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A Shortest-Path Algorithm for the Departure Time and Speed Optimization Problem

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Abstract. We present a shortest-path algorithm for the Departure Time and Speed Optimization Problem (DSOP) under traffic congestion. The objective of the problem is to determine an optimal schedule for a vehicle visiting a fixed sequence of customer locations in order to minimize a total cost function encompassing emissions cost and labor cost. We account for the presence of traffic congestion which limits the vehicle speed during peak hours. We show how to cast this problem as a shortest path problem by exploiting some structural results of the optimal solution. We illustrate the solution method and discuss some properties of the problem.

Keywords: Speed optimization, shortest path, scheduling.

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1 Introduction

In recent years there has been a growing interest in models and algorithms for routing and scheduling problems in which fuel consumption and related CO_{2e} emissions are taken into account. This line of research is rooted in the works of Fagerholt et al. (2010), Norstad et al. (2011) and Hvattum et al. (2013) in the field of maritime transportation, and in those of Bektaş and Laporte (2011) and Demir et al. (2012) in the field of road transportation. The first set of papers seek to optimize the routing and scheduling of a vessel visiting a set of ports on a path subject to arrival time restrictions. In particular, Hvattum et al. (2013) proposed an exact algorithm to optimize the cruise speed of the vessel in order to minimize the fuel consumption while ensuring that ships arrive within the time windows. This problem is referred to as the *Speed Optimization Problem* (SOP). The second set of papers tackles a similar problem of optimizing the routing and scheduling of a fleet of vehicles that must serve a set of customers on a path with hard time windows for the start of service at customer locations. In this variant, the vehicles may arrive before the opening of the time windows. These authors consider a more global objective function that includes labor and fuel costs. In this context, Demir et al. (2012) adapted the algorithm proposed by Hvattum et al. (2013) to optimize the travel speed of a vehicle on each arc of the path. More recently Kramer et al. (2015) developed a quadratic-time algorithm to solve this problem to optimality. Franceschetti et al. (2013) extended the model from Demir et al. (2012) to allow for traffic congestion, which limits the vehicle speed during peak hours. They refer to the problem as the *Departure Time and Speed Optimization Problem* (DSOP). The consideration of traffic congestion greatly complicates the analysis since it may now be optimal for the vehicle to wait idly at the depot or at customer locations in order to mitigate the negative impact of slow congestion speed on emission costs. These authors proposed a polynomial-time heuristic algorithm which builds upon the closed-form solution they develop for the single-arc version of the problem.

In this paper, we develop a shortest-path based algorithm for the DSOP under traffic congestion, which is exact and efficient. Hence we extend the work from Kramer et al. (2015) by considering traffic congestion and that of Franceschetti et al. (2013) by computing an optimal solution. Note that our algorithm can be used a subroutine to optimize individual routes in some vehicle routing and scheduling problems with speed considerations.

The remainder of the article is organized as follows. In §2 we present our model and discuss feasibility conditions. In §3 we establish key structural results of the optimal solution, which we exploit to build our shortest path formulations, presented in §4. We illustrate our solution method in §5 and conclude with a summary of our contributions in §6. All proofs are presented in the Appendix.

2 Model

A single vehicle departs from an origin location called the *depot* to visit a number of customer locations according to a fixed route. For example, a delivery vehicle leaves from a central warehouse to visit retail locations and deliver merchandise, or a plumber leaves her office to visit customer homes in order to make repairs. Let 0 denote the depot, let locations 1 to n represent customers, and let $n+1$ denote a copy of the original location, which represents the return of the vehicle to its starting point.[1] Hence, the fixed route is $(0, 1, ..., n+1)$. We refer to arc $(i, i+1)$ as arc i for $i = 0, ..., n$. Let d_i denote the distance on arc i and h_i denote the service time at location i for $i = 0, ..., n$. In the delivery vehicle example, the service time would correspond to the time spent loading the vehicle with merchandise and the service time at each customer location would be the unloading and delivery time. Each customer location $i \in \{1, ..., n\}$ has a hard time window $[l_i, u_i]$ within which service must start. The vehicle may arrive at the customer location earlier than l_i but service may not start until that time. Let μ_i be the arrival time of the vehicle at location i for $i = 0, ..., n+1$. We set $\mu_0 = 0$, which corresponds to the start of the planning horizon. Note that this value does not necessary correspond to the time at which the driver arrives to the depot. We set $l_0 = l_{n+1} = 0$ and $u_{n+1} = \infty$ for notational convenience. We refer to the time between the arrival time at a customer location and the start of service as the *pre-service waiting time*. This waiting time can be divided into the *mandatory pre-service waiting time*, which is equal to $(l_i - \mu_i)^+$, and the *voluntary pre-service waiting time*, which is denoted by y_i. If the vehicle arrives at the customer location before the lower time window limit, i.e., $\mu_i < l_i$, then the *mandatory pre-service waiting time* is $l_i - \mu_i$, and the *voluntary pre-service waiting time* is equal to the difference between the start of service at location i and l_i. If the vehicle arrives at location i after the lower time window limit, i.e., $\mu_i \geq l_i$, then the *mandatory pre-service waiting time* is zero, and the *voluntary pre-service waiting time* is

[1] All of our results would also apply with minor modifications if the last node is also a customer node and there is no return to the depot or if the vehicle must return to the depot within a certain time period.

equal to the difference between the start of service and the arrival time μ_i. Let y_0 denote the time elapsed between the start of service at the depot and the start of the planning horizon. Upon completion of service at a location, the driver may decide to wait before leaving; let z_i denote the *post-service waiting time* at location $i = 0, ..., n$. The trip ends with the arrival at location $n+1$, that is, with the return to the depot. Finally, let ρ_i denote the departure time from location i for $i = 0, ..., n$. We have $\rho_0 = \mu_0 + y_0 + h_0 + z_0$ and $\rho_i = \max\{\mu_i, l_i\} + y_i + h_i + z_i$ for $i = 1, ..., n$.

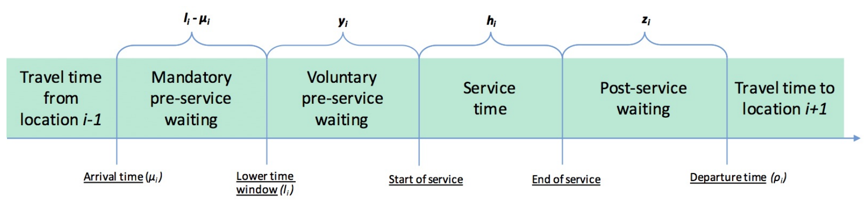

Figure 1: Sequence of events at customer location i.

Figure 1 depicts the sequence of events at customer i. In this example, the vehicle arrives before l_i, so that there is a positive mandatory pre-service waiting time. Also, because service does not start immediately at l_i, there is a positive voluntary pre-service waiting time. Finally the vehicle does not leave the location immediately after the completion of service, so there is also a positive post-service waiting time.

As in Franceschetti et al. (2013) and Jabali et al. (2012), we assume that the planning horizon starts with an initial period of traffic congestion lasting a time units during which the vehicle travels at a constant speed v_{con}. After this period of traffic congestion ends, the driver can choose its vehicle's speed between the limits v^{min} and v^{max}. We refer to this period as the *free-flow period*. Let v_i denote the chosen free-flow speed on arc i for $i = 0, ..., n$. Note that v_i is only defined if location $i+1$ is reached past the end of the congestion period, that is, if $\mu_{i+1} \geq a$.

Let $T_i(\rho_i, v_i)$ denote the total travel time to traverse arc i if the vehicle leaves location i at time ρ_i and chooses a free-flow speed v_i (if applicable). We divide this time between the time spent driving in congestion and the time spent driving in free-flow as follows: we set $T_i(\rho_i, v_i) = T_i^{con}(\rho_i) + T_i^{free}(\rho_i, v_i)$ where $T_i^{con}(\rho_i) = \min\{(a - \rho_i)^+, d_i/v_{con}\}$ is the time spent traveling in congestion on arc i and $T_i^{free}(\rho_i, v_i) = [d_i - (a - \rho_i)^+ v_{con}]^+ / v_i$ is the time spent traveling at the free-flow speed v_i on arc i when departing at time ρ_i from location i. The arrival time at location $i+1$ is then calculated as $\mu_{i+1} = \rho_i + T_{i+1}(\rho_i, v_i)$.

The decision maker makes two types of decision: how fast to drive on each arc during the free-flow period and how much to wait at each location, either pre- or post-service. We assume that waiting can only take place at the locations, not between them. Hence an optimal schedule is fully characterized by three vectors: $\mathbf{y} = (y_0, ..., y_n)$, $\mathbf{z} = (z_0, ..., z_n)$ and $\mathbf{v} = (v_0, ..., v_n)$, which are the voluntary pre-service waiting time vector, the post-service waiting time vector and the free-flow speed vector, respectively. Table 2 summarizes our notation.

Notation	Description
d_i	arc i distance
l_i	lower time window limit at location i
u_i	upper time window limit at location i
U_i	effective upper time window limit at location i (see §2.3)
h_i	service time at location i
y_i	voluntary pre-service waiting time at location i
z_i	post-service waiting time at location i
μ_i	arrival time of the vehicle at location i
ρ_i	departure time from location i
$\underline{\mu}_i$	earliest possible arrival time at location i (see §2.3)
v_i	chosen free-flow speed on arc i
v_{con}	congestion speed
v^{min}	minimum free-flow speed
v^{max}	maximum free-flow speed

Table 1: Notation

2.1 Objective function

The decision maker aims at minimizing total cost, which is the sum of: (i) the labor cost, i.e. the cost of paying the driver of the vehicle and (ii) the cost of carbon dioxide equivalent (CO_{2e}) emissions, which are directly proportional to the amount of fuel used during the trip. As such, our objective function is identical to that of Franceschetti et al. (2013).

In line with Franceschetti et al. (2013), we consider two ways of calculating the total time for which the driver is paid. In the *early policy* the driver is paid starting from the start of the planning horizon, i.e., time zero, until the arrival time at location $n+1$; in the *late policy* the driver is paid starting from the start of the service at the depot until the arrival time at location

$n+1$. Under the *early policy*, the driver reports to the depot at the start of the planning horizon, which coincides with the start of a typical work day, but may have to wait before starting her service and driving duties (during this time she may be asked to perform some administrative duties). Under the *late policy*, the driver arrives at the depot right on time to start service.

We assume that the labor cost on arc i is measured from the arrival time at customer location i to the arrival time at location $i+1$, i.e., from μ_i to μ_{i+1}. Formally, the labor cost, L_i, for traversing arc i, given an arrival time of μ_i at customer location i, a pre-service waiting y_i, a post-service waiting z_i and a free-flow speed v_i is given by

$$L_i(\mu_i, y_i, z_i, v_i) \quad = \quad D\left[(l_i - \mu_i)^+ + y_i + h_i + z_i + T_i(\max\{\mu_i, l_i\} + y_i + h_i + z_i, v_i)\right],$$

where D is the labor cost. For arc 0, the labor cost is calculated differently depending on the driver wage policy. Under the early policy, we have

$$L_0(0, y_0, z_0, v_0) \quad = \quad D\left[y_0 + h_0 + z_0 + T_0(y_0 + h_0 + z_0, v_0)\right].$$

Otherwise, under the late policy, we have

$$L_1(0, y_0, z_0, v_0) \quad = \quad D\left[h_0 + z_0 + T_0(y_0 + h_0 + z_0, v_0)\right].$$

The difference between the two driver wage policies is the pre-service waiting time at the depot Dy_0, which is paid under the early policy but not under the late policy.

To measure the emissions cost, we use the model of Barth et al. (2005) and Scora and Barth (2006) who assume that the amount of CO_{2e} emissions produced by a vehicle is directly proportional to the amount of fuel consumed. According to their model, the emissions costs E_i, for traversing arc i given a chosen free-flow speed v_i and a departure time from location i of ρ_i, is given by:

$$E_i(\rho_i, v_i) \quad = \quad Ad_i + BT_i(\rho_i, v_i) + C\left[T_i^{con}(\rho_i)(v_{con})^3 + T_i^{free}(\rho_i, v_i)v_i^3\right],$$

where A, B and C are non-negative constants which depend on features of the vehicle such as frontal surface area and curb weight as well as on the road conditions. (see Franceschetti et al. (2013) for how to calculate these values).

The total cost for traversing arc i, measured from the arrival time at location i to the arrival time at location $i+1$, i.e., from μ_i to μ_{i+1}, is denoted by g_i and is given by

$$g_i(\mu_i, y_i, z_i, v_i) \quad = \quad E_i(\max\{\mu_i, l_i\} + y_i + h_i + z_i, v_i) + L_i(\mu_i, y_i, z_i, v_i).$$

The decision maker's problem is to determine the speed vector $\mathbf{v}$ and waiting time vectors $\mathbf{y}$ and $\mathbf{z}$ in order to minimize the total cost $\mathcal{C}$ incurred over the entire vehicle trip. The problem can be written as follows:

$$\text{minimize}_{\mathbf{v},\mathbf{z},\mathbf{y}} \quad \mathcal{C}(\mathbf{v},\mathbf{y},\mathbf{z}) = \sum_{i=0}^{n} g_i(\mu_i, y_i, z_i, v_i)$$

$$\begin{aligned} &\text{subject to } \mu_0 = 0, \\ \mu_i = \max\{l_{i-1}, \mu_{i-1}\} + y_{i-1} + h_{i-1} + z_{i-1} & \\ +T_{i-1}\left(\max\{l_{i-1}, \mu_{i-1}\} + y_{i-1} + h_{i-1} + z_{i-1}, v_{i-1}\right) & \quad \text{for } i = 1, ..., n+1, \\ \mu_i \leq u_i & \quad \text{for } i = 1, ..., n, \\ v^{min} \leq v_i \leq v^{max} & \quad \text{for } i = 0, ..., n. \end{aligned}$$

The first two sets of constraints follow from the definition of the arrival time into a location. The third set corresponds to the upper time windows constraints and the last set is where we impose the minimum and maximum free-flow speed constraints.

Note that, for fixed $\mathbf{v}, \mathbf{z}$ and $\mathbf{y}$, the total cost under the early policy is higher than under late policy by a quantity equal to Dy_0, which corresponds to the cost of paying the driver during the pre-service waiting time at the depot.

2.2 Important speed values

Given a fixed departure time ρ_i from node i, the speed that minimizes the emissions costs on arc i, i.e., $E_i(\rho_i, v_i)$, is equal to $\underline{v} = (\frac{B}{2C})^{1/3}$. Similarly, given fixed values for μ_i, y_i and z_i, the speed value that minimizes the total cost of traversing arc i, i.e., $g_i(\mu_i, y_i, z_i, v_i)$ is equal to $\overline{v} = (\frac{B+D}{2C})^{1/3}$. We always have $\underline{v} \leq \overline{v}$ and we further assume that $v^{min} \leq \underline{v}$ and $\overline{v} \leq v^{max}$. These two speed values are useful in our subsequent analysis and have previously been identified by Demir et al. (2012).

2.3 Feasibility conditions and effective upper time window limits

In this section we establish the conditions under which the problem is feasible and we introduce the concept of *effective* upper time window limits, which is useful in our subsequent derivations. Let $\underline{\mu}_i$ denote the earliest possible arrival time at location i, which is calculated by assuming the vehicle drives at the maximum free-flow speed whenever possible until reaching location i

and never waits (pre- or post-service) at any location, i.e., setting $v_j = v^{max}$, $z_j = 0$ for $j = 0, ..., i-1$ and $y_j = 0$ for $j = 0, ..., i-1$. We obtain $\underline{\mu}_i$ for $i = 0, ..., n+1$, recursively starting from location 0 as follows:

$$\begin{aligned} \underline{\mu}_0 &= 0, \\ \underline{\mu}_i &= \max\left\{\underline{\mu}_{i-1}, l_{i-1}\right\} + h_{i-1} + T_{i-1}(\max\{\underline{\mu}_{i-1}, l_{i-1}\} + h_{i-1}, v^{max}) \qquad \text{for } i = 1, ..., n+1. \end{aligned}$$

The problem is feasible if $\underline{\mu}_i \leq u_i$ for $i = 1, ..., n$. In what follows we assume that these conditions are satisfied. For $i = 1, ..., n+1$, let U_i denote the latest possible arrival time at location i so that it is still possible to meet all the time windows at locations $i+1, ..., n+1$. By construction, we always have $U_i \leq u_i$. We call U_i the *effective* upper time window limit at location i: if the vehicle was to arrive at location i between U_i and u_i, then it would not violate the upper time window limit of location i, but it would certainly violate the upper time window limit of at least one of the subsequent locations. We set $U_{n+1} = u_{n+1} = +\infty$ and calculate $U_1, ..., U_n$ recursively starting from location n, assuming maximum speed and no post-service waiting time, as follows:

$$\begin{aligned} U_n &= u_n, \\ U_i &= \min\left\{u_i, U_{i+1} - \min\{(U_{i+1} - a)^+, d_i/v^{max}\} - [d_i - (U_{i+1} - a)^+ v^{max}]^+/v_{con} - h_i\right\} \\ & \qquad \text{for } i = n-1, ..., 1. \end{aligned} \tag{1}$$

Note that the feasibility of the problem implies that $l_i \leq U_i$ for $i = 1, ..., n$. We use these effective upper time window limit U_i, rather than u_i, throughout our analysis, and we refer to $[l_i, U_i]$ as the *effective* time window at location $i \in \{1, ..., n+1\}$.

3 Structural Results

In this section we present some structural properties of the optimal solution to the DSOP with traffic congestion. When there is traffic congestion, the optimal driving schedule may be such that it is optimal for the vehicle to wait idly at the depot or at customer locations because doing so can reduce the driving time in congestion, thereby possibly reducing GHG emissions. However, waiting may increase labor costs: the idle time at a customer location is always costly, while the idle time at the depot is only costly if the driver is paid from the start of the planning horizon, that is, under the early policy. Our first result is about the voluntary pre- and post-service waiting times at the depot and customer locations under both driver wage policies.

Proposition 1. *There exists an optimal solution such that there is no voluntary pre-service waiting time at any of the customer locations, that is,* $y_i = 0$ *for* $i = 1, ..., n$. *Further, when the driver is paid from the start of the planning horizon (early policy), this optimal solution also has no voluntary pre-service waiting time at the depot, that is,* $y_0 = 0$. *In contrast, when the driver is paid from the start of service at the initial location (late policy), the optimal solution also has no post-service waiting time at the depot, that is,* $z_0 = 0$.

Based on Proposition 1, one can ignore voluntary pre-service waiting time at the customer locations in the optimization process and focus solely on post-service waiting times. This is because, at customer locations, the driver is paid equally for pre- and post-service waiting times, and hence there is no difference in costs if, given a fixed amount of total waiting time, all the waiting takes place after service is completed. But the situation is different at the depot as it depends on the driver wage policy: pre-service waiting is free if the driver is paid from the start of service at the depot (late policy), therefore postponing service at the depot then leaving immediately (that is, setting $y_0 \geq 0$ and $z_0 = 0$) may be an effective strategy to reduce GHG emissions costs without increasing labor costs. When the driver is paid from the start of the planning horizon (early policy), waiting at the depot is equally costly whether it takes place pre- or post-service, therefore concentrating all the waiting after the service has ended (that is, setting $y_0 = 0$ and $z_0 \geq 0$) does not affect total costs. [2] Given Proposition 1, the decision maker only needs to determine $(v_0, ..., v_n)$ and $(z_0, ..., z_n)$ under the early policy, and $(v_0, ..., v_n)$, y_0 and $(z_1, ..., z_n)$ under the late policy.

Any solution to the DOSP can be seen as a subsequence $(j_1, ..., j_m)$ of the location indices $\{0, 1, ..., n+1\}$ with $0 \leq j_1 \leq .... \leq j_m = n + 1$ as follows. Arcs 0 to $j_1 - 1$ are entirely traversed during the congestion period, that is, $\mu_i \leq a$ for $i = 1, ..., j_1$ and arcs j_k to $j_{k+1} - 1$ are traversed at the same free-flow speed for $k = 1, ..., m$, that is, $v_{j_k} = v_{j_k+1} = ... = v_{j_{k+1}-1}$. In other words the set of arcs is partitioned into adjacent segments on which the driver keeps a constant free-flow speed: the first segment consists of arcs 0 to $j_1 - 1$, the second one consists of arcs j_1 to $j_2 - 1$, and the last segment consists of arcs j_{m-1} to n. Also, arc j_1 is the first arc to be traversed during the free-flow period (if all the arcs are traversed during the congestion

[2] Alternatively, we could assume that all the waiting takes place *before* service at the depot when the driver is paid from the start of the planning horizon. However, this alternative complicates the exposition of Proposition 2, which is why we have opted for the present statement of Proposition 1.

period, we have $m = 1$). Our next result, which holds under both driver wage policies, uses this notation to establish important properties of the optimal solution.

Proposition 2. *There exists an optimal solution such that*

(i) *all the post-service waiting (if there is any) takes place at the last location that is reached before the end of the congestion period, i.e., $z_{j_1} \geq 0$ and $z_i = 0$ for $i \neq j_1$ where j_1 is the largest index in $\{1, ..., n+1\}$ such that $\mu_{j_1} \leq a$;*

(ii) *if the vehicle reaches a customer location $i > j_1$ strictly within its time window, i.e., $\mu_i \in (l_i, U_i)$, then the free-flow speed on arcs $i-1$ and i must be the same.*

Proposition 2(i) implies that there exists an optimal solution such that the vehicle does not wait idly post-service after the end of the congestion period, which is intuitive since doing so would increase labor costs without reducing GHG emissions. Proposition 2 (ii) implies that changes in the optimal free-flow speed can only occur when the vehicle reaches a customer location by its lower time window, i.e., $\mu_i \leq l_i$, or exactly at its upper time window, i.e., $\mu_i = U_i$. In other words, the speed on two adjacent arcs must be the same unless the start of service at the common location is exactly equal to its lower or upper time window limit.

Figure 2 depicts an example of scheduling that satisfies the conditions of Proposition 2, where the x-axis represents the geographical distance and the y-axis represents time. The schedule is represented by discontinuous black line, where the points of discontinuity occur at the customer locations. The slope of the increasing portions corresponds to the time elapsed per unit of distance, which is the inverse of vehicle the speed, while the vertical portions of the line correspond to periods of service and waiting (with waiting being represented by dotted lines). Time windows limits for each locations are marked on the y-axis. In this example, the vehicle travels on arc 0 during the congestion period (therefore at a speed of v_{con}), then waits at location 1 following the completion of service. The congestion period ends while it is traversing arc 1, so the driver switches to a free-flow speed which we denote by v^1. The vehicle keeps the same speed on arc 2, arriving at location 3 exactly at time U_3. Finally it drives on arcs 3 and 4 at a constant speed, denoted by v^2. Formally, in this example, we have $m = 3$, $j_1 = 1, j_2 = 3$ and $j_3 = 5$. This schedule satisfies Proposition 2(i) since the only post-service waiting occurs at location 1, which is the last customer location that is reached before the end of the congestion period. It also satisfies (ii) since the vehicle arrives at locations 2 and 4 strictly within their time window and keeps the same free-flow speed before and after reaching these locations. Further, at location 3, which marks a change in free-flow speed, the vehicle's arrival time exactly matches the effective upper time window limit.

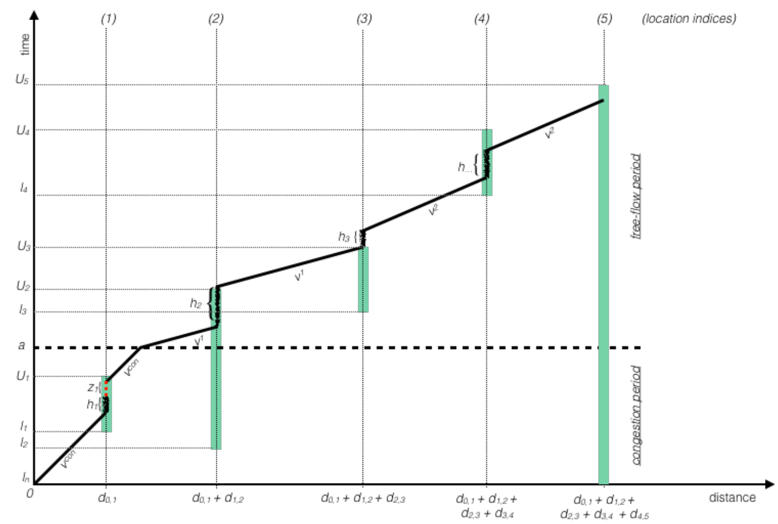

Figure 2: Example of scheduling satisfying the conditions of Proposition 2. Note that $\hat{v} = \sqrt{\frac{B+Cv_c}{3Cv_c}}$ (see §A)

Given Proposition 2, there exists an optimal sequence $\{j_1, ..., j_{m-1}, n+1\}$ of locations such that service starts service at locations $j_2, ..., j_{m-1}$ exactly at their lower or upper time window limit. Therefore we can obtain an optimal schedule by considering all possible sequences in combination with the two possible values for start of service at locations which mark a speed change. So suppose we fix a sequence $\{j_1, ..., j_{m-1}, n+1\}$ and fix the start of service at locations $j_2, ..., j_{m-1}$. Optimizing the pre- and post service waiting times and vehicle speeds can be done independently for each segment of locations: 0 to j_1, j_1 to j_2, etc. By definition, we know the free-flow speed is constant on each such segment and by Proposition 2, we know that the arrival time at the intermediate nodes is strictly within their time window. Lemma 1 further establishes that the optimization of each segment amounts to solving a single-arc DSOP problem, which implies that there is no waiting time at the intermediate nodes.

Lemma 1. *Minimizing the total cost of traversing arcs $i, ..., j$ at a constant free-flow speed, ignoring the time windows at locations $i+1, ..., j-1$ such that service at location j starts exactly at time $\eta \in \{l_j, U_j\}$ is equivalent to minimizing the total cost of traversing a single arc with distance $d = \sum_{k=i}^{j-1} d_k$ such that service at the destination location starts exactly at time $\hat{\eta} = \eta - \sum_{k=i+1}^{j-1} h_k$.*

In §A we show how to solve a single-arc problem with a fixed start of service time. Note that there results are different from those of Franceschetti et al. (2013) since they do not assume a fixed start of service time at the final location.

Because the final arc represents the return to the initial location and there is no time window at location $n+1$, (i.e., $l_{n+1} = 0$ and $u_{n+1} = \infty$) the last segment must be dealt with differently. In this case the results from Franceschetti et al. (2013) can be used directly to show that if the vehicle drives at least partially in free-flow, then the last part of the trip should be driven at speed $\overline{v}$ as defined in §2.2.

Lemma 2. *If the optimal solution is such that the vehicle drives at least part of the trip in free-flow (i.e., $m > 1$), then it should drive at speed $\overline{v}$ on the last arcs, i.e., $v_{j_{m-1}} = v_{j_{m-1}+1} = ... = v_n = \overline{v}$.*

When the driver is paid from the start of service at the depot (late policy), we have the following extra result.

Proposition 3. *Consider the case where the driver is paid from the start of service at the initial location. If at least one customer location has a finite upper time window, then there exists a **pivot** location $\hat{j} \in \{1, ..., n\}$ defined as follows: $\hat{j}$ is the lowest index such that $\mu_{\hat{j}} \in \{l_{\hat{j}}, U_{\hat{j}}\}$ and $\mu_i \in (l_1, U_i)$ for $i = 1, ..., \hat{j}-1$. Further, if $\mu_{\hat{j}} \leq a$ and $\mu_{\hat{j}} = U_{\hat{j}}$ then $\hat{j} = 1$. Also the optimal solution is such that $z_0 = ... = z_{\hat{j}-1} = 0$.*

When at least one customer location has a finite upper time window, Proposition 3 establishes that, under the late policy, there always exists a customer location that is reached exactly at its lower or effective upper time window, without any post-service waiting time at any of the preceding locations. We refer to the earliest customer location satisfying this condition as the *pivot customer location* and denote it by $\hat{j}$. Intuitively, the role of the pivot customer location is to determine the optimal amount of pre-service waiting time at the depot. When the driver is paid from the start of service at the depot, postponing service at the depot is an effective way to reduce the total cost. Given a fixed arrival time at the pivot location, one can optimize the pre-service waiting time at the depot and the (constant) free-flow speed on arcs 0 to $\hat{j}-1$ by treating the problem as a single-arc DSOP with fixed arrival time (see §A for how to solve such a problem under the late policy).

Note that the pivot customer location can be reached either before or after the end of the congestion period. When it is reached after congestion had ended, i.e., $\mu_{\hat{j}} > a$, Proposition 3 together with Proposition 2(ii) imply that there is no post-service waiting time at any location. In contrast, if the pivot customer location is reached during the congestion period, i.e., $\mu_{\hat{j}} \leq a$, there could be post-service waiting time at one of the customer locations (either location $\hat{j}$ or a later one). Further, if the arrival time at the pivot location matches its effective upper time window and occurs before congestion ends, then the pivot location is location 1.

The structural properties of the optimal solution established in Propositions 1, 2 and 3 allow us to recast the DSOP with traffic congestion as a shortest path problem, which we present in detail in the next section. Before doing so, we provide a stronger characterization of the optimal solution in a special case.

Proposition 4. *Suppose that each customer location has an infinite upper time window, i.e., $u_i = \infty$ for $i = 1, ..., n+1$. If the driver is paid from the start of service at the initial location (late policy), it is optimal to set y_0 so that the vehicle leaves the initial location past the congestion period and avoids any mandatory pre-service at any customer location. Further, the vehicle drives in free-flow at speed $\overline{v}$ on all the arcs, without any post-service waiting. Specifically, setting y_0 to a value greater or equal to* $\max\left\{(a-h_0)^+, \max_{i=1,...,n}\left(l_i - \sum_{j=0}^{i-1} h_j - \frac{\sum_{j=0}^{i-1} d_j}{\overline{v}}\right)\right\}$ *and $z_0 = z_1 = ... = z_n = 0$ is optimal. The total cost achieved is equal to* $\mathcal{C} = A\sum_{i=0}^{n} d_i + (B + C\overline{v}^2 + D)\frac{\sum_{i=0}^{n} d_i}{\overline{v}} + D\sum_{i=0}^{n} h_i$.

Intuitively, when there are no upper time windows and the driver is paid from the start of service at the depot, the decision maker can postpone service indefinitely without bearing any cost consequences, and therefore she waits until the congestion period has ended so as to avoid driving at the GHG emissions-causing congestion speed. Also she waits long enough to make sure the vehicle never arrives at any customer location earlier than its lower time window limit. By doing so she is able to avoid any mandatory pre-service wait time. As a result, the total time for which the driver is paid for is the sum of her travel time and service time and, in this case, the total cost is minimized when using free-flow speed $\overline{v}$ defined in §2.2 on every arc. Note that total cost of the optimal solution is a lower bound on all possible values achievable.

4 Shortest Path Formulation

In this section we show how to turn the DSOP with traffic congestion into a shortest path problem. By doing so we exploit two main results proven in the previous section. First, we use the property that an optimal schedule can be broken down into segments of arcs where the vehicle keeps a constant free-flow speed, arrives at each intermediate location strictly within its locations time windows limits and only waits post-service (if at all) at the starting point location of the segment. Second, we exploit the property that service at the locations that mark a change in speed starts either exactly at their lower or effective upper time window limits. The design of the shortest path network differs for the two driver wage policies, so we present the solutions separately in two subsections below.

4.1 Shortest path formulation under the early policy

Propositions 1 and 2 (i) together imply that when the driver is paid from the start of the planning horizon, locations 0 to j_1 are reached as early as possible given the congestion speed, that is, $\mu_i = \underline{\mu}_i$ for $i = 0, ..., j_1$, where $\underline{\mu}_i$ is defined in §2.3. This is because there is no pre- or post-service waiting at the depot ($y_0^* = z_0^* = 0$) and no post-service waiting at locations 1 to $j_1 - 1$ ($z_1^* = ... = z_{j_1-1}^* = 0$). Hence, under the early policy, the start of service time at location j_i for $i = 1, ..., m-1$ can only take three possible values: $\underline{\mu}_{j_i}$, l_{j_i} or U_{j_i}. Note that the arrival time at location j_i can be earlier than l_{j_i} but the start of service time cannot be.

For use in the design of the shortest path, we define location $k \in \{0, ..., n+1\}$ as the highest index location that can be reached before the end of the congestion period (assuming no waiting), i.e., k is the largest value of i such that $\underline{\mu}_i \leq a$ (we set $k = 0$ if it is impossible to reach location 1 before the end of the congestion period, that is, if $h_0 + d_0/v_c > a$).

From the original network with $n+1$ arcs and $n+2$ nodes we construct a shortest path network with $2n+k$ nodes as follows: node 0 (corresponding to the depot), nodes $\underline{i}$ for $i = 1, ..., k$ (i.e., one such node for each of the first k customer locations), nodes i_l and i_u for $i = 1, ..., n$ (i.e., two such nodes for each customer locations) and location $n+1$ (corresponding to the return to the depot). Let V denote the set of nodes in the shortest path network, i.e., $V = \{0, \underline{1}, ..., \underline{k}, 1_l, \ldots, n_l, 1_u, ..., n_u, n+1\}$. For $i = 1, ..., k$, location $\underline{i}$ corresponds to the event "the vehicle starts service at time $\max\{\underline{\mu}_i, l_i\}$." For $i = 1, ..., n$, node i_l corresponds to the event "the vehicle arrives at location i at the latest at the lower time window limit l_i (and therefore starts service at time l_i), and node i_u corresponds to the event "the vehicle arrives at location i exactly at the effective upper time window limit U_i." Finally, node $n+1$ corresponds to the event "the vehicle returns to the depot". Each arc in the shortest path network correspond to a segment of adjacent arcs in the original network. For example, the arc in the shorhest path network from location 0 to location $\underline{i}$ (or i_l or i_u) for $i = 1, ..., n$ corresponds to arc 0 to $i-1$ in the original network. The length of each arc is equal to the minimum cost of traversing the corresponding segment given constraints on the arrival time at each location. This cost can be calculated by treating each segment as a single-arc DSOP as explained in Lemma 1 and is set to infinity if meeting the constraints is not feasible.

More precisely, the length of the arcs in the shortest path network are set as follows:

- The length of arcs $(0, \underline{i})$ for $i = 1, ..., k$ is set equal to the (minimum) cost of driving in congestion from the depot to location i, without any waiting at locations 0 to $k-1$.

- The length of arcs $(0, j_l)$ (resp. $(0, j_u)$) for $j = 1, ..., n$ is set equal to the minimum cost of starting service at the depot at time zero and starting service at location j at time l_j (resp. U_j), while reaching locations $i+1$,...,$j-1$ strictly within their time window (and is infinite if this is not possible);

- The length of arcs (i_l, j_l) (resp. $(i_l, j_u), (i_u, j_l), (i_u, j_u)$) for $1 \leq i < j \leq n$ is set equal to the minimum cost of starting service at location i at time l_i (resp. l_i, U_i, U_i) and starting service at location j at time l_j (resp. U_i, l_i, U_i), while reaching locations $i+1$,...,$j-1$ strictly within their time window (and is infinite if this is not possible);

- The length of arcs $(\underline{i}, j_l)$ (resp. $(\underline{i}, j_u)$) for $1 \leq i < j \leq n$ is set equal to the minimum cost of starting service at location i at time $\underline{\mu}_i$ and starting service at location j at time l_j (resp. U_j), while reaching locations $i+1$,...,$j-1$ strictly within their time window (and is infinite if this is not possible);

- The length of arcs $(i_l, n+1)$ (resp. $(i_u, n+1)$ and $(\underline{i}, n+1)$) for $1 \leq i \leq \min\{k, n\}$ is set equal to the minimum cost of starting service at location i at time l_i (resp. U_i and $\underline{\mu}_i$) and driving to location $n+1$, while reaching locations $i+1$,...,$j-1$ strictly within their time window (and is infinite if this is not possible);
- The length of arc $(0, n+1)$ is set equal to the minimum cost of starting service at the depot at time zero, reaching locations 1 to n within their time window and returning back to the depot (and is infinite if this is not possible).
- If $k = n+1$, the length of arc $(\underline{n+1}, n+1)$ is set equal to zero.
- All other arcs have infinite lengths.

In particular, all arcs from an "i-node" (i.e., either $\underline{i}, i_l$ or i_u) to a "j-node" (i.e., either $\underline{j}, j_l$ or j_u) with $j < i$ have infinite length. So do arcs $(i_l, \underline{j}), (i_u, \underline{j})$ and $(\underline{i}, \underline{j})$ for $j > i$.

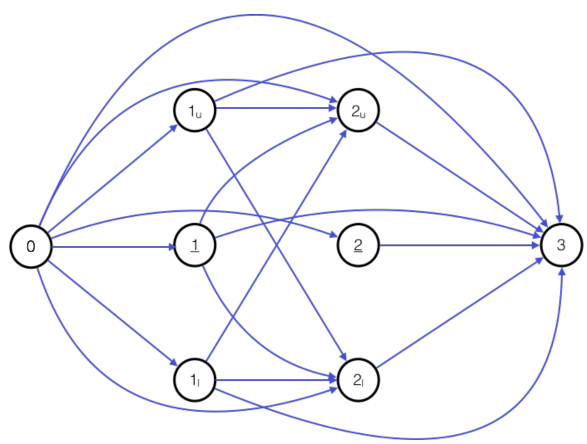

Figure 3: Shortest path network for an example with $n = 2$ and $k = 2$.

From this is follows that the problem of finding the optimal vectors **v**, **z** and **y** when the driver is paid from the start of the planning horizon reduces to a shortest path from node 0 to node $n+1$ as described above.

4.2 Shortest path formulation under the late policy

When the driver is paid from the start of service at the depot, we know from Proposition 1 that it may be optimal for the decision maker to postpone the start of service at the initial location in order to save on labor costs, that is, set $y_0^* > 0$. Further, there can be some additional waiting, which, if it exists, takes place post-service at the last location visited before the end of the congestion period (by Proposition 2(i)). This will be the case if the pivot customer location (i.e., the $\hat{j}$ index defined in Proposition 3) is visited before the end of the congestion period. When the pivot customer location is visited exactly at its upper time window, Proposition 3 establishes that it must be the first location; in practice this means that the driver postpones the start of service at the depot so as to arrive at the first customer location exactly at time U_1. In that case, the arrival time at subsequent locations $i = 2, ...j_1$, denoted by $\hat{\mu}_i$, are calculated recursively as:

$$\begin{aligned}\hat{\mu}_1 &= U_1\\ \hat{\mu}_i &= \max\{\hat{\mu}_{i-1}, l_{i-1}\} + h_{i-1} + T_{i-1}(\max\{\hat{\mu}_{i-1}, l_{i-1}\} + h_{i-1}, v^{max}) \qquad \text{for } i = 2, ..., j_1.\end{aligned}$$

In contrast, when arrival at the pivot customer location is at its lower time window and occurs before the end of the congestion period, the arrival time at subsequent locations $\hat{j}+1, ..., j_1$ is given by $\underline{\mu}_i$ since the vehicle will start service at location $\hat{j}$ at the earliest possible time and there is no waiting until possibly at location j_1. Hence, under the late policy, the start of service time at location j_i for $i = 1, ..., m-1$ can only take four possible values: $\hat{\mu}_{j_i}, \underline{\mu}_{j_i}$, l_{j_i} or U_{j_i}. In other words, compared to the early policy, there is one more possible value for the arrival time at a location which marks the end of a segment. We exploit this structure to modify the shortest path formulation presented in the previous section.

As in §4.1, we define $k \in \{0, ..., n+1\}$ be the highest index location that can be reached before the end of the congestion period (assuming no waiting), i.e., k is the largest value of i such that $\underline{\mu}_i \leq a$ (again, we set $k = 0$ if even location i cannot be reached before the end of the congestion period). Further, we also define $p \in \{0, ..., n+1\}$ to be the highest index location that can be reached before the end of the congestion period, *when the vehicle waits pre-service at the depot so as to arrive at location 1 exactly at time* U_1, i.e., p is the largest value of i such that $\hat{\mu}_i \leq a$ (we set $p = 0$ if $U_1 > a$). Note that we always have $k \geq p$ since $\underline{\mu}_i \leq \hat{\mu}_i$ for $i = 0, ..., n+1$.

From the original network with $n+1$ arcs and $n+2$ locations, we construct an SP network with $2n+k+p$ nodes. The first $2n+k$ nodes are the same as in the SP network from §4.1 and have the same interpretation. The extra p nodes are labeled $\hat{i}$ for $i = 1, ..., p$ and correspond to the event "the vehicle arrives at location i at time $\hat{\mu}_i$". All the arcs which exist in the SP network from §4.1 continue to exist and have the same length. On top of these we have the following extra arcs:

- $(0, \hat{i})$ for $i = 1, ..., p$:

  The length of arc $(0, \hat{i})$ is set equal to the (minimum) cost of driving from the depot to location i when the vehicle waits at the depot so as to arrive at location 1 exactly at time U_1 and does not wait at locations $1, ..., p-1$;

- $(\hat{i}, j_l), (\hat{i}, j_u)$ for $1 \leq i < j \leq n$:

  The length of arc $(\hat{i}, j_l)$ (resp. $(\hat{i}, j_u)$) for $1 \leq i < j \leq n$ is set equal to the minimum cost of starting service at location i at time $\hat{\mu}_i$ and starting service at location j at time l_j (resp. U_j), while reaching locations $i+1$,...,$j-1$ strictly within their time window (and is infinite if this is not possible);

- $(\hat{i}, n+1)$ for $1 \leq i \leq \min\{p, n\}$:

  The length of arc $(\hat{i}, n+1)$ for $1 \leq i < j \leq n$ is set equal to the minimum cost of starting service at location i at time $\hat{\mu}_i$ and driving to location $n+1$, while reaching locations $i+1$,...,$j-1$ strictly within their time window (and is infinite if this is not possible);

- $(n \hat{+} 1, n+1)$ if $p = n+1$:

  The length of this arc is set equal to zero.

From this it follows that the problem of finding the optimal vectors $\mathbf{v}$, $\mathbf{z}$ and $\mathbf{y}$ when the driver is paid from the start of service at the depot reduces to a shortest path from node 0 to node $n+1$ as described above.

5 Numerical illustrations

In this section we present the optimal solution for a numerical example under both driver wage policies. We do so in order to illustrate the workings of the shortest path and to present some structural properties of the optimal solution. Consider the route depicted in Figure 4 with 4 customer locations.

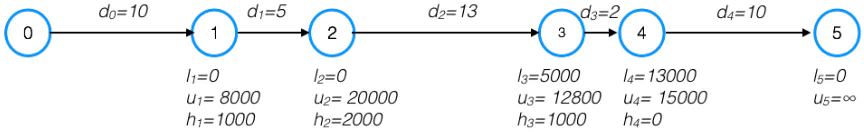

Figure 4: Example of a route with 4 customer locations

All distances reported in Figure 4 are in km and times are in seconds.

We assume a congestion period a of 10000 seconds, a congestion speed v_{con} of 10 km/h, a minimum free flow speed v^{min} of 10 km/h and a maximum free flow speed v^{max} of 90 km/h. Regarding the cost parameters, we use the same values as in Franceschetti et al. (2013), namely $A = 7.47047 * 10^{-5}$, $B = 0.0014$, $C = 1.977 * 10^{-7}$ and $D = 0.0022$.

Using (1) we calculate the effective upper time window limits for each customer location and see that it matches the actual upper time window limit for each of them, except for location 2, where $U_2 = 10280 < u_2$. Given the cost parameters, we have $\underline{v} = 55.19$km/h and $\overline{v} = 75.48$km/h. The furthest location that can be reached before the end of the congestion period is location 2, i.e., we have $k = 2$ where k is defined in §4.1. Also, the furthest location that can be reached before the end of the congestion period when the vehicle arrives at location 1 at time U_1 is location 1, i.e. we have $p = 1$ where p is defined in §4.2. Figures 5 and 6 depict the SP graph under the early policy and the late policy, respectively. Both figures only show the arcs which have finite length. The shortest path on each graph is marked in bold.

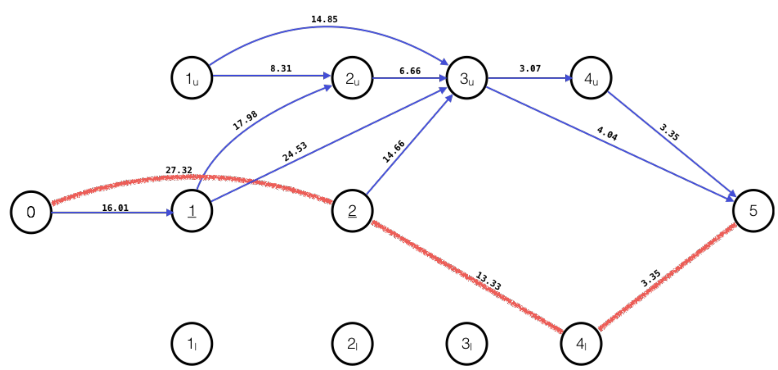

Figure 5: SP graph under the early policy.

The optimal scheduling under the early and late policies are also depicted on Figures 7 and 8, respectively.

From Figure 5, we see that the shortest path from 0 to 5 under the early policy goes through nodes $\underline{2}$ and 4_l. This means that the optimal solution has three segments with speed changes occurring when reaching locations 2 and 4, that is, we have $m = 3$, $j_1 = 2, j_2 = 4$ and $j_3 = 5$. It also means that the optimal solution is such that the vehicle drives in congestion on arcs 0 and 1, arrives at location 2 at time $\underline{\mu}_2$, and starts service at location 4 at

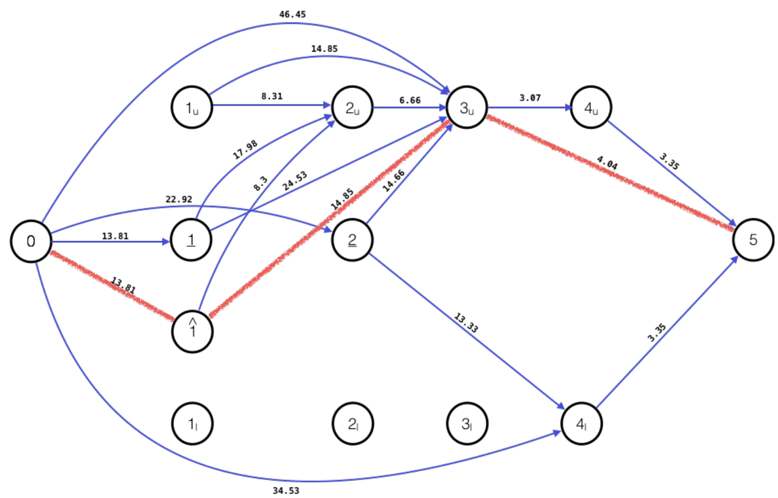

Figure 6: SP graph under the early policy.

time l_4. Figure 7 provides further information about the optimal schedule: we see that the vehicle waits post service at location 2 until the end of the congestion period, then drives on arcs 2 and 3 at speed $\underline{v}$, reaching location 4 strictly before its lower time window limit, and causing some mandatory pre-service waiting time. Upon completion of service at location 4, the vehicle returns to the depot driving at a speed of $\overline{v}$. The total cost of this solution is 43.99, which is equal to the sum of the arc lengths on the shortest path.

From Figure 6, we see that the shortest path from 0 to 5 under the late policy goes through nodes $\hat{1}$ and 3_u. This means that the optimal solution has three segments with speed changes occurring when reaching locations 1 and 3, that is, we have $m = 3$, $j_1 = 1, j_2 = 3$ and $j_3 = 5$. It also means that the optimal solution is such that the vehicle drives in congestion on arc 0, arrives at location 1 at time U_1 (making location 1 the pivot customer location), and at location 3 at time U_3. From Figure 8 we further learn that service at the depot is delayed until after 4400 seconds. Also, we see that, the vehicle waits post-service at location 1 until the end of the congestion period, then drives on arcs 1 and 2 at the required speed in order to arrive at location 3 exactly at time U_3. After completing the service at location 3 the vehicle drives on arcs 3 and 4 at speed $\overline{v}$. The total cost of this solution is 32.67, which is equal to the sum of the arc lengths on the shortest path.

Tables 3, 4 and 2 respectively display a comparison of the optimal cost values, speed values, waiting times between the two driver wage policies.

	Driver wage policy	
	early policy	late policy
Driving time in congestion	5399.99	3600
Driving time in free-flow	1456.27	1373.38
Total driving time	6856.26	4973.38
Paid waiting time	2621.5	1000
Total labor time	13,477.77	9973.38
Return to depot time	13477.77	14373.38
Labor cost	29.65	21.94
Emissions cost	13.63	10.73
Total cost	43.28	32.67

Table 2: Times (in seconds) and costs for the optimal solutions.

	Driver wage policy	
	early policy	late policy
v_0^*	10	10
v_1^*	10	81
v_2^*	55.19	81
v_3^*	55.19	75.48
v_4^*	75.48	75.48

Table 3: Optimal speeds values (in km/h).

	Driver policy			
	early policy		late policy	
Location	pre-service	post-service	pre-service	post-service
0	0	0	4400	0
1	0	0	0	1000
2	0	2621.5	0	0
3	0	0	0	0
4	1869.49	0	0	0

Table 4: Optimal (mandatory) pre- and (voluntary) post-service waiting times (in seconds).

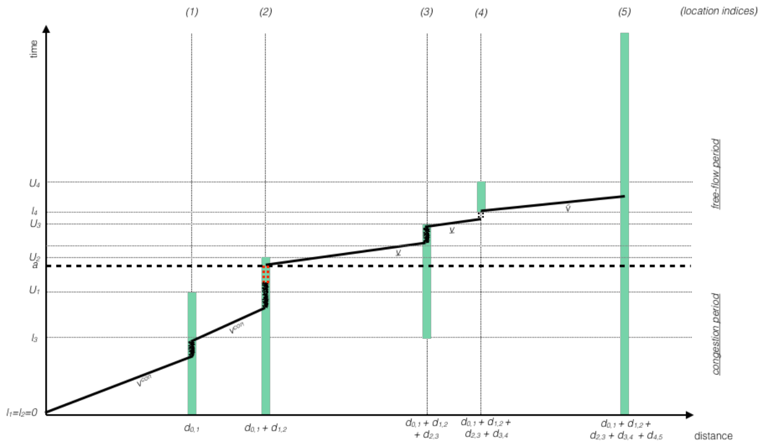

Figure 7: Optimal of scheduling under the early policy

We see that both the labor and emission costs are lower under the late policy. This can be explained by the following factors. First, because service time at the depot is delayed under the late policy, the vehicle spends significantly more time driving in congestion under the early policy than under the late policy. Second, the optimal vehicle speed is (weakly) larger on each arc under the late policy. Third, the driver is paid for significantly more waiting time under the early policy (specifically an extra 3490.90 minutes). In other words, by delaying the arrival (and therefore the payment) of the driver at the depot under the late policy the decision maker is able to mitigate the negative impact of congestion, increase driving speed and reduce in-transit waiting times. Note that the driver returns sooner to the depot under the early policy than under the late policy.

6 Conclusions

We have studied the DSOP under traffic congestion, which consists in finding the optimal schedule for a vehicle visiting a fixed sequence of customer locations in order to minimize the sum of labor and emissions costs. We have provided an efficient shortest path formulation to solve this problem optimally in polynomial time. Because traffic congestion creates incentives for voluntary waiting times in the schedule, existing methodologies for the DSOP without traffic congestion cannot be directly applied or easily adapted. Beside being efficient, our solution methodology provides an intuitive visual

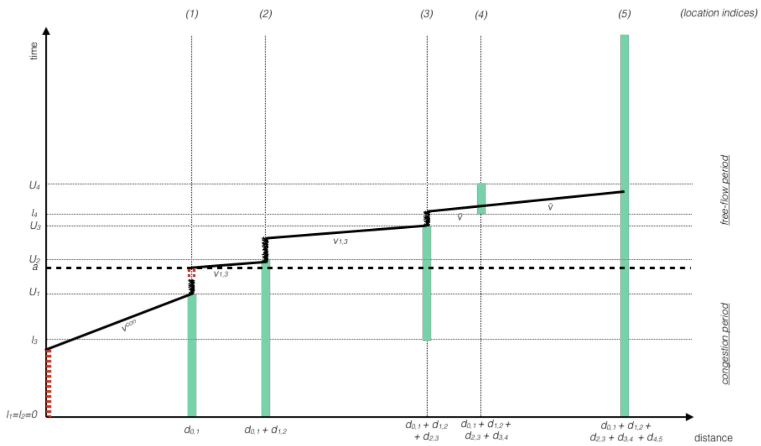

Figure 8: Optimal of scheduling under the late policy

representation of the optimal solution through the shortest path network, which represents the breakdown of the optimal schedule into adjacent segments of arcs on which the optimal speed is kept constant. In our model formulation, we have considered two different driver wage policies and we have illustrated through a numerical example how their optimal solutions differ. Our solution methodology can be embedded as a subroutine within an algorithm to solve a more general vehicle routing and scheduling problem such as the pollution-routing problem with traffic congestion (Franceschetti et al. (2013)).

Acknowledgements

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A Single-arc DSOP optimization

Given Lemma 1, the length of each arc in the SP network can be computed by considering a single-arc problem with a fixed start of service time. We show how to solve such a problem in this section. Let 0 denote the initial node (depot) and 1 denote the arrival node (customer location) with time window $[l_1, u_1]$. To simplify the notation let d denote the length of the arc and v denote the free-flow speed on the arc. All other notation is the same as in §2.

First we consider the problem of minimizing the total cost (i.e., labor and emissions costs) under the constraint that the arrival time at location 1 must be exactly equal to μ. The decision maker needs to decide how long the vehicle should wait at the depot (pre- and post-service, i.e., y_0 and z_0) and how fast to drive on the arc (i.e., speed v), so as to minimize the total cost, which is measured until the arrival time at location 1, that is, we do not consider the return trip to the depot. For the problem to be feasible we need $\mu \in \left[h_0 + \min\left\{\frac{d}{v_{con}}, (a-h_0)^+\right\} + \frac{(d-(a-h_0)^+ v_{con})^+}{v^{max}}, u_1\right]$.

We provide conditions on the optimal solution to this problem in Propositions 5 (early policy) and 6 (late policy).

Proposition 5. *Consider a single-arc DSOP where the driver is paid from the start of the planning horizon (early policy) and must arrive at location 1 exactly at time* $\mu \in \left[h_0 + \min\left\{\frac{d}{v_{con}}, (a-h_0)^+\right\} + \frac{(d-(a-h_0)^+ v_{con})^+}{v^{max}}, u_1\right]$. *The optimal solution is such that* $y_0^* = 0$. *Further, if* $\mu \leq a$, *then* $z_0^* = \mu - h_0 - \frac{d}{v_{con}}$ *and the entire arc is driven in congestion. If* $\mu > a$, *then the optimal free-flow speed* v^* *takes one of the following 4 values: (i)* $\frac{d-(a-h_0)^+ v_{con}}{\mu - \max\{a, h_0\}}$, *(ii)* $\frac{d}{\mu - a}$, *(iii)* $\underline{v}$ *or (iv)* $\hat{v} \equiv \sqrt{\frac{B + C v_{con}^3}{3 C v_{con}}}$. *The optimal value of* z_0 *can be found by solving* $h_0 + z_0 + T_0(h_0 + z_0, v^*) = \mu$.

Since $y_0^* = 0$ and the arrival time at location 1 is fixed, the problem reduces to a single decision variable problem: given a chosen free-flow speed v, it is possible to calculate the amount of post-service waiting time at the depot so as to arrive exactly at μ.

When $\mu > a$, the first possible optimal speed value, i.e. $\frac{d-(a-h_0)^+ v_{con}}{\mu - \max\{a, h_0\}}$, corresponds to the case where the vehicle leaves the depot immediately upon completion of service while the other three involve a positive amount of post-service waiting time at the depot. In particular, with the second possible optimal speed value, i.e. $\frac{d}{\mu - a}$, the vehicle waits until the end of the congestion period, leaving the depot exactly at time a.

Proposition 6. *Consider a single-arc DSOP where the driver is paid from the start of service at location* 0 *(early policy) and must arrive at location* 1 *exactly at time* $\mu \in \left[h_0 + \min\left\{\frac{d}{v_{con}}, (a-h_0)^+\right\} + \frac{(d-(a-h_0)^+ v_{con})^+}{v^{max}}, u_1\right]$*. The optimal solution is such that* $z_0^* = 0$*. Further, if* $\mu \leq a$*, then* $y_0^* = \mu - h_0 - \frac{d}{v_{con}}$ *and the entire arc is driven in congestion. If* $\mu > a$*, then the optimal free-flow speed* v^**, takes one of the following 4 values: (i)* $\frac{d-(a-h_0)^+ v_{con}}{\mu - \max\{a,h_0\}}$*, (ii)* $\frac{d}{\mu - a}$*, (iii)* $\overline{v}$ *or (iv)* $\tilde{v} \equiv \sqrt{\frac{B+D+Cv_{con}^3}{3Cv_{con}}}$*. The optimal value of* y_0 *can be found by solving* $y_0 + h_0 + T_0(y_0 + h_0, v^*) = \mu$*.*

The first possible speed value when $\mu \leq a$, i.e. $\frac{d-(a-h_0)^+ v_{con}}{\mu - \max\{a,h_0\}}$, corresponds to the case where the vehicle leaves the depot immediately upon completion of service while the other three involve a positive amount of post-service waiting time at the depot. In particular, with the second speed value, i.e. $\frac{d}{\mu-a}$, the vehicle waits until the end of the congestion period, leaving exactly after a time units.

Next we consider the problem of minimizing total cost under the constraint that service at location 1 must start at a fixed time η. The decision maker needs to decide how long the vehicle should wait at the depot (pre- and post-service, i.e. y_0 and z_0), how fast to drive on the arc (i.e., speed v) and how long to wait pre-service at location 1 (i.e., y_1) in order to minimize the total cost which is measured until the fixed start of service at location 1. For this problem to be feasible we need $\eta \in [l_1, u_1]$.

Proposition 7. *Consider a single-arc DSOP where service at location* 1 *must start exactly at time* $\eta \in [l_1, U_1]$*. Under the early policy, there exists an optimal solution such that the arrival time at location* 1 *is equal to* $\min\{\max\{a, h_0\} + d/\underline{v}, \eta\}$*. Under the late policy, there exists an optimal solution such that the arrival time at location* 1 *is equal to* η*.*

According to Proposition 7, there will be some pre-service waiting time at location 1 under the early policy if it is possible for the vehicle to arrive by η at location 1 after waiting at location 0 until the end of traffic congestion and driving on the arc at speed $\underline{v}$. This waiting time at location 1 may be mandatory or voluntary In contrast, under the late policy, it is always optimal to start service immediately upon arrival into location 1.

In our shortest path formulation, we calculate the arc lengths by fixing the start of service time at the destination location. By Lemma 1, we know that these values can be calculated by considering a single arc-problem. Hence the results in this section can be used as follows: first we use Proposition 7 to obtain the optimal arrival time at location 1 given a fixed start of service time, then we use Proposition 5 under the early policy or Proposition 6 under the late policy in order to obtain the optimal wait times and speed value, and ultimately the optimal costs of traversing the arc, given a fixed arrival time.[3]

B Propositions and Proofs

For use in this section let w_i denote the end of service time at location i and let $\underline{w}_i$ denote the earliest end of service time at location i, namely $\underline{w}_0 = 0$ and $\underline{w}_i = \max\{\underline{\mu}_i, l_i\} + h_i$ for $i = 1, ..., n$. =Also remember that μ_i denotes the arrival time into node i. The two sets of variables are connected by the following formulas: $w_i = \max\{\mu_i, l_i\} + y_i + h_i$ for $i = 1, ..., n$, and $\mu_i = w_{i-1} + z_{i-1} + T_{i-1}(w_{i-1} + z_{i-1}, v_{i-1})$ for $i = 1, ..., n + 1$.

B.1 Proposition 1

Proof. Proof First we prove that there exists an optimal solution such that $y_i = 0$ for $i = 1, ..., n$. The proof is by contradiction. Consider a solution S with $y_i > 0$ for some $i \in \{1, ..., n\}$. There are two cases (i) either the vehicle arrives at location i before its lower time window limit, i.e., $\mu_i \leq l_i$ (ii) or it arrives after its lower time window limit $\mu_i > l_i$. In Case (i) consider an alternate solution S' with $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$ and $\mathbf{z}' = \mathbf{z}$ except that $z'_{i-1} = l_i + y_i - w_{i-1} - \min\{(l_i + y_i - a)^+, d_{i-1}/v_{i-1}\} - [d_{i-1} - (l_i + y_i - a)^+ v_{i-1}]^+ / v_{con}$ and $y'_i = 0$, which implies that the vehicle arrives at location i exactly at $l_i + y_i$ and starts service immediately at that time. Note that $z'_i \geq z_i$. In both solutions S and S' the end of service time at location i is $w_i = l_i + y_i + h_i = w_{i-1} + z'_{i-1} + T_{i-1}(w_{i-1} + z'_{i-1}, v_i) + h_i$ so the costs on arcs other than $i-1$ are the same; however, the cost of traversing arc i is less or equal in solution S because, while the driver cost remains the same, the emission cost is (weakly) lower. This is because, in solution S, postponing the departure time may reduce the emissions cost as the vehicle travels in congestion for a shorter period of time. Therefore the total cost of solution S' cannot be strictly more than that of solution S, which is a contradiction. In Case (ii) consider an alternate solution S' with $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$ and $\mathbf{z}' = \mathbf{z}$ expect that z'_{i-1} is set such that $z'_{i-1} = T_{i-1}(w_{i+1} + z_{i+1}, v_i) + y_i - w_{i-1} - \min\{(T_{i-1}(w_{i+1} + z_{i+1}, v_i) + y_i - a)^+, d_{i-1}/v_{i-1}\} - [d_{i-1} - (T_{i-1}(w_{i+1} + z_{i+1}, v_i) + y_i - a)^+ v_{i-1}]^+ / v_{con}$.

[3] Note that Proposition 7 is more general than needed. It tells us what the optimal arrival time should be for any possible value for the start of service time. However, in our implementation of the DP, we only need this information when the start of service time is either l_1 or U_1.

In both solutions S and S' the end of service time at location i is $w_i = w_{i-1} + z_{i-1} + T_{i-1}(w_{i+1} + z_{i+1}, v_i) + y_i + h_i = w_{i-1} + z'_{i-1} + T_{i-1}(w_{i+1} + z'_{i+1}, v_i) + h_i$ so the costs on arcs other than $i-1$ are the same. However, as in Case (i), the emissions cost from traversing arc $i-1$ may be lower in solution S' than in S due to the postponed departure time from location $i-1$. Therefore the total cost of solution S' cannot be strictly more than that of solution S, which is a contradiction.

Next we prove by contradiction that there exists an optimal solution such that $y_0 = 0$ under the early policy. Suppose all optimal solutions have a positive voluntary pre-service waiting time at the depot. Let S be one such solution with voluntary pre-service waiting time at the depot $y_0 > 0$. Consider an alternate solution S' with $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$ and $\mathbf{z}' = \mathbf{z}$ expect that $y'_0 = 0$ and $z'_0 = z_0 + y_0$. In this solution, we have $w'_0 = y_0 + h_0 + z_0 = h_0 + z'_0$ and $w'_i = w_i$ for $i = 0, ..., n$. Since the free-flow speed on arc and the departure time from each location is the same, the two solutions have the same total cost.

Finally we prove by contradiction that there exists an optimal solution such that $z_0 = 0$ under the late policy. Suppose all optimal solutions have a positive voluntary post-service waiting time at the depot. Let S be one such solution with voluntary pre-service waiting time at the depot $z_0 > 0$. Consider an alternate solution S' with $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$ and $\mathbf{z}' = \mathbf{z}$ expect that $y'_0 = y_0 + z_0$ and $z'_0 = 0$. In this solution, we have $w'_0 = y_0 + h_0 + z_0 = h_0 + y'_0$ and $w'_i = w_i$ for $i = 0, ..., n$. The free-flow speed on arc and the departure time from each location is the same, therefore the two solutions have the same emissions cost, however since in S' the driver starts being paid at time y'_0, the total cost in S' must be lower to that in S, by an amount equal to Dz_0. Hence we have a contradiction. □

B.2 Proposition 2

Proof. Proof *Proof of part (i).* First we prove that $z_i = 0$ for $i < j_1$. Suppose we have an optimal solution S with $y_i = 0$ (by Proposition 1) for $i = 1, ..., n$ but $z_i \geq 0$ for some $i < j_1$. By definition of j_1, arcs 0 to $j_1 - 1$ are traversed entirely in congestion. Consider an alternate solution S' such that $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$, and $\mathbf{z}' = \mathbf{z}$ except that $z'_i = 0$ for $i = 1, ..., j_1 - 1$ and $z'_{j_1} = w_{j_1} - w'_{j_1}$. The departure time from location j_1 is the same in both solutions as $w_{j_1} + z_{j_1} = w'_{j_1} + z'_{j_1}$. Since the travel speed on every arc remain the same both solutions have the same total cost.

Next we prove that there exists an optimal solution with $z_i = 0$ for $i > j_1$. Suppose we have an optimal solution S with $y_i = 0$ for $i = 1, ..., n$ but $z_i > 0$

for some $i \in \{j_1 + 1, ..., n\}$. By definition of j_1, it must be that $w_i \geq a$ for $i = j_1 + 1, ..., n$, therefore arcs $j_1 + 1, ..., n$ are travelled during the free flow period and $T_i(w_i + z_i, v_i) = T_i^{free}(w_i + z_i, v_i) = d_i/v_i$ for $i = j_1 + 1, ..., n$.

There are two cases (i) either the vehicle arrives at location $i+1$ before its lower time window limit, i.e., $w_i + z_i + d_i/v_i \leq l_{i+1}$ (ii) or it arrives after the lower time window limit, i.e., $w_i + z_i + d_i/v_i \geq l_{i+1}$. In Case (i), the vehicle has to wait at location $i + 1$ until time l_{i+1} before starting service so that the sum of post-service waiting time at location i and mandatory pre-service waiting time at location $i+1$ is $z_i + (l_{i+1} - w_i - z_i - d_i/v_i) = l_{i+1} - w_i - d_i/v_i$. Now consider an alternate solution S' such that $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$ and $\mathbf{z}' = \mathbf{z}$ expect that $z_i' = 0$. In this case, the departure time from location i is $w_i' = w_i$ and the arrival time at location $i+1$ at time is $w_i' + d_i/v_i = w_i + d_i/v_i \leq l_{i+1}$, so that the vehicle has to wait $l_{i+1} - w_i - \frac{d_i}{v_i}$ at location $i + 1$. Hence, in this alternate solution the sum of post-service waiting time at location i and mandatory pre-service waiting time at location $i + 1$ is $l_{i+1} - w_i - d_i/v_i$, which is the same as in the optimal solution. Since the driver is paid equally for any type of waiting time and the speed driven on all the arcs has not changed, both solutions have the same total cost. In Case (ii), the vehicle arrives at location $i + 1$ at time $w_i + z_i + \frac{d_i}{v_i}$, which is greater than l_{i+1}, so it does not wait at location $i + 1$ before starting service. Consider an alternative solution S' such that $\mathbf{v}' = \mathbf{v}, \mathbf{y}' = \mathbf{y}$ and $\mathbf{z}' = \mathbf{z}$ expect that with $z_i' = 0$ and z_{i+1}' is set as explained below. In this case, the departure time from location i is $w_i' = w_i$ and the arrival time at location $i + 1$ is $w_i' + d_i/v_i = w_i + d_i/v_i$. If $i = n$, then the vehicle arrives at location $n + 1$ in solution S' at time $w_n + d_n/v_n$, which is earlier than the arrival time of $w_n + z_n + d_n/v_n$ in the solution S. Since the voluntary waiting time is decreased and speed values are the same, it must have a lower cost, which is a contradiction. If $i < n$, there are 2 subcases: (a) $w_i + d_i/v_i \leq l_{i+1}$ and (b) $w_i + d_i/v_i > l_{i+1}$. In sub-case (a), the vehicle in solution S' waits for a duration of $l_{i+1} - w_i - d_i/v_i$ at location $i+1$ before starting service; then we set $z_{i+1}' = w_i + z_i + d_i/v_i + z_{i+1} - l_{i+1}$. In sub-case (b), the vehicle arrives at location $i+1$ after l_{i+1} so it does not wait before starting service. In this case we set a post-service waiting time of $z_{i+1}' = z_{i+1} + z_i$ at location $i + 1$. In either sub-case, solution S' has no post-service waiting time at location i and it has the same total cost as the original solution since the total waiting time and the speed values are the same. The same argument can be used for the following locations: either we eliminate the post-service waiting time (as in Case (i)) or we transfer it to the next location (as in Case (ii)), etc., until we reach the last arc.

Proof of part (ii).

Given $i > j_1$ the vehicle reaches locations $i, ..., n+1$ past the end of the congestion period, therefore, by Part (i) we have $z_i = ... = z_n = 0$. The cost of driving on arcs $i-1$ and i, denoted $TC_{i-1,i+1}$, measured from the departure time from location $i-1$ until the arrival time at location $i+1$, is

$$\begin{aligned} TC_{i-1,i+1}(v_{i-1}, v_i) &= A(d_{i-1} + d_i) + (B + D)\left[(a - w_{i-1} - z_{i-1})^+ + \frac{d_{i-1} - (a - w_{i-1} - z_{i-1})^+ v_{con}}{v_{i-1}}\right] + \\ &\quad C\left[(a - w_{i-1} - z_{i-1})^+ {v_{con}}^3 + (d_{i-1} - (a - w_{i-1} - z_{i-1})^+ v_{con})v_{i-1}^2\right] \\ &\quad + Dh_i + (B + D)\frac{d_i}{v_i} + Cd_i v_i^2 \\ &= A(d_{i-1} + d_i) + (B + D)(a - w_{i-1} - z_{i-1})^+ + C(a - w_{i-1} - z_{i-1})^+ {v_{con}}^3 + Dh_i \\ &\quad + (B + D)\left(\frac{d_{i-1} - (a - w_{i-1} - z_{i-1})^+ v_{con}}{v_{i-1}} + \frac{d_i}{v_i}\right) \\ &\quad + C((d_{i-1} - (a - w_{i-1} - z_{i-1})^+ v_{con})v_{i-1}^2 + d_i v_i^2). \end{aligned}$$

The arrival time at location $i+1$, i.e., μ_{i+1}, is equal to $\max\{a, w_{i-1} + z_{i-1}\} + \frac{d_i - (a - w_{i-1} + z_{i-1})^+ v_{con}}{v_{i-1}} + h_i + \frac{d_i}{v_i}$. Given a speed of v_{i-1} on arc $i-1$, the vehicle needs to travel at speed

$$v_i = \frac{d_i}{\mu_{i+1} - \max\{a, w_{i-1} + z_{i-1}\} - h_i - ((d_{i-1} - (a - w_{i-1} + z_{i-1})^+ v_{con})/v_{i-1}}$$

so as to arrive at location $i+1$ exactly at time μ_{i+1}. Using this expression we rewrite the cost function as a function of v_{i-1} as:

$$\begin{aligned} TC_{i-1,i+1}(v_{i-1}) &= A(d_{i-1} + d_i) + (B + D)(a - w_{i-1} - z_{i-1})^+ + C(a - w_{i-1} - z_{i-1})^+ {v_{con}}^3 + Dh_i \\ &\quad + (B + D)\left(\mu_{i+1} - \max\{a, w_{i-1} + z_{i-1}\} - h_i\right) + \\ &\quad C\Big[(d_{i-1} - (a - w_{i-1} + z_{i-1})^+ v_{con})v_{i-1}^2 + \\ &\quad \frac{d_i^3}{(\mu_{i+1} - \max\{a, w_{i-1} + z_{i-1}\} - h_i - (d_{i-1} - (a - w_{i-1} - z_{i-1})^+ v_{con})/v_{i-1})^2}\Big]. \end{aligned}$$

This function is convex in v_{i-1} and achieves a maximum at

$$v_{i-1} = \frac{(d_{i-1} - (a - w_{i-1} + z_{i-1})^+ v_{con}) + d_i}{\mu_{i+1} - \max\{a, w_{i-1} + z_{i-1}\} - h_i}$$

which implies that $v_{i-1} = v_i$.

□

B.3 Lemma 1

Proof. Proof From Proposition 1 it is $y_i = 0$ for $i + 1, ..., n$. Furthermore, since arcs $i, i + 1, ..., j$ are traversed in free-flow it is $i + 1 > j_1$, therefore from Proposition 2 part (i) it is $z_i = 0$ for $i+1, ..., n$. This implies that there exists an optimal solution to this problem with zero waiting time (pre- or post- service) at locations $i+1, ..., j-1$ and therefore the problem can be reduced to a single arc problem by summing the lengths of the arcs and by setting the start of service time to $\hat{\eta} = \eta - \sum_{k=i+1}^{j-1} h_k$. □

B.4 Proposition 3

Proof. Proof From Proposition 1 we only consider optimal solutions such that $y_i = 0$ for $i = 1, ..., n+1$.

First we prove the existence of $\hat{j}$ by contradiction. Suppose $\hat{j}$ does not exist. There can be two cases: (1) there exists a location $\tilde{i} \in \{1, ..., \hat{k}-1\}$ with $\mu_{\tilde{i}} < l_{\tilde{i}}$, where $\hat{k}$ is the lowest index location such that $\mu_{\hat{k}} \in \{l_{\hat{k}}, U_{\hat{k}}\}$; (2) all the customer locations are such that $l_i < \mu_i < U_i$ or $\mu_i < l_i$ for $i = 1, ..., n+1$.

Case (1): Let $\tilde{i}$ be the first location where this holds so that $\mu_i \in (l_i, U_i)$ for $i = 1, ..., \tilde{i}-1$. There are two sub-cases depending on whether the vehicle arrives at location $\tilde{i}$ before or after the end of the congestion period: (1.1) $\mu_{\tilde{i}} < a$ and (1.2) $\mu_{\tilde{i}} \geq a$.

In sub-case (1.1), consider an alternate solution S' with $z'_i = z_i$, $v'_i = v_i$ for $i = 0, ..., n$ but $y'_0 = y_0 + \epsilon$ where $\epsilon = \min\left\{\min_{i=1,...,\tilde{i}-1}\{U_i - \mu_i\}, l_{\tilde{i}} - \mu_{\tilde{i}}, a - \mu_{\tilde{i}}\right\}$. As a result, the vehicle arrives at location $\tilde{i}$ with a ϵ-delay but since service does not start at location $\tilde{i}$ until $l_{\tilde{i}}$, the departure time from location $\tilde{i}$ and the arrival time at locations $\tilde{i}+1, ..., n+1$ are unaffected, that is, $\mu'_i = \mu_i + \epsilon$ for $i = 1, ..., \tilde{i}$ and $\mu'_i = \mu_i$ for $i = \tilde{i}+1, ..., n+1$. Since the total time spent driving in congestion and in free-flow, as well as the free-flow speeds are the same in both solutions, the difference in costs between S and S' is equal to the difference in total labor costs, which is $D\epsilon \geq 0$, therefore we have a contradiction.

In sub-case (1.2) let j denote the last location such that $\mu_j < a$ (set $j = 0$ if $\mu_1 \geq a$). By Proposition 2 Part (ii) it must be $v_j = v_{j+1} = ... = v_{\tilde{i}-1} \equiv v$ since the arrival time at locations j to $\tilde{i}-1$ is within their respective time windows. We first prove that this speed v must all be equal to $\underline{v}$ if S is an optimal solution. Let $d_{j+1,\tilde{i}} = \sum_{i=j+1}^{\tilde{i}-1} d_i$ denote the total distance from location $j+1$ to location $\tilde{i}$. The cost of driving from location j to location $\tilde{i}$, leaving location j at time $w_j + z_j$ and arriving at $\tilde{i}$ at time $\mu_{\tilde{i}}$ strictly less than $l_{\tilde{i}}$ (measured from time $w_j + z_j$ until the start of service at location $\tilde{i}$, which is $l_{\tilde{i}}$) can be written as

$$\begin{aligned} TC_{j+1,\tilde{i}-1}(v) &= B\left((a-w_j-z_j)^+ + \frac{(d_j-(a-w_j-z_j)^+v_{con})}{v} + \frac{d_{j+1,\tilde{i}}}{v}\right) \\ &+C\left((a-w_j-z_j)^+ {v_{con}}^3 + ((d_j-(a-w_j-z_j)^+v_{con}))v^2 + d_{j+1,\tilde{i}}v^2\right) \\ &+D(l_{\tilde{i}}-w_j-z_j). \end{aligned}$$

This function is convex and minimized at $\underline{v}$. Hence, if S is optimal, it must be that $v = \underline{v}$. Now consider an alternative solution S' with $v'_i = v_i$ for $i = 0, ..., n$, $y'_0 = y_0 + \epsilon$, $z'_i = z_i$ for $i = 0, ..., n$, where

$$\epsilon = \begin{cases} \min\left\{\min_{\{i=1,\dots,\tilde{i}-1\}}\{U_i-\mu_i\}, a-\mu_j, l_{\tilde{i}}-\mu_{\tilde{i}}\right\} & \text{if } h_0+y_0<a; \\ \min\left\{\min_{\{i=1,\dots,\tilde{i}-1\}}\{U_i-\mu_i\}, l_{\tilde{i}}-\mu_{\tilde{i}}\right\} & \text{otherwise.} \end{cases}$$

In other words, the driver waits pre-service an extra ϵ units of time at the initial location,

Note that we have $w'_i = w_i + \epsilon$ for $i = 1, ..., j$ and $w'_i \leq w_i + \epsilon$ for $i = j+1, ..., n+1$ because, thanks to the delay at the initial location, a greater (or equal) proportion of arc j is driven in free-flow speed in solution S' compared to S. Arcs 1 to $j-1$ are traversed entirely in congestion and arcs $j+1$ to $\tilde{i}-1$ are traversed entirely in free flow (at speed $\underline{v}$) in both solutions. Also, just like in solution S, the vehicle arrives at location $\tilde{i}$ before $l_{\tilde{i}}$ in solution S'; hence, the service completion times at location $\tilde{i}, ..., n$ is unaffected, i.e., $w'_i = w_i$ for $i = \tilde{i}, ..., n$. Therefore the difference in costs between solution S and S' is equal to the difference in costs of traveling arc j only (measured from the departure time from location j until the arrival time at location $j+1$) plus the difference in total labor cost, which is $D\epsilon$. If $w_j - z_j < a$, it is equal to

$$\begin{aligned} TC-TC' &= B(a-w_j-z_j) + B\frac{d_j-(a-w_j-z_j)v_{con}}{\underline{v}} + Cv_{con}^3(a-\underline{v}) \\ &+C\underline{v}^2(d_j-(a-w_j-z_j)v_{con}) - B(a-w_j-z_j-\epsilon) \\ &-B\frac{d_j-(a-w_j-z_j-\epsilon)v_{con}}{\underline{v}} - Cv_{con}^3(a-w_j-z_j-\epsilon) \\ &-C\underline{v}^2(d_j-(a-w_j-z_j-\epsilon)v_{con}) + D\epsilon \\ &= \epsilon\left(B\frac{\underline{v}-v_{con}}{\underline{v}} + C(v_{con}^3-\underline{v}^2v_{con})\right) + D\epsilon \\ &= \epsilon C(\underline{v}-v_{con})^2(2\underline{v}+v_{con}) + D\epsilon, \end{aligned}$$

where the last inequality was obtained using $B = 2C\underline{v}^3$. This expression is positive since $\underline{v} \geq v_{con}$ and $\epsilon > 0$, therefore we have a contradiction. If $w_j + z_j > a$ (i.e., the end of the congestion period happens during the service time at location $j+1$) then arc j is traversed entirely in free flow in both solutions so that $TC_{j,j+1} - TC'_{j,j+1} = D\epsilon > 0$. Therefore in both cases we have a contradiction.

Case (2) Again, we consider two sub-cases depending on whether or not there exists a location $i \in \{1, ..., n+1\}$ such that $\mu_i < l_i$. In sub-case (2.1), let $\tilde{i}$ be the first customer location such that $\mu_{\tilde{i}} < l_{\tilde{i}}$. Here, we can construct an alternative solution which is better than S exactly as we did in Case (1).

In sub-case (2.2), all customer locations are such that $\mu_i \in (l_i, U_i)$. Let j denote the last customer location such that $\mu_j < a$ (set $j = 0$ if $\mu_1 \geq a$). Since the vehicle arrives at each location strictly within the time window limits, by Proposition 2 it must be that $v_j = v_{j+1} = ... = v_n \equiv v$. Let $d_{j+1,n+1} = \sum_{i=j+1}^{n} d_i$ denote the total distance from location $j+1$ to location $n+1$. The cost of driving from location j to location $n+1$, leaving location j at time $w_j + z_j$, reaching each location strictly within their time window (measured from time $w_j + z_j$ until the end of service time at location $n+1$) can be written as:

$$
\begin{aligned}
TC_{j,n+1}(v) \quad = \quad & (B+D)\left((a - w_j - z_j)^+ + \frac{(d_j - (a - w_j - z_j)^+ v_{con})}{v} + \frac{d_{j+1,n+1}}{v}\right) \\
& + C\left((a - w_j - z_j)^+ {v_{con}}^3 + ((d_j - (a - w_j - z_j)^+ v_{con}))v^2 + d_{j+1,n+1}v^2\right) \\
& + D \sum_{i=j+1}^{n+1} h_i.
\end{aligned}
$$

This function is convex and minimized at $\overline{v}$. Hence, if S is optimal, it must be that $v = \overline{v}$. The rest of the proof is the same as in Case (1.2) except that $\underline{v}$ is replaced with $\overline{v}$. Hence, we have proven the existence of location $\hat{j}$.

Next we prove that $z_0 = ... = z_{\hat{j}-1} = 0$. By Proposition 2, there exist optimal solutions where all the post-service waiting time (if there is any) is concentrated at location j_1, defined as the largest index such that $\mu_{j_1} < a$. In other words we have $z_i = 0$ for $i \neq j_1$. Let consider one such optimal solution S. The proof is by contradiction: suppose that $j_1 < \hat{j}$, such that $z_{j_1} > 0$. Consider an alternate solution S' where $v'_i = v_i$ for $i = 0, .., n$, $z'_i = z_i = 0$ for $i \neq j$ but $y'_0 = y_0 + \epsilon$ and $z'_j = z_j - \epsilon$, where $\epsilon = \min\{\min_{i=1,...,j_1-1}\{U_i - \mu_i\}, z_{j_1}, a - \mu_{j_1}\}$. In other words we shift some of the post-service waiting time from location j_1 to a pre-service waiting time at the depot. As a result we have $w'_i = w_i + \epsilon$ for $i = 0, ..., j_1 - 1$ and $w'_i = w_i$ for $i = j_1, ..., n+1$. Since the total time spent driving in congestion and in free flow, as well as the free-flow speeds are the same in both solutions, the difference in costs between S and S' is equal to the difference in total labor costs, which is $D\epsilon \geq 0$, therefore we have a contradiction.

Finally, we prove that if $\mu_{\hat{j}} \leq a$ and $\mu_{\hat{j}} = U_{\hat{j}}$ then $\hat{j} = 1$. Suppose (contradiction) that $\hat{j} > 1$, given the definition of $U_{\hat{j}}$ and the fact that $z_0 = ... = z_{\hat{j}-1} = 0$ it follows $\mu_i = U_i$ for $i = 1, ..., \hat{j}$ and therefore we have a contradiction. □

B.5 Proposition 4

Proof. Proof Consider the problem of minimizing total cost in the absence of any time window constraint, i.e., assuming $l_i = 0$ and $u_i = \infty$ for $i = 1, ..., n$ and without any traffic congestion, i.e. assuming $a = 0$. In this case, we have $\mathcal{C} = A\sum_{i=0}^{n} d_i + \sum_{i=0}^{n}(B + Cv_i^2 + D)\frac{d_i}{v_i} + D\sum_{i=0}^{n} h_i$, which is minimized by setting $v_i = \overline{v}$ for $i = 0, ..., n$. The minimum total cost in this problem constitutes a lower bound for the problem with time windows and traffic congestion. Since the total cost of the schedule described in Proposition 4 is equal to this lower bound, it must be optimal. □

B.6 Proposition 5

Proof. Proof

Under the early policy, the driver is paid equally for her pre- or post-service waiting time. Therefore there must exist an optimal solution such that $y_0^* = 0$ and $z_0^* \geq 0$, that is, all the waiting occurs post-service at the depot.

Given this, the total cost expression under the early policy simplifies to

$$TC(v, z_0) = Ad +$$
$$\begin{cases} (B + D + C{v_{con}}^3)\frac{d}{v_{con}} + D\left(h_0 + z_0 + \left(l_1 - h_0 - z_0 - \frac{d}{v_{con}}\right)^+\right) & \text{if } z_0 \leq a - \frac{d}{v_{con}} - h_0 \\ (B + D)\left((a - h_0 - z_0)^+ + \frac{(d - (a - h_0 - z_0)^+ v_{con})}{v}\right) & \\ +C\left((a - h_0 - z_0)^+ {v_{con}}^3 + (d - (a - h_0 - z_0)^+ v_{con})^+ v^2\right) & \\ +D\left(h_0 + z_0 + \left(l_1 - h_0 - z_0 - (a - h_0 - z_0)^+ - \frac{d - (a - h_0 - z_0)^+ v_{con}}{v}\right)^+\right) & \text{if } z_0 > a - \frac{d}{v_{con}} - h_0. \end{cases}$$

If $\mu \leq a$, then the vehicle drives the entire arc in congestion, so the free-flow speed v is irrelevant and in order to arrive at μ, we must have $h_0 + z_0 + \frac{d}{v_{con}} = \mu$, that is, $z_0 = \mu - h_0 - \frac{d}{v_{con}}$.

If $\mu > a$, then the vehicle drives (at least) part of the arc in free-flow. In order to arrive at μ, we must have $h_0 + z_0 + T_0(h_0 + z_0, v) = \mu$, that is, $h_0 + z_0 + (a - h_0 - z_0)^+ + \frac{d - (a - h_0 - z_0)^+ v_{con}}{v} = \mu$, so we can write the free-flow speed as a function of z_0 as $v(z_0) = d - (a - h_0 - z_0)^+ v_{con}\mu - h_0 - z_0 - (a - h_0 - z_0)^+$. If $z_0 < (a - h_0)^+$, the vehicle drives partially in congestion and partially in free-flow and $v(z_0) = \frac{d-(a-h_0-z_0)v_{con}}{\mu-a}$. If $z_0 \geq (a - h_0)^+$, then the vehicle drives entirely in free-flow and we have $v(z_0) = \frac{d}{\mu-h_0-z_0}$.

There are two cases: (i) $a < h_0 < \mu$ (i.e., congestion period ends before service time at the depot) and (ii) $a > h_0$ (i.e., the congestion period ends after the service at the depot).

In Case (i), the total cost as a function of $z_0 \in \left[0, \mu - h_0 - \frac{d}{v^{max}}\right]$:

$$\begin{aligned} TC(z_0) &= Ad + (B+D)\frac{d}{v(z_0)} + Cd[v(z_0)]^2 + D(h_0 + z_0 + (l_1 - \mu)^+) \\ &= Ad + (B+D)(\mu - h_0 - z_0) + C\frac{d^3}{(\mu - h_0 - z_0)^2} + D(h_0 + z_0 + (l_1 - \mu)^+) \\ &= Ad + B(\mu - h_0 - z_0) + C\frac{d^3}{(\mu - h_0 - z_0)^2} + D(\mu + (l_1 - \mu)^+). \end{aligned}$$

This expression is convex in z_0 and minimized at $z_0 = \mu - h_0 - \frac{d}{\underline{v}}$, which corresponds to a speed $v(z_0) = \underline{v}$. This value of z_0 is positive if and only if $\mu > h_0 + \frac{d}{\underline{v}}$. So if $\mu > h_0 + \frac{d}{\underline{v}}$, then the driver should wait $z_0 = \mu - h_0 - \frac{d}{\underline{v}}$ at the depot and drive at $\underline{v}$ (which is the third possible optimal speed value), otherwise he or she should leave right away and drive at $\frac{d}{\mu - h_0}$ (which, in this case, corresponds to the first possible optimal speed value).

In Case (ii), the total cost as a function of z_0 is:

$$TC(z_0) = \begin{cases} Ad + B(\mu - h_0 - z_0) & \\ +C\left((a - h_0 - z_0)v_{con}{}^3 + \frac{(d-(a-h_0-z_0)v_{con})^3}{(\mu-a)^2}\right) & \text{if } z_0 < a - h_0 \\ +D(\mu + (l_1 - \mu)^+) & \\ Ad + B(\mu - h_0 - z_0) + C\frac{d^3}{(\mu-h_0-z_0)^2} & \text{if } a - h_0 \leq z_0 \leq \mu - h_0 - \frac{d}{v^{max}} \\ +D(\mu + (l_1 - \mu)^+) & \end{cases}$$

Note that if $\mu \leq a + \frac{d}{v^{max}}$ then the second piece is not defined as we must have $z_0 \leq a - h_0 - \frac{(d-(\mu-a)v^{max})^+}{v_{con}}$, which in that case is less than $a - h_0$.

The first piece is minimized at $z_0 = a - h_0 - \frac{d-(\mu-a)\hat{v}}{v_{con}}$ where $\hat{v} = \sqrt{\frac{B+Cv_{con}^3}{3Cv_{con}}}$, which corresponds to a speed of $v(z_0) = \hat{v}$. The minimum value is to the left of $a - h_0$ if $\mu < a + \frac{d}{\hat{v}}$, otherwise it is at $a - h_0$. Also the minimum value of z_0 is positive if and only if $\mu > a + \frac{d-(a-h_0)v_{con}}{\hat{v}}$. The second piece is minimized at $z_0 = \mu - h_0 - \frac{d}{\underline{v}}$, which corresponds to a speed $v(z_0) = \underline{v}$. The minimum value is always to the left of $\mu - h_0 - \frac{d}{v^{max}}$ and is to the right of $a - h_0$ if and only if $\mu > a + \frac{d}{\underline{v}}$.

It is easy to show that we always have $\hat{v} > \underline{v}$. Hence, $a + \frac{d-(a-h_0)v_{con}}{\hat{v}} \leq a + \frac{d}{\hat{v}} < a + \frac{d}{\underline{v}}$ and we have following cases:

- If $\mu < a + \frac{d-(a-h_0)v_{con}}{\hat{v}}$, then TC is increasing and hence minimized at $z_0 = 0$, where the corresponding optimal speed is $\frac{d-(a-h_0)v_{con}}{\mu - a}$ (which is, in this case corresponds to the first possible optimal speed value). The driver leaves the depot right away and drives the necessary speed as to be able to arrive at the customer at μ.

- If $a + \frac{d-(a-h_0)v_{con}}{\hat{v}} \leq \mu \leq a + \frac{d}{\hat{v}}$, then TC is minimized at $z_0 = a - h_0 - \frac{d-(\mu-a)\hat{v}}{v_{con}}$. The driver waits at the depot the required amount of time so as to be able to arrive at the customer at μ driving at speed $\hat{v}$ (which is the fourth possible optimal speed value).

- If $a + \frac{d}{\hat{v}} \leq \mu \leq a + \frac{d}{\underline{v}}$, then TC is minimized at $z_0 = a - h_0$. The driver waits at the depot until the end of the congestion period, then drives in free-flow and arrives at the customer location at μ driving at speed $\frac{d}{a-\mu}$ =(which is the second possible optimal speed value).

- If $a + \frac{d}{\underline{v}} < \mu$, then TC is minimized at $z_0 = \mu - h_0 - \frac{d}{\underline{v}}$. The driver waits at the depot the required amount of time so as to be able to arrive at the customer at μ driving at speed $\underline{v}$ (which is the third possible optimal speed value).

From this we see that there are only four possible values for the optimal speed, which are the ones listed in the Proposition. □

B.7 Proposition 6

Proof. Proof

The proof is the same as for the early policy except (Proposition 5) that z_0 is replaced with y_0 and we subtract Dy_0 from the cost function. As a result, $\underline{v}$ gets replaced by $\overline{v}$ and $\hat{v}$ is replaced with $\tilde{v} = \sqrt{\frac{B+D+Cv_{con}^3}{3Cv_{con}}}$. The optimal value of y_0 can be found by solving $y_0 + h_0 + T_0(h_0 + z_0, v^*) = \mu$. □

B.8 Proposition 7

Proof. Proof First, we prove the result under the early policy. If $\eta \leq a$, then the vehicle drives the entire arc in congestion, so the free-flow speed v is irrelevant and the arrival time at location 1 is $\mu = h_0 + z_0 + \frac{d}{v_{con}}$, in other words, $z_0 = \mu - h_0 - \frