer i, we determine its due date d_i as $r_i + \alpha$. The penalty cost p_i is set to $10h_i$. This value takes into account the inventory cost beyond H plus the fact that the customer still needs to be served and this will incur in an additional transportation cost.

The purpose of the computational experiments is twofold. In Section 3.1 we compare the performance of the three proposed formulations. In Section 3.2 we then perform several analyses examining the potential impact of relaxing and tightening the experimental parameters.

3.1 Model selection

The performance of the three formulations (presented in Section 2) is evaluated on a base case, in which for each instance the number of vehicles was set to one with $\alpha = 1$. Aside from the rounded capacity constraints, all valid inequalities are added at the route node. The results for the flow based formulation (F1), the flow based formulation with assignment variables (F2) and the load based formulation (F3), are presented in Tables 1, 2, and 3, respectively.

The number of instances solved by F1 is 101 out of 180 while F2 solved 126 instances, including all those solved by F1. Moreover, the optimality gaps and the runtimes for F2 on average are lower than those of F1.

Num. of customers	Num. of solved	Average gap	Average runtime (sec)
10	20	0.0%	0
15	20	0.0%	2.1
20	20	0.0%	27.4
25	20	0.0%	966.5
30	12	3.0%	5727.9
35	6	5.4%	8416.5
40	3	11.8%	10030.5
45	0	22.7%	10800
50	0	27.5%	10800
Total	101		
Average			5196.8

Table 1: Computational results for F1 with valid inequalities

F3 solved 167 instances, including all those solved by F1. For instances with 20 customers and more the average runtimes for F3 are substantially lower than those of F2. Therefore,

Num. of customers	Num. of solved	Average gap	Average runtime (sec)
10	20	0.0%	0
15	20	0.0%	0.9
20	20	0.0%	12.1
25	20	0.0%	180
30	19	0.2%	2014.1
35	18	0.2%	3698.3
40	7	5.6%	7663.8
45	2	12.4%	10211.2
50	0	19.8%	10800
Total	126		
Average			3842.3

Table 2: Computational results for F2 with valid inequalities

we conclude that F3 outperforms F2 and F1, on the chosen instances.

Num. of customers	Num. of solved	Average gap	Average runtime (sec)
10	20	0.0%	0
15	20	0.0%	1.1
20	20	0.0%	6.6
25	20	0.0%	28.1
30	20	0.0%	133.2
35	20	0.0%	238.8
40	20	0.0%	1477.2
45	16	1.3%	3296.8
50	11	2.0%	3975.2
Total	167		
Average			1017.4

Table 3: Computational results for F3

Num. of customers	Num. of solved	Average gap	Average runtime (sec)
10	20	0.0%	0
15	20	0.0%	2.1
20	20	0.0%	7.8
25	20	0.0%	33.4
30	20	0.0%	84.5
35	20	0.0%	382.5
40	18	0.1%	2068.4
45	13	1.6%	4705.5
50	8	3.2%	7489.9
Total	159		
verage			1641.6

Table 4: Computational results for F3 without valid inequalities

In order to assess the added value of the valid inequalities, we ran the experiments with F3 without the valid inequalities. These results can be found in Table 4. Without the valid inequalities F3 solved 159 instances, all of which were solved by F3 with the valid inequalities. In 76% of the instances solved by F3 without valid inequalities, the runtimes were longer than their corresponding value when solved by F3 with the valid inequalities. Moreover, the average gaps are higher for F3 without the valid inequalities.

The above discussion leads us to the conclusion that F3 with the valid inequalities is superior to the other presented formulations. Therefore, we opted to use this model in Section 3.2 to further analyze the behaviour of the problem when we relax and tighten some of the parameters.

3.2 Analysis

We explore here the structure of the solutions obtained. First, we analyse the distribution of costs on the tested instances. Then, we evaluate the impact of changing the parameter values with respect to the base case. We study the impact of the range of flexibility on the due dates, the impact of having more or less vehicles and, finally, the impact of having larger or smaller capacity vehicles. In what follows we focus on the 90 instances with $H = 6$, as the longer horizon allows a more careful analysis of the effects of parameter changes. In this case the number solved instances is 86 out of 90.

