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Comparing Regret Minimization and Utility Maximization for Route Choice using the Recursive Logit Model†

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Abstract. This paper focuses on the comparison of estimation and prediction results between the random utility maximization (RUM) and random regret minimization (RRM) frameworks for the route choice recursive logit (RL) model (Fosgerau et al., 2013). The RL model is originally based on the RUM principle. We propose different versions of the RL model based on the RRM by adapting and extending the model proposed by Chorus (2014). We report estimation results and a cross-validation study for a real network with more than 3000 nodes and 7000 links. The cross-validation results show that one of the proposed extended versions of the RRM-based model has the best out-of-sample fit. While this observation favors the RRM framework, we note that the RRM-based models are computationally more time consuming to estimate and the parameter estimates are less straightforward to interpret than the RUM-based ones.

Keywords: Route choice modeling, recursive logit, random utility maximization, random regret minimization, maximum likelihood estimation, cross-validation.

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1 Introduction

Discrete choice models are generally used for analyzing path choices in real networks based on revealed preference (RP) data. Following the discussion in Fosgerau et al. (2013) the route choice models in the literature can be grouped into three approaches. The classical approach corresponds to path logit (PL) models where choice sets of paths are generated and treated as the actual choice sets. The second approach, proposed by Frejinger et al. (2009), is based on the idea that the choice set can be sampled and the estimation can be consistent if sampling corrections are added to the choice probabilities. Recently, Fosgerau et al. (2013) proposed a third approach, called the recursive logit (RL) model which can be consistently estimated based on RP data and used for prediction without sampling any choice sets of paths. Another extension of the RL model, the nested RL (NRL) model, has been proposed by Mai et al. (2015) that allows to relax the independence of irrelevant alternatives (IIA) property. These models are based on the Random Utility Maximization (RUM) framework.

Recently, Chorus (2010, 2012, 2014) proposed the Random Regret Minimization (RRM) framework. It is based on a regret minimization-based decision rule postulating that when decision makers choose between alternatives, they try to avoid the situation where a non-chosen alternative outperforms a chosen one in terms of attributes. We base this paper on the so-called Generalized Random Regret Minimization (GRRM) model proposed by Chorus (2014), and we adapt and compare the estimation and prediction results of the RL model using RRM and RUM decision rules.

Prato (2012) analyses the estimation results of path based models using the model proposed by Chorus (2010) in a route choice context. He focuses on the two well-known challenges associated with route choice modeling, namely, choice set generation and correlation. He finds that the RRM performs well on real data, but in an experimental setting, he finds that the parameter estimates of the RRM models have the wrong signs when irrelevant alternatives are included in the choice sets. The RL model does not need choice set generation, being based on the universal choice set of all paths connecting an origin-destination pair. We investigate whether RL model presents similar issues to path based models and we analyze the out-of-sample fit.

This paper makes a number of contributions. First, we adapt and propose two specifications for random regret. The first model (called Extended Random Regret Minimization - ERRM) extends the GRRM model by adding factors that allow to capture the impact of the non-chosen alternatives in a more flexible way and the second model, called Average Random Regret Minimization (ARRM) model, modifies the first one by adding a normalization. We prove that by specifying some parameters, the regret given by the ARRM model model has a linear-in-parameters form $R_{ni} = -\beta^T x_{ni}$ and the probability is equivalent to a RUM-based model with utility $V_{ni} = \beta^T x_{ni}$. This model therefore generalizes the RUM-based RL model. Second, we show how the RRM-based RL models can be estimated. Third, we provide estimation and cross-validation results for a real network with over 3000 nodes and 7000 links. The estimation code for the RRM-based RL models is implemented in MATLAB and is freely available upon request.

The paper is structured as follows. In Section 2 we review the RUM- and RRM-based models. Section 3 proposes the RL model using the regret decision rules with two different formulas for the regrets. In Section 6 we give details about the model estimation by maximum likelihood. Model specifications as well as estimation and cross-validation results are presented in Section 7, and finally, Section 8 concludes.

2 Random utility maximization and random regret minimization models

In the context of the RUM-based discrete choice models, we assume that an individual n associates a utility U_{ni} with an alternative i within a choice set C_n. The utility consists of two additive parts $U_{ni} = V_{ni} + \epsilon_{ni}$: a deterministic V_{ni} part, observed by the modeler, and a random part ϵ_{ni}. Typically, a linear-in-parameters formula associated with a vector of attributes is used i.e. $V_{ni} = \beta^T x_{ni}$, where β is a vector of parameters to be estimated and x_{ni} is a vector of attributes with respect to individual n and alternative i. A decision maker chooses the alternative that maximizes his/her utility

$$i^* = \text{argmax}_{i \in C_n} \{V_{ni} + \epsilon_{ni}\}.$$

The well-known multinational logit (MNL) model assumes that the random terms ϵ_{ni} are independently and identically distributed (i.i.d.) extreme value type I, and the probability of choosing an alternative i is $P_n(i) = \frac{e^{V_{ni}}}{\sum_{j \in C_n} e^{V_{nj}}}$. The MNL model however retains the IIA property, so other models may be preferred in order to better capture the correlations between random terms e.g. the nested logit model (Ben-Akiva, 1973), cross-nested logit model (Vovsha and Bekhor, 1998) or network multivariate extreme value model (Daly and Bierlaire, 2006). Mai et al. (2015) propose a nested version of the RL model that relaxes the IIA property. However, in route choice applications, MNL based models such as path size logit (Ben-Akiva and Bierlaire, 1999) are often used.

The RRM-based models are based on the assumption that when decision makers choose between alternatives, they try to avoid the situation where a non-chosen alternative outperforms a chosen one in terms of one or more attributes. This translates into a regret function for a considered alternative that by definition features all attributes of all competing alternatives. The random regret RR_{ni} can be written as the sum of a systematic part R_{ni} and a random error term ϵ_{ni},

$$RR_{ni} = R_{ni} + \epsilon_{ni} = \sum_{j \neq i, j \in C_n} \sum_t \ln\left(1 + e^{\beta_t (x_{nj}(t) - x_{ni}(t))}\right) + \epsilon_{ni} \tag{1}$$

where t is an attribute. So the regret R_{ni} is computed based on two sums, the first sum is over all other alternatives in the choice set and the second over all the attributes. We note that (1) is proposed by Chorus (2012). Other alternative formulations for the regret have been presented in the literature, for instance Chorus (2014) that we present in the following. Contrary to the RUM-based models, a decision maker aims to minimize the random regret

$$i^* = \text{argmin}_{i \in C_n} \{R_{ni} + \epsilon_{ni}\} = \text{argmax}_{i \in C_n} \{-R_{ni} - \epsilon_{ni}\}. \tag{2}$$

Under the assumption that the random terms $-\epsilon_{ni}$ are i.i.d extreme value type I, the choice probability is given by the MNL

$$P_n(i) = \frac{e^{-R_{ni}}}{\sum_j e^{-R_{ni}}}.$$

It is important to note that even though this is the logit model, the IIA property is relaxed since the regrets are not alternative specific. Chorus (2014) recently presented a generalization of the RRM model given in (1), called Generalized Random Regret Minimization - GRRM, where the random regret can be expressed as

$$GRR_{ni} = \sum_{j \neq i, j \in C_n} \sum_t \ln\left(\lambda_t + e^{\beta_t (x_{nj}(t) - x_{ni}(t))}\right) + \epsilon_{ni}.$$

Indeed, we can recover the RRM model (1) by fixing λ_t to 1 for all t. Moreover, as pointed out in Chorus (2014), if $\lambda_t = 0 \; \forall t$ the resulting regret becomes linear-in-parameters

$$GRR_{ni} = \sum_{j \neq i, j \in C_n} \sum_t \beta_t (x_{nj}(t) - x_{ni}(t)) = \sum_{j \in C_n} \beta^T x_{nj} - |C_n| \beta^T x_{ni},$$

where $|\cdot|$ is the cardinality operator. The term $\sum_{j \in C_n} \beta^T x_{jn}$ being the same whatever the considered alternative i, it does not affect the choice (2). The regret has a linear-in-parameter formulation but it is different from the RUM-based model because of $|C_n|$.

A disadvantage of the RRM or GRRM model, highlighted in Chorus (2012), is that the running time for computing the choice probabilities increases exponentially as the choice sets become larger. Indeed, every alternative is compared with every other in terms of each attribute. This can hence be a problem for route choice applications which are characterized by large choice sets. The RL model is based on the universal choice set (of infinite size) but the choice set at each choice stage is small (outgoing links at a node). This is therefore not an issue for the RL model.

Finally, we note that the random regrets GRR_{in} and RR_{in} are undefined when the choice set C_n is singleton. This is an issue for the RL model if the transport network contains only one outgoing link for some nodes. As we explain in the following, we deal with this issue by summing over all alternatives.

3 Recursive logit with regret-based models

The RUM-based RL (RL-RUM) model formulates the path choice problem as a sequence of link choices, represented in a dynamic discrete choice framework. A utility is associated with each link pair in the network, and is the sum of a deterministic and a random term. Fosgerau et al. (2013) consider a linear-in-parameters formulation of the deterministic utility. A traveler maximizes his/her value function, defined as the sum of the instantaneous link utility at the current decision stage and the expected maximum utility from the sink node of outgoing links to the destination. In the following we present the RRM-based RL model. The derivation is similar to Fosgerau et al. (2013) but the utilities and value functions are different since they are based on random regret minimization.

A directed connected graph (not assumed acyclic) $\mathcal{G} = (\mathcal{A}; \mathcal{V})$ is considered, where $\mathcal{A}$ and $\mathcal{V}$ are the set of links and nodes, respectively. For each link $k \in \mathcal{A}$, we denote the set of outgoing links from the sink node of k by $A(k)$. We extend the network with a dummy link d, without successors, per destination, that is, an absorbing state. The set of all links for a given destination is hence $\tilde{\mathcal{A}} = \mathcal{A} \cup \{d\}$. Given two links $a, k \in \tilde{\mathcal{A}}$, $a \in A(k)$, we associate the following instantaneous random regret for individual n

$$rr_n(a|k) = r_n(a|k) + \mu\epsilon_n(a)$$

where $r_n(a|k)$ is the deterministic regret of link a given k, $-\epsilon_n(a)$ are i.i.d. extreme value type I distributed error terms and μ is a strictly positive scale parameter. We ensure that $\epsilon_n(a)$ have zero mean by subtracting Euler's constant. For notational simplicity, we omit from now on the index for individual n but note that the regrets and random terms can be individual specific.

At each current state k the traveler observes the realizations of the random terms $\epsilon(a)$, $a \in A(k)$. He/she then chooses link a that minimizes the sum of instantaneous random regret $rr(a|k)$ and expected downstream regret. The latter, denoted by $R^d(k)$, is defined as the expected minimum regret from state k to the

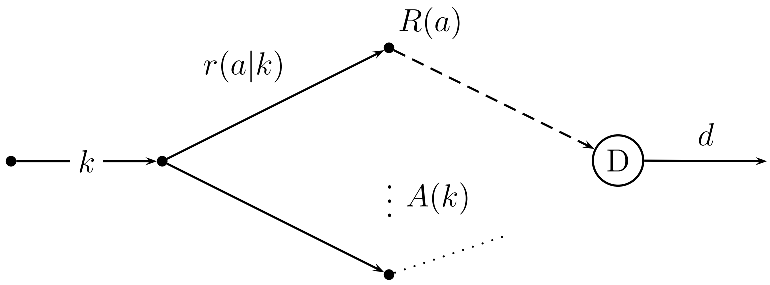

Figure 1: Illustration of notation

destination (see Figure 1). The superscript d indicates that the expected minimum regrets are destination specific (through dummy link d). $R^d(k)$ is recursively defined by Bellman's equation as

$$R^d(k) = \mathbb{E}\left[\min_{a\in A(k)} \left\{r(a|k) + R^d(a) + \mu\epsilon(a)\right\}\right], \quad \forall k \in \mathcal{A}. \tag{3}$$

We note that $R^d(k)$ and $r(a|k)$ may be conditional on the model parameters so they can be written as $R^d(k) = R^d(k;\beta)$ and $r(a|k) = r(a|k;\beta)$ where β is the parameters to be estimated. We however omit β for notational simplicity. Equation (3) can be written as

$$\begin{aligned} R^d(k) &= \mathbb{E}\left[-\max_{a\in A(k)} \left\{-r(a|k) - R^d(a) - \mu\epsilon(a)\right\}\right] \\ &= -\mathbb{E}\left[\max_{a\in A(k)} \left\{-r(a|k) - R^d(a) + \mu(-\epsilon(a))\right\}\right], \quad \forall k \in \mathcal{A}, \end{aligned}$$

or equivalently

$$\frac{1}{\mu}R^d(k) = -\mathbb{E}\left[\max_{a\in A(k)} \left\{\frac{1}{\mu}(-r(a|k) - R^d(a)) + (-\epsilon(a))\right\}\right], \quad \forall k \in \mathcal{A} \tag{4}$$

Since $-\epsilon(a)$ are i.i.d. standard extreme value type I by assumption, the probability of choosing link a given k is given by the MNL model

$$P^d(a|k) = \frac{\xi(a|k)e^{-\frac{1}{\mu}(r(a|k)+R^d(a))}}{\sum_{a'\in A(k)} e^{-\frac{1}{\mu}(r(a'|k)+R^d(a'))}}, \quad \forall a, k \in \tilde{\mathcal{A}}. \tag{5}$$

Note that we include $\xi(a|k)$ that equals one if $a \in A(k)$ and zero otherwise so that the probability is defined for all $a, k \in \tilde{\mathcal{A}}$ (we recall that $\tilde{\mathcal{A}} = \mathcal{A} \cup \{d\}$). Since the choice at each state is the MNL model, the expected minimum regrets in this case are given recursively by the logsum

$$-\frac{1}{\mu}R^d(k) = \mathbb{E}\left[\max_{a\in A(k)}\left\{\frac{1}{\mu}(-r(a|k) - R^d(a)) + (-\epsilon(a))\right\}\right]$$
$$= \ln\left(\sum_{a\in A(k)} e^{\frac{1}{\mu}\left(-r(a|k)-R^d(a)\right)}\right), \forall k \in \mathcal{A}, \tag{6}$$

and $R^d(d) = 0$ by assumption. We define a matrix M^d of size $|\tilde{\mathcal{A}}| \times |\tilde{\mathcal{A}}|$ and a vector z of size $|\tilde{\mathcal{A}}|$ with entries

$$M^d_{ka} = \xi(a|k)e^{-\frac{1}{\mu}r(a|k)},\ z^d_k = e^{-\frac{1}{\mu}R^d(k)}, \forall k, a \in \tilde{\mathcal{A}}. \tag{7}$$

So from (6) we have

$$z^d_k = \begin{cases} \sum_{a\in\mathcal{A}} M^d_{ka} z^d_a & k \in \mathcal{A} \\ 1 & k = d \end{cases}. \tag{8}$$

The system in (8) can be written in matrix form as

$$z^d = M^d z^d + b \text{ or } z^d = (I - M^d)^{-1} b, \tag{9}$$

where b is a vector of size $|\tilde{\mathcal{A}}|$ with zeros values for all states except for the destination d that equals 1 and I is the identity matrix. So similar to Fosgerau et al. (2013) we obtain a system of linear equations which can be solved in short computational time. Fosgerau et al. (2013) discuss the existence of a solution to the Bellman's equation for the RL-RUM model and this can be applied in the context of the RRM-based models. In essence, the existence of a solution depends on the size of the scaled instantaneous regrets and on the balance between the number of paths connecting the nodes in the network. It is easy to find a feasible solution by using large enough magnitude of the model parameters. Note that if the scales μ are different over links, the system in (8) becomes nonlinear-in-parameters, similar to the one given in Mai et al. (2015).

Using (5), the probability of choosing link a given a state k can be written as

$$P^d(a|k) = \xi(a|k)e^{-\frac{1}{\mu}(r(a|k)+R^d(a)-R^d(k))}, \quad \forall k, a \in \tilde{\mathcal{A}}$$

and the probability of a path defined by a sequence of links $\sigma = [k_0, \ldots, k_J]$ is

$$P^d(\sigma) = \prod_{i=0}^{J-1} P^d(k_{i+1}|k_i) = e^{\frac{1}{\mu}R^d(k_0)} \prod_{i=0}^{J-1} e^{-\frac{1}{\mu}r(k_{i+1}|k_i)} = e^{\frac{1}{\mu}R^d(k_0)} e^{-\frac{1}{\mu}r(\sigma)},$$

where $r(\sigma) = \sum_{i=0}^{J-1} r(k_{i+1}|k_i)$. Given two paths σ_1 and σ_2, the ratio between two probabilities is

$$\frac{P(\sigma_2)}{P(\sigma_2)} = e^{\frac{1}{\mu}(r(\sigma_2)-r(\sigma_1))}$$

and it does not depend only on the attributes of links on paths σ_1, σ_2. Hence, the IIA property does not hold for the RRM-based models. In the following section we discuss different formulations for the link regrets.

4 Link regret formulations

We define the regret $r(a|k)$ of link $a \in A(k)$ conditional on link $k \in \mathcal{A}$, based on the GRRM model (Chorus, 2014). It is important to consider that $A(k)$ may contain only one link. Existing random regret models would in this case assign a regret zero which could cause numerical issues for the RL model. We therefore define the regret based on all outgoing links and the slightly modified GRRM is

$$r^{GRRM}(a|k) = \sum_{a' \in A(k)} \sum_t \ln\left(\lambda_t + e^{\beta_t (x(a'|k)_t - x(a|k)_t)}\right), \ \forall k \in \mathcal{A}, a \in A(k), \tag{10}$$

where $x(a|k)$ is a vector of attributes associated with link a given k, λ and β are vector of parameters to be estimated. The only difference here with respect to the model in Chorus (2014) is that the first sum is over all alternatives.

We also define a new formulation for regret that we call Extended Random Regret Minimization (ERRM) to capture the impact of non-chosen alternatives in a more flexible way

$$r^{ERRM}(a|k) = \sum_{a' \in A(k)} \sum_t \ln\left(\lambda_t + e^{\beta_t (x(a'|k)_t - x(a|k)_t) + \delta_t x(a'|k)_t}\right), \ \forall k \in \mathcal{A}, a \in A(k). \tag{11}$$

The difference lies in the term $\delta_t x(a'|k)_t$. If $\delta_t > 0$, the impact of the non-chosen alternatives becomes larger and if $\delta_t < 0$, it is smaller, compared to the GRRM model. Moreover, if $\delta_t = 0$ we obtain the GRRM formulation.

By specifying $\lambda_t = 0$ and $\delta_t = -\beta_t$, for all attributes t, the regret given by the ERRM model becomes

$$r^{ERRM}(a|k) = -|A(k)|\beta^T x(a|k) = -|A(k)|v(a|k),$$

where $v(a|k)$ are the linear-in-parameters utilities as in Fosgerau et al. (2013). So the regret is also linear-in-parameters but different from the RUM based model with a factor $|A(k)|$. This factor appears because the sum in the regret formula is over all the outgoing links from the sink node of k. We propose an alternative of the ERRM model where a normalization factor is used so that the regret is averaged over all the alternatives

$$r^{ARRM}(a|k) = \frac{1}{|A(k)|} r^{ERRM}(a|k),\ \forall k \in \mathcal{A}, a \in A(k). \tag{12}$$

We refer to this model as Averaged Random Regret Minimization (ARRM). Accordingly, by specifying $\lambda_t = 0$ and $\delta_t = -\beta_t$, $\forall t$, we obtain $r^{ARRM}(a|k) = -v(a|k)$. Based on (7) the entries of matrix M^d becomes

$$M^d_{ka} = \xi(a|k) e^{\frac{1}{\mu} v(a|k)}, \forall k, a \in \tilde{\mathcal{A}}.$$

We refer to the definition of the matrix M^d in Fosgerau et al. (2013) and note that z^d is a solution to the system of linear equations $z^d = (I - M^d)^{-1} b$, therefore it is straightforward to show that

$$z^d_k = e^{-\frac{1}{\mu} R^d(k)} = e^{\frac{1}{\mu} V^d(k)}, \quad \forall k \in \tilde{A},$$

where $V^d(k)$ is the expected maximum utility from state k to the destination. The probability of choosing a link a given link k can be written as

$$\begin{aligned} P^d(a|k) &= \xi(a|k) e^{-\frac{1}{\mu}(r(a|k) + R^d(a) - R^d(k))} \\ &= \xi(a|k) e^{\frac{1}{\mu}(v(a|k) + V^d(a) - V^d(k))}. \end{aligned}$$

This choice probability is equivalent to the one given by the RL-RUM model. So the RL model based on ARRM model generalizes the RL-RUM model.

5 Illustrative example

In this section, similar to several studies in the literature (for instance Ben-Akiva and Bierlaire, 1999, Mai et al., 2015), we use a simple three path network shown in Figure 2 to illustrate how the RRM-based RL model relax the IIA property. The network consists of three paths connecting link o (origin) and dummy link d (destination), namely $[o, a, d]$, $[o, b, e, d]$, $[o, c, e, d]$. We number these paths 1, 2 and 3 and the corresponding path probabilities are $P1$, $P2$ and $P3$, respectively. Only a link length attribute is included in the deterministic regrets and the values are given in the parentheses on each arc.

First, we note that both the RL-RUM and path-based MNL RUM and RRM models assign the probability 1/3 to the three paths. We illustrate the choice probabilities given by the GRRM, ERRM and ARRM RL models in Figure 3. We vary λ over the interval $[0, 1]$ for the three models. For the GRRM model, β varies

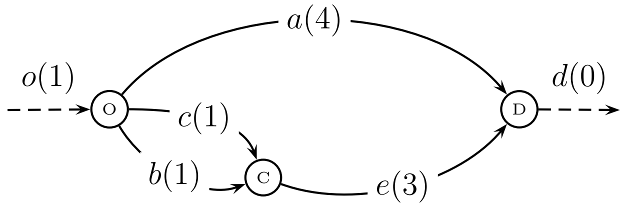

Figure 2: Classic three paths example network

over $[-2, 3]$ and for the ERRM and the ARRM models we vary δ over the interval $[-1, 5]$ keeping β fixed to -1. First we note that the path probabilities are indeed no-longer equal. If the value of β or δ is small, $P2$ and $P3$ are close to $1/2$ and $P1$ approaches zero. On the contrary, if β or δ increases, $P2$ and $P3$ tend to 0 and $P1$ tends to 1. Recall that ARRM becomes the RUM-based model when $\delta = -\beta$ and $\lambda = 0$. This is indeed the case since $P1 = P2 = P3 = 1/3$ when $\delta = -\beta = 1$ and $\lambda = 0$. Moreover, the results from this example suggest that the impact of parameter λ on the path probabilities is small, compared to other parameters.

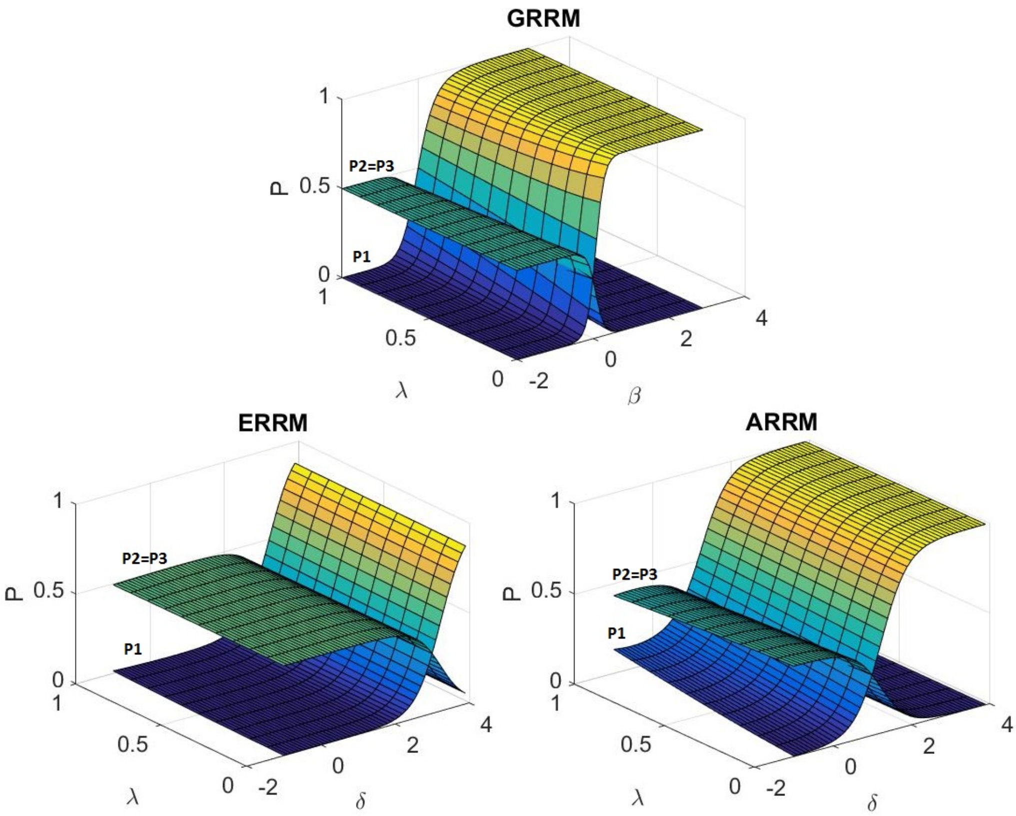

Figure 3: Path probabilities

Finally, we note that the regrets given by the RRM-based models are nonlinear function of the model parameters. This could add more complexity to the path probabilities, compared to the linear-in-parameters RUM models. This becomes even clearer when we plot them as functions of β and δ (see Figure 4). This leads to the fact that RRM-based models are more complicated and time consuming to estimate than linear-in-parameters RUM-based models, which is confirmed by our numerical results.

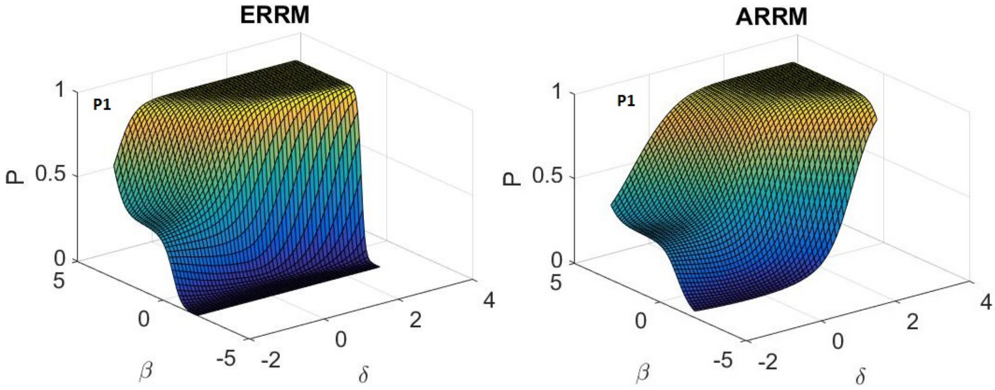

Figure 4: $P1$ as function of β and δ

6 Maximum likelihood estimation

There are different ways of estimating a dynamic discrete choice model (see for instance Aguirregabiria and Mira, 2010). Similar to Fosgerau et al. (2013) and Mai et al. (2015) we use the nested fixed point algorithm proposed by Rust (1987). This algorithm combines an outer iterative nonlinear optimization algorithm for searching over the parameter space with an inner algorithm for solving the expected minimum regrets. The expected minimum regrets can be solved quickly using the system of linear equations in (9). We therefore turn our attention to the definition of the log-likelihood function as well as its derivatives.

The log-likelihood function defined for N observations $\sigma_1, \ldots, \sigma_N$ with respect to the vector of model parameters β is

$$LL(\beta) = \sum_{n=1}^{N} \ln P(\sigma_n) = \frac{1}{\mu} \sum_{n=1}^{N} \sum_{i=0}^{J_n - 1} \left(R(k_0^n) - r(\sigma_n) \right).$$

For notational simplicity we omit the superscript d indicating the destinations but note that the choice probabilities $P(\sigma_n)$ and expected minimum regrets $R(k_0^n)$ depend on the destination of path σ_n. Efficient nonlinear techniques for the problem require analytical derivatives of the log-likelihood function. We therefore derive the gradient of $LL(\beta)$ with respect to a parameter β_i as

$$\frac{\partial LL(\beta)}{\partial \beta_i} = \frac{1}{\mu}\sum_{n=1}^{N}\sum_{i=0}^{J_n-1}\left(\frac{\partial R(k_0^n)}{\partial \beta_i} - \frac{\partial r(\sigma_n)}{\partial \beta_i}\right),$$

which requires the derivatives of $R(k_0^n)$. We differentiate (9) which yields

$$\frac{\partial z}{\partial \beta_i} = (I - M)^{-1}\frac{\partial M}{\partial \beta_i}z \quad \text{and using} \quad \frac{\partial R(k)}{\partial \beta_i} = -\mu\frac{\partial z_k}{z\partial \beta_i}. \tag{13}$$

The gradient of the regret value function $R(k)$, $k \in \tilde{A}$ can be quickly computed using the system of linear equations (13). The value of $r(\sigma)$ for a given path σ is nonlinear-in-parameters, so that $\frac{\partial r(\sigma)}{\partial \beta_i}$ has a complicated form but is easy to derive. We note that from (10), (11) and (12) the regret-based models have three vector of parameters to be estimated i.e. λ, β and δ. The GRRM model requires $0 \leq \lambda_t \leq 1$ for all attributes t. This implies that the MLE becomes a constrained optimization problem as in the following

$$\max_{\substack{\lambda,\beta,\delta \\ 0\leq\lambda_t\leq 1,\,\forall t}} LL(\lambda, \beta, \delta).$$

We use the interior point algorithm with BFGS to solve this constrained problem. The code is implemented in MATLAB (available upon request) and we use the function *fmincon* for solving the problem. More precisely, the following MATLAB commands are used to maximize the log likelihood function

```
options =
    optimoptions(@fmincon,'Algorithm','interior-point','GradObj','on');
[x,fval] = fmincon(@f,x,[],[],[],[],lb,ub,[],options);
```

where f is the objective function, which is the opposite of the log-likelihood, x is the vector of the parameters to be estimated and lb, ub are two vectors of the lower and upper bounds of the model parameters.

7 Numerical results

In order to have comparable numerical results with previous studies, we use the same data as Fosgerau et al. (2013) (also used in Frejinger and Bierlaire, 2007, Mai et al., 2014, 2015), collected in the city of Borlänge, Sweden. This network is composed of 3077 nodes and 7459 links and it is uncongested so travel times are assumed static and deterministic. There are 1832 observations containing 466 destinations, 1420 different origin-destination (OD) pairs and more than 37,000 link choices. Moreover, as we explained in the following we specify the link regret functions using the same attributes as Fosgerau et al. (2013) and Mai et al. (2015).

7.1 Model specifications

Four attributes are included in the regret function: travel time $TT(a)$ of link a, number of left turn $LT(a|k)$ that equals one if the turn angle from k to a is larger than 40 degrees and less than 177 degrees, link constant $LC(a)$ that equals one except the dummy link which equals zero and U-turn $UT(a|k)$ that equals one if the turn angle is larger than 177.

For the sake of comparison we report the estimation and prediction results for the RUM-based RL (Fosgerau et al., 2013) and NRL (Mai et al., 2015) models, their deterministic utility specifications are

$$\begin{aligned} v^{\text{RL}}(a|k;\beta) = v^{\text{NRL}}(a|k;\beta) &= \beta_{TT}TT(a) + \beta_{LT}LT(a|k) + \beta_{LC}LC(a) \\ &\quad + \beta_{UT}UT(a|k) \\ v^{\text{RL-LS}}(a|k;\beta) = v^{\text{NRL-LS}}(a|k;\beta) &= \beta_{TT}TT(a) + \beta_{LT}LT(a|k) + \beta_{LC}LC(a) \\ &\quad + \beta_{UT}UT(a|k) + \beta_{LS}LS(a) \end{aligned}$$

where LS is the link size attribute (for a detailed description see Fosgerau et al., 2013). It has been computed using a linear-in-parameters formulation of the aforementioned four attributes using parameters $\tilde{\beta}_{TT} = -2.5$, $\tilde{\beta}_{LT} = -1$, $\tilde{\beta}_{LC} = 0.4$, $\tilde{\beta}_{UT} = -4$. This attribute can be considered as a correction for the utilities in order to relax with the IIA property from the RL model. The NRL model has the same instantaneous utility but the IIA is relaxed by allowing the random terms to have link specific scale parameters.

The regret specifications for the RRM-based models can be defined based on (10), (11) and (12), respectively, using the same four attributes as the RUM-based models. There is however an important difference related to the LC attribute. In the RUM-based model, the rationale behind using $LC(a)$ in the instantaneous utilities is to penalize paths with many crossings (links). In the regret context, the link constant equals one except for the dummy link which equals zero. So this attribute cancels out when comparing two outgoing links except when comparing a link in $\mathcal{A}$ with dummy link d. More precisely, for each link $k \in \mathcal{A}$, the regret for the ERRM model can be expressed as

$$r^{\text{ERRM}}(a|k) = \sum_{t, t \neq LC} \sum_{a' \in A(k)} \left(\lambda_t + e^{\beta_t (x(a'|k)_t - x(a|k)_t) + \delta_t x(a'|k)_t} \right) + \psi(a|k)_{LC}, \quad (14)$$

where

$$\psi(a|k)_{LC} = \begin{cases} \sum_{a' \in A(k)} \ln(\lambda_{LC} + e^{\delta_{LC}}),\ \forall a \in A(k), a \neq d, d \notin A(k) \\ \sum_{\substack{a' \in A(k) \\ a' \neq d}} \ln(\lambda_{LC} + e^{\delta_{LC}}) + \ln(\lambda_{LC} + e^{-\beta_{LC}+\delta_{LC}}),\ \text{if } d \in A(k), a \neq d \\ \sum_{a' \in A(k)} \ln(\lambda_{LC} + e^{\beta_{LC}+\delta_{LC}}),\ \text{if } a = d \end{cases} \tag{15}$$

Equations (14) and (15) indicate that the value of β_{LC} only affects the regret $r^{\text{ERRM}}(a|k)$ if link k connects directly to dummy link d. The GRRM and ARRM regrets can be written in a similar way. Consequently, the link constant in the RRM-based models plays a different role from the one in the RUM-based models; it is an attraction factor at the destination. Such a factor is actually important for the RRM-based models to ensure that the probability of choosing the destination link (once arriving at the destination) is close to one. Such an attraction attribute is not needed (and does not affect the probabilities) in the RUM-based models since the instantaneous utilities are negative except for the dummy link that is zero. In order to make the distinction clear between these attributes, we call it *destination constant* (DC) in the RRM-based models. Accordingly, the regrets for the three RRM models are

$$\begin{aligned}
r^{\text{ERRM}}(a|k) = \sum_{a' \in A(k)} \Big\{ & \ln(\lambda_{TT} + e^{\beta_{TT}(TT(a') - TT(a)) + \delta_{TT} TT(a')}) \\
& + \ln(\lambda_{LT} + e^{\beta_{LT}(LT(a'|k) - LT(a|k)) + \delta_{LT} LT(a'|k)}) \\
& + \ln(\lambda_{DC} + e^{\beta_{DC}(DC(a') - DC(a)) + \delta_{DC} DC(a')}) \\
& + \ln(\lambda_{UT} + e^{\beta_{UT}(UT(a'|k) - UT(a|k)) + \delta_{UT} UT(a'|k)}) \Big\} \\
& \forall k \in \mathcal{A}, a \in A(k), \\
r^{\text{ARRM}}(a|k) = {} & \frac{1}{|A(k)|} r^{\text{ERRM}}(a|k),\ \forall k \in \mathcal{A}, a \in A(k), \\
r^{\text{GRRM}}(a|k) = \sum_{a' \in A(k)} \Big\{ & \ln(\lambda_{TT} + e^{\beta_{TT}(TT(a') - TT(a))}) \\
& + \ln(\lambda_{LT} + e^{\beta_{LT}(LT(a'|k) - LT(a|k)}) \\
& + \ln(\lambda_{DC} + e^{\beta_{DC}(DC(a') - DC(a))}) \\
& + \ln(\lambda_{UT} + e^{\beta_{UT}(UT(a'|k) - UT(a|k))}) \Big\}, \\
& \forall k \in \mathcal{A}, a \in A(k).
\end{aligned} \tag{16}$$

7.2 Estimation results

The estimation results for the three models are presented in Table 2. The $\widehat{\beta}$ are significantly different from zero except for the parameter associated with u-turns in the ARRM model. Moreover, they are, as expected, negative for travel time, left turns and u-turns. Based on the discussion in the previous section, it is also expected that $\widehat{\beta}_{DC}$ are positive and with large magnitudes so that $P(d|k)$ are close to one.

We now turn our attention to the δ estimates. Note that the parameters δ are designed only for the ERRM and ARRM models. If $\widehat{\delta_t} > 0$, the impact of non-chosen alternatives is larger than if $\widehat{\delta_t} < 0$. The estimation results show that $\widehat{\delta_t}$ are either not significantly different from zero, or they are significant and positive ($\widehat{\delta}_{TT}$ in the ARRM and $\widehat{\delta}_{TT}$, $\widehat{\delta}_{UT}$ in the ERRM model). It means that the impacts of the non-chosen alternatives in the ERRM is larger than the GRRM model in terms of travel time and u-turns.

We recall that if $\lambda_t = 0$ the regret associated with attribute t is linear-in-parameters and if $\lambda_t = 1$ the regret becomes the original RRM model proposed by Chorus (2012). The four last rows of Table 2 show the λ estimates. We do not provide standard errors and t-tests for the estimates that are on the bounds (close to 0 or 1) since the respective gradient values are not close to zero. For the GRRM model, the $\widehat{\lambda}_t$ are only significantly different from zero for the parameters associated with u-turns and destination constant. The others are very to zero (on the bound). However, for the ERRM or ARRM models, the $\widehat{\lambda}_t$ are either on the bounds ($\widehat{\lambda}_{LT}$, $\widehat{\lambda}_{UT}$ for the ARRM and $\widehat{\lambda}_{TT}$, $\widehat{\lambda}_{LT}$, $\widehat{\lambda}_{UT}$ for the ERRM), or not significantly different from zero ($\widehat{\lambda}_{TT}$, $\widehat{\lambda}_{DC}$ for the ARRM and $\widehat{\lambda}_{DC}$ for the ERRM). Chorus (2014) provide more detailed discussions on how the regrets change when parameters λ vary in the interval $[0, 1]$.

We report the final log-likelihood values in Table 1. In general, the differences in in-sample fit between the RUM-based and RRM-based models cannot be statistically compared with a likelihood ratio test since they are not nested. In the RUM-based, the NRL models have significantly better fit than the other RL models, and the models with LS attribute are better than the ones without. Among the RRM-based models, the ERRM performs better than the GRRM. Moreover, since the RL-RUM model is a restricted model of the ARRM model, the results show that the ARRM has significantly better fit than the RL-RUM. Finally, we note that the ERRM has the highest and the GRRM has the lowest final log-likelihood value.

Before discussing the out-of-sample fit of these models in the following section, we make some remarks about the computational time for estimation. The RRM-based models are more difficult to estimate than the RUM-based models due to the nonlinearity in the regrets. The nonlinear optimization algorithm needs approximate 30 iterations to converge for the RUM-based models while from 300 to 500 iterations are needed for the RRM-based models.

	RL	RL-LS	NRL	NRL-LS	GRRM	ARRM	ERRM
# parameters	4	5	7	8	8	12	12
Final log-likelihood	-6303.9	-6045.6	-6187.9	-5952.0	-7931.6	-5661.6	-5500.4

Table 1: Final log-likelihood values

7.3 Prediction results

In this section we report results from a cross-validation study. The objective is to compare the out-of-sample fit of the models which is useful to detect over-fitting and assess prediction performance.

Similar to Mai et al. (2015), the sample of observations is repeatedly divided into two sets by drawing observations at random with a fixed probability: one set contains 80% of the observations and is used for estimation and the other (20%) is used as holdout samples to evaluate the predicted probabilities by applying the estimated model. We generate 40 holdout samples of the same size by reshuffling the real sample and use the log-likelihood loss as the loss function to evaluate the prediction performance.

For each holdout sample i, $0 \leq i \leq 40$ we estimate the parameters $\widehat{\beta}_i$ of the corresponding training sample and these parameters are used to compute the test errors err_i

$$err_i = -\frac{1}{|PS_i|} \sum_{\sigma_j \in PS_i} \ln P(\sigma_j, \widehat{\beta}_i)$$

where PS_i is the size of the prediction sample i. We then compute the average of err_i over samples in order to have unconditional test error values

$$\overline{err}_p = \frac{1}{p} \sum_{i=1}^{p} err_i \quad \forall 1 \leq p \leq 40. \tag{17}$$

For comparison we also report the predictions performances of the four RUM-based models.

The values of $\overline{err}_p$, $1 \leq p \leq 40$ are plotted in Figure 5 and Table 3 reports the average of the test error values over 40 samples. As expected, the value of $\overline{err}_p$ for each model stabilizes as p increases. The results show that the ERRM model performs best (lowest value of the loss function). The performance of the ERRM model is very different from GRRM that has the worst performance. Interestingly, the ARRM has a final log-likelihood value (in-sample fit) that is almost 300 units

Parameters	GRRM	ARRM	ERRM
$\widehat{\beta}_{TT}$	-0.15	-1.92	-0.37
Rob. Std. Err.	0.01	0.21	0.09
Rob. t-test(0)	-11.46	-8.98	-4.05
$\widehat{\beta}_{LT}$	-0.34	-1.80	-0.31
Rob. Std. Err.	0.02	0.41	0.08
Rob. t-test(0)	-15.36	-4.43	-3.84
$\widehat{\beta}_{UT}$	-5.89	-7.32	-5.32
Rob. Std. Err.	0.57	65.32	1.87
Rob. t-test(0)	-10.32	-0.11	-2.85
$\widehat{\beta}_{DC}$	12.92	99.99	23.18
Rob. Std. Err.	1.66	36.03	3.79
Rob. t-test(0)	7.77	2.77	6.11
$\widehat{\delta}_{TT}$	-	3.75	1.22
Rob. Std. Err.	-	0.46	0.21
Rob. t-test(0)	-	8.18	5.69
$\widehat{\delta}_{LT}$	-	0.12	0.09
Rob. Std. Err.	-	0.70	0.10
Rob. t-test(0)	-	0.17	0.89
$\widehat{\delta}_{UT}$	-	7.16	4.75
Rob. Std. Err.	-	53.66	1.31
Rob. t-test(0)	-	0.13	3.62
$\widehat{\delta}_{DC}$	-	-7.16	-1.44
Rob. Std. Err.	-	63.49	1.73
Rob. t-test(0)	-	-0.11	-0.84
$\widehat{\lambda}_{TT}$	8.13e-6	0.37	1.00
Rob. Std. Err.	-	0.31	-
Rob. t-test(0)	-	1.20	-
$\widehat{\lambda}_{LT}$	7.26e-6	1.00	8.29e-5
Rob. Std. Err.	-	-	-
Rob. t-test(0)	-	-	-
$\widehat{\lambda}_{UT}$	0.76	0.01	1.04e-4
Rob. Std. Err.	0.04	-	-
Rob. t-test(0)	17.86	-	-
$\widehat{\lambda}_{DC}$	0.46	0.58	0.48
Rob. Std. Err.	0.02	37.06	0.81
Rob. t-test(0)	21.62	0.02	0.59

Table 2: *Estimation results*

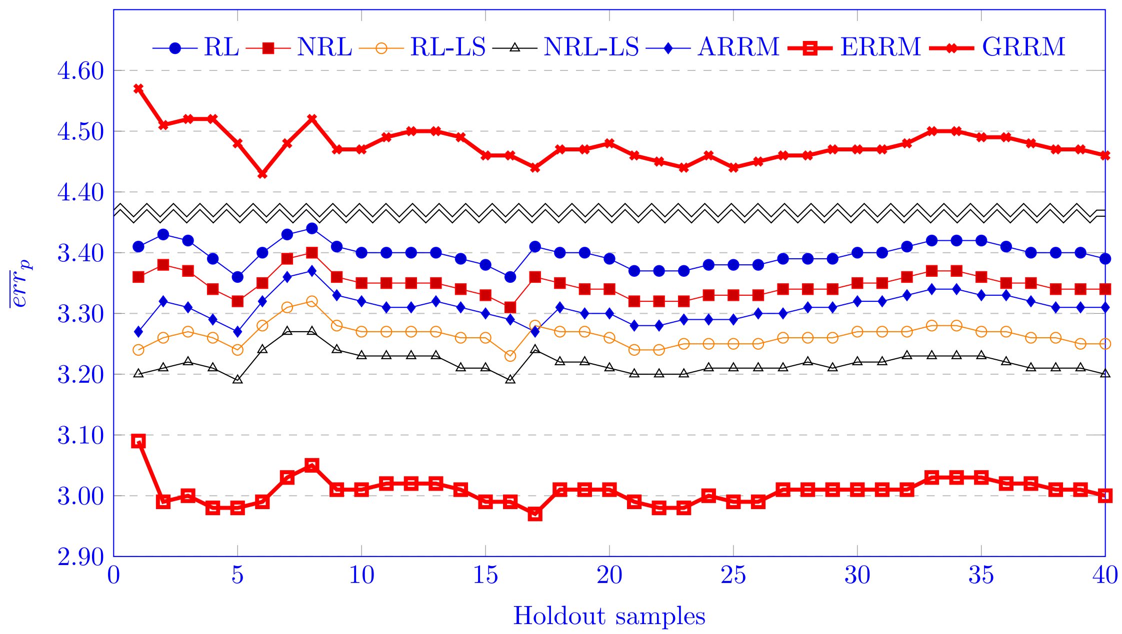

Figure 5: Average of the test error values over holdout samples

better than the best RUM-based model (NRL-LS) but the prediction performance is worse than both NRL-LS and RL-LS.

We note that all the RRM-based models considered in this paper are based on the sums over all alternatives due to the fact that there can be only one outgoing at a node, which could cause numerical issues in the RL model. This is, however, not the case for the Borlänge network.

RL	RL-LS	NRL	NRL-LS	GRRM	ARRM	ERRM
3.39	3.25	3.36	3.20	4.46	3.31	3.00

Table 3: Average of test error values over 40 holdout samples

8 Conclusion

In this paper we have compared estimation results and prediction performance between RUM– and RRM–based RL route choice models. We adapted the GRRM model proposed by Chorus (2014) and propose two variants: ARRM and ERRM models. We provided numerical results and a cross-validation study using real data and a network with more 3000 nodes and 7000 links. The cross-validation results indicated that the ERRM model performs the best (it also has a higher final log-likelihood value in the estimation) and the performance of the GRRM model is worse. The superiorities of the ARRM and ERRM models, compared to the GRRM, suggested that it is important to flexibly capture the impacts of the non-chosen alternatives in the regrets of the GRRM.

The numerical results indicated that RRM rule may be an interesting avenue for route choice modeling. Moreover, the results from the ARRM model (the in-sample and out-of-sample fit is better than the RL-RUM model) showed that non-linear utility specifications should be investigated for RUM-based RL models. It is however important to note that the estimation and application of the RRM-based models are more complicated and time consuming than the RUM ones. Moreover, the interpretation of the parameter estimates are less straightforward. Finally, other specifications of the RRM-based models have been recently proposed (for instance van Cranenburgh et al., 2015), which could be interesting to investigate.

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