Cost components

We first explore the cost structure, with respect to the cost components of the objective function. Considering the solved instances for each category, Table 5 shows the percentages of transportation cost in columns two and three and the inventory cost in columns four and five. The instances are classified in two categories: the ones with high inventory cost ('high') and the ones with low inventory cost ('low'). The transportation cost is the major component of the total cost, also for the instances where the inventory cost is higher. This confirms that the instances sensibly simulate real cases.

On average the transportation costs constitute a lower portion of the total costs for instances with higher inventory costs. Naturally, the inventory costs constitute a higher portion of the total costs for instances with higher inventory costs.

Num. of customers	Transp. cost	Transp. cost	Inv. cost	Inv. cost
Category	high	low	high	low
10	86.9%	98.5%	13.0%	1.5%
15	90.4%	98.4%	9.4%	1.6%
20	91.3%	98.3%	8.5%	1.7%
25	91.3%	97.8%	8.5%	2.2%
30	91.0%	97.3%	8.7%	2.7%
35	88.7%	97.7%	11%	2.3%
40	90.1%	96.5%	9.6%	3.4%
45	86.8%	96.8%	12.9%	3.1%
50	88.7%	97.0%	10.9%	2.9%
Average	89.5%	97.7%	10.3%	2.3%

Table 5: Percentage of transportation and inventory cost

Due date flexibility

As strict due dates entail a high total cost, in this section we aim to quantify the potential saving that might be obtained by extending the due dates. Table 6 shows the results of setting $\alpha = 2$, i.e., setting the due date two days after the release date. The results from this experimental setting are compared against the base case, in which $\alpha = 1$. Similarly, Table 7 shows the results of setting $\alpha = 3$ and its comparison with the base case.

In Tables 6 and 7 the fifth column indicates the average savings with respect to the base case, as measured on instances solved by the base case as well as the setting with $\alpha = 2$ or 3, respectively. The percentage of savings is also indicated in the fifth column. The last three columns report the average percentage of savings for each of the three cost components of the objective function: transportation, inventory and penalty.

Num. of customers	Num. of solved	Average gap	Average runtime (sec)	Savings (%)	% savings		
					Transp.	Inv.	Pen.
10	10	0%	0	497.8 (14.0%)	96.4%	14.7%	-11.1%
15	10	0%	5.8	573.0 (15.9%)	132.5%	-3.3%	-29.3%
20	10	0%	34.7	841.6 (20.0%)	120.4%	-1.2%	-19.2%
25	10	0%	351.5	840.8 (17.8%)	133.6%	-11.8%	-21.9%
30	10	0%	340	871.1 (18.6%)	129.9%	-1.1%	-28.8%
35	8	1.3%	3324	890.3 (17.0%)	129.7%	7.6%	-37.4%
40	7	1.5%	4999.1	744.0 (14.2%)	146.3%	-0.5%	-45.8%
45	5	5.4%	5993.7	701.0 (11.4%)	169.6%	15.9%	-85.5%
50	3	3.8%	7936.9	638.6 (9.4%)	200.1%	2.7%	-102.8%
Total	73						
Average			1680.6	739.6 (16.2%)	132.8%	0.9%	-33.7%

Table 6: Base case with $\alpha = 2$

Table 6 indicates that the average cost savings that can be achieved by increasing due dates by one day is 16.9%, when compared to the base case. Such savings largely stem from a substantial reduction in transportation costs. Further savings are observed in inventory costs, while penalty costs are increasing. Thus, we infer that extending due dates reduces the total transportation costs by grouping clients into more efficient routes.

Num. of customers	Num. of solved	Average gap	Average runtime (sec)	Savings (%)	% savings Transp.	Inv.	Pen.
10	10	0%	1.3	859.8 (%24.9)	121.2%	9.0%	-30.3%
15	10	0%	7.3	797.4 (%22.9)	154.7%	-2.2%	-52.6%
20	10	0%	30.2	1182.4 (%28.6)	146.7%	2.5%	-49.2%
25	10	0%	339.2	1296.9 (%28.1)	150.6%	3.9%	-54.4%
30	10	0%	359.4	1228.4 (%26.6)	166.2%	3.6%	-69.8%
35	10	0%	988.5	1425.3 (%29.0)	169.0%	3.2%	-72.2%
40	7	1.6%	3902.1	647.7 (%13.3)	282.2%	30.6%	-212.8%
45	5	2.2%	5593.6	280.3 (%4.5)	770.9%	60.8%	-731.7%
50	5	3.7%	6355.3	606.3 (%8.9)	399.2%	55.7%	-354.9%
Total	77						
Average			1354.8	998.3 (%22.9)	181.9%	8.1%	-90.0%

Table 7: Base case with $\alpha = 3$

The results observed for $\alpha = 2$ are further amplified for the case of $\alpha = 3$. These results indicate that increasing due dates can have a large influence on the distance travelled by vehicles.

Vehicle flexibility

A potential cost reduction can be achieved if we increase the number of vehicles. Thus, we performed experiments with $m = 2$ and $m = 3$, the results of which are reported in Tables 8 and 9, respectively.

Num. of customers	Num. of solved	Average gap	Average runtime (sec)	Savings (%)	% savings Transp.	Inv.	Pen.
10	10	0%	0	171.9 (%4.8)	26.4%	73.6%	0.0%
15	10	0%	0.3	130.6 (%3.3)	68.2%	31.8%	0.0%
20	10	0%	4.9	59.0 (%1.6)	82.4%	17.6%	0.0%
25	10	0%	13.4	62.9 (%1.3)	160.1%	-60.1%	0.0%
30	10	0%	221.7	69.9 (%1.4)	159.7%	-59.7%	0.0%
35	10	0%	121.2	75.7 (%1.6)	105.7%	-5.7%	0.0%
40	10	0%	1129.8	66.7 (%1.3)	185.3%	-85.3%	0.0%
45	7	1%	3890.1	51.4 (%1.1)	104.9%	-4.9%	0.0%
50	7	1.1%	5943.2	29.7 (%0.5)	124.6%	-24.6%	0.0%
Total	84						
Average			997	82.6 (%2)	95.5%	4.5%	0.0%

Table 8: Base case with $m = 2$

The average saving achieved by having two vehicles is 2%, which is mainly due to savings in transportation costs and increases in inventory costs. A total of 84 instances are also solved to optimality for the setting with two vehicles, 83 of which were solved by the setting with one vehicle. A total of 86 instances are also solved to optimality for the setting with three vehicles, 85 of which were solved by the with one vehicle. Furthermore, 84 instances were solved to optimality for the setting with two vehicles as well as the setting with three vehicles. In these 84 instances, the savings achieved by using two or three vehicles were identical. Furthermore, in 44 out of the 84 instances no cost reduction was observed, with respect to the base case. These results indicate that while using two vehicles may yield a cost reduction, there is no apparent advantage of using three vehicles.

					% savings		
Num. of customers	Num. of solved	Average gap	Average runtime (sec)	Savings (%)	Transp.	Inv.	Pen.
10	10	0%	0	171.9 (%4.8)	26.4%	73.6%	0.0%
15	10	0%	0.9	130.6 (%3.3)	68.2%	31.8%	0.0%
20	10	0%	3.8	59.0 (%1.6)	82.4%	17.6%	0.0%
25	10	0%	12.4	62.9 (%1.3)	160.1%	-60.1%	0.0%
30	10	0%	163	69.9 (%1.4)	159.7%	-59.7%	0.0%
35	10	0%	80.6	75.7 (%1.6)	105.7%	-5.7%	0.0%
40	10	0%	1164.6	66.7 (%1.3)	185.3%	-85.3%	0.0%
45	9	40%	2519.6	40.0 (%0.8)	104.8%	-4.8%	0.0%
50	7	120%	1721.2	29.7 (%0.5)	124.6%	-24.6%	0.0%
Total	86						
Average			569.5	80.6 (%1.9)	95.5%	4.5%	0.0%

Table 9: Base case with $m = 3$

As previously mentioned, extending due dates yields substantial cost reduction. This reduction is much greater than that achieved by using two vehicles. In order to observe the potential cost reduction that can be achieved by extending due dates and using more vehicles, we experimented with the setting with $\alpha = 2$ and $m = 2$, the results of which are reported in Table 10. We note that the difference between the average savings reported in Table 6 and those reported in Table 10 is 1.2%. This result implies that while extending due dates warrants a crucial cost reduction, using additional vehicles entails little added value.

Num. of customers	Num. of solved	Average gap	Average runtime (sec)	Savings (%)	% savings		
					Transp.	Inv.	Pen.
10	10	0%	0.2	599.3 (%17)	85.9%	23.4%	-9.2%
15	10	0%	7.2	631 (%17.3)	125.0%	1.6%	-26.6%
20	10	0%	160.4	884.7 (%21)	120.5%	-2.3%	-18.3%
25	10	0%	928.9	911.6 (%19.4)	129.8%	-9.7%	-20.2%
30	10	0%	401.4	910.1 (%19.4)	131.2%	-3.7%	-27.5%
35	7	150.3%	3215.7	903.5 (%16.9)	130.4%	10.5%	-40.9%
40	7	110.2%	2108.4	764.4 (%14.3)	146.4%	-2.7%	-43.8%
45	5	170%	1135.5	701 (%11.4)	169.6%	15.9%	-85.5%
50	4	121.4%	3594.5	809.2 (%12.6)	161.3%	1.6%	-62.9%
Total	73						
Average			990.5	791.6 (%17.4)	129.3%	2.1%	-31.4%

Table 10: Base case with $\alpha = 2$ and $m = 2$

Vehicle capacity flexibility

The use of the smaller vehicles should increase cost. However, this may be offset by the advantages of using lighter vehicles, e.g., reducing emissions. To examine the potential cost increase due to a decrease in vehicle capacity we experimented with three different settings, consisting of 10%, 20% and 30% reduction in vehicle capacity. The details of the experiments are in the Appendix.

Table 11 summarizes the results for the three experimental settings. More instances are likely to become infeasible as the capacity decreases. In columns two, four and six we report the number of solved instances and the number of feasible instances for each of the three settings. Columns three, five, and seven contain the average increase in cost, in value and as percentage, when compared to the base case. The cost increase is computed only on instances that were solved in the base case as well as in the reduced capacity setting.

The number of feasible instances decreases from 90 to 88 if vehicle capacity is reduced by 10%, and to 73 if vehicle capacity is reduced by 30%. We observe that the infeasible instances tend to appear in instances with a low number of customers. A 10% reduction in vehicle capacity causes an average cost increase of 3.6%, whereas a 30% reduction in vehicle capacity causes an average cost increase of 15.8%.

Finally, we experimented with capacity reductions for the base case when extending due dates with $\alpha = 2$. We compare this case with the base case in order to validate if the flexibility in due dates can balance the decrease in vehicle capacity. The results of these experiments are summarized in Table 12 and the details of the experiments are in the Appendix. Despite the reduced vehicle capacity, average savings are achieved. This is explained by the fact that extending the deadlines offsets the decrease in vehicle capacity. The number of feasible instances decreases from 90 to 84 only if vehicle capacity is reduced by 30%. A 10% reduction in vehicle capacity, coupled with $\alpha = 2$, causes an average saving of 12.4%, whereas a 30% reduction in vehicle capacity, coupled with $\alpha = 2$, causes a saving of 1.1%.

	10% capacity reduction		20% capacity reduction		30% capacity reduction	
Num. of customers	Num. of solved (feasible)	Cost increase (%)	Num. of solved (feasible)	Cost increase(%)	Num. of solved (feasible)	Cost increase (%)
10	8(8)	124.9(3.7)	2(2)	287.3(7.7)	0(0)	0(0)
15	10(10)	296.5(8.5)	10(10)	572.3(15.7)	2(2)	970.7(31.6)
20	10(10)	198.3(4.7)	10(10)	485.3(11.3)	10(10)	1017.6(23.6)
25	10(10)	146.5(3.1)	10(10)	347.7(7.5)	10(10)	899(18.6)
30	10(10)	123.7(2.8)	10(10)	276.9(6.1)	10(10)	533(11.7)
35	10(10)	169.9(3.4)	10(10)	431.3(8.7)	5(10)	886.1(16.7)
40	8(10)	110.3(2.2)	5(10)	276.7(5.4)	3(10)	688.9(12.3)
45	7(10)	67.8(1.2)	7(10)	122.2(2.2)	3(10)	433.1(7.1)
50	5(10)	28.7(0.4)	5(10)	107.4(1.5)	4(10)	220.2(3.2)
Total	78(88)		69(82)		48(73)	
Average		151.9(3.6)		354.9(8.1)		731.6(15.8)

Table 11: Base case with 10%, 20% and 30% capacity reduction

	10% capacity reduction		20% capacity reduction		30% capacity reduction	
Num. of customers	Num. of solved (feasible)	Cost increase (%)	Num. of solved (feasible)	Cost increase(%)	Num. of solved (feasible)	Cost increase (%)
10	10(10)	-371.7(-10.6)	10(10)	-62.6(-1.5)	4(4)	267(8.7)
15	10(10)	-371.6(-10.2)	10(10)	-122.1(-3.2)	10(10)	227.7(6.6)
20	10(10)	-622.4(-15.1)	10(10)	-477.7(-11.7)	10(10)	-153(-4.1)
25	10(10)	-665.9(-14.3)	10(10)	-506.4(-10.9)	10(10)	-148.6(-3.6)
30	10(10)	-759.4(-16.1)	10(10)	-607.6(-12.8)	8(10)	-445.3(-9.1)
35	7(10)	-612.1(-12.1)	9(10)	-569.5(-11.5)	4(10)	36.6(0.6)
40	6(10)	-496.4(-9.1)	6(10)	-354.2(-6.6)	2(10)	139(2.8)
45	4(10)	-573(-9.6)	5(10)	-500.2(-8.2)	3(10)	-87.2(-1.6)
50	4(10)	-735.9(-10.7)	4(10)	-708.1(-10.3)	2(10)	-379.2(-5.8)
Total	71(90)		74(90)		53(84)	
Average		-569.1(-12.4)		-410.1(-8.5)		-72.2(-1.1)

Table 12: Base case with $\alpha = 2$, 10%, 20% and 30% capacity reduction

4. Conclusions

The MVRPD captures the operations of many distributors that deliver products from a CDC to customers, within predetermined due dates. The distribution activities are planned over several days, the main decisions relate to which customers to visit on each day and in what order. Products kept at the warehouse entail inventory holding costs. The MVRPD aims to balance transportation cost, inventory cost, as well as penalty costs, incurred as a result of unserved demand within the planing horizon. As such, the MVRPD balances the flexibility of choosing to serve a customer within an interval of consecutive days with the cost of keeping the its demand at the CDC.

We proposed three formulations for the MVRPD: a flow based formulation, a flow based formulation with assignment variables and a load based formulation. For each of the three formulations we developed a series of valid inequalities. We generated a test bed for the MVRPD, which we used to examine the performance of each of the formulations. The load based formulation substantially outperformed the two other formulations, in terms of the number of solved instances and of the average runtimes.

We performed a number of analyses with the aim of understanding how the MVRPD solutions would be altered to accommodate changes in the input parameters. Through a series of experiments we demonstrated how the model may provide managerial insights with respect to such changes. Based on the results of the considered instances, we conclude that substantial cost savings can be achieved by extending customer due dates. Such additional flexibility allows for improved routing decisions, that yield a considerable reduction in the travelled distance. Despite the fact that the due dates are exogenous parameters, in reality distribution companies periodically negotiate contracts with their customers. Therefore, the MVRPD is paramount in quantifying the added value of extending due dates. Our experiments also showed that using additional vehicles does not yield a substantial reduction in the considered operational costs. Finally, our results indicated that reducing vehicle capacity causes a cost increase but this can be balanced by due dates flexibility. Future research could incorporate environmental factors in the model, while accounting for more complex distribution networks.

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Appendix

Num. of customers	Num. of solved (feasible)	Average gap	Average runtime (solved)(sec)
110	8(8)	0%	0
15	10(10)	0%	0.2
20	10(10)	0%	6.1
25	10(10)	0%	22.3
30	10(10)	0%	96.2
35	10(10)	0%	621.5
40	8(10)	0.6%	4236.1
45	7(10)	0.8%	4460.6
50	5(10)	1.5%	6135
Total	78(88)		
Average			1323.7

Table 13: Computational results for base case with 10% capacity reduction

Num. of customers	Num. of solved (feasible)	Average gap	Average runtime (solved)(sec)
10	2(2)	0%	0
15	10(10)	0%	0
20	10(10)	0%	7.3
25	10(10)	0%	23.9
30	10(10)	0%	211.2
35	10(10)	0%	1126.6
40	5(10)	1.7%	6462.9
45	7(10)	2.4%	4088.6
50	5(10)	2.4%	6552.2
Total	69(82)		
Average			1556.3

Table 14: Computational results for base case with 20% capacity reduction

Num. of customers	Num. of solved (feasible)	Average gap	Average runtime (solved)(sec)
10	0(0)		0
15	2(2)	0%	0
20	10(10)	0%	7.1
25	10(10)	0%	130.9
30	10(10)	0%	423
35	5(10)	1.4%	6992
40	3(10)	4%	9389
45	3(10)	3.2%	7965.7
50	4(10)	4.1%	8091.2
Total	48(73)		
Average			2604.1

Table 15: Computational results for base case with 30% capacity reduction

Num. of customers	Num. of solved (feasible)	Average gap	Average runtime (solved)(sec)
10	10(10)	0%	0
15	10(10)	0%	6
20	10(10)	0%	54.3
25	10(10)	0%	384.1
30	10(10)	0%	455.6
35	7(10)	1%	5060.1
40	6(10)	2.4%	5365.2
45	4(10)	6.1%	7049.6
50	4(10)	4.9%	7088.5
Total	71(90)		
Average			1875.6

Table 16: Computational results for base case with 10% capacity reduction and $\alpha = 2$

Num. of customers	Num. of solved (feasible)	Average gap	Average runtime (solved)(sec)
10	10(10)	0%	0
15	10(10)	0%	5.8
20	10(10)	0%	29.5
25	10(10)	0%	202.5
30	10(10)	0%	738
35	9(10)	0.3%	4517.3
40	6(10)	3.2%	6115.5
45	5(10)	4.9%	6400.2
50	4(10)	6.8%	6781.2
Total	74(90)		
Average			1976.1

Table 17: Computational results for base case with 20% capacity reduction $\alpha = 2$

Num. of customers	Num. of solved (feasible)	Average gap	Average runtime (solved)(sec)
10	4(4)	0%	0
15	10(10)	0%	5.6
20	10(10)	0%	78.6
25	10(10)	0%	458.4
30	8(10)	0.5%	2915.2
35	4(10)	2.9%	7948.8
40	2(10)	6.7%	9114.2
45	3(10)	6.6%	9099.8
50	2(10)	8.2%	8889.1
Total	53(84)		
Average			2336.8

Table 18: Computational results for base case with 30% capacity reduction $\alpha = 2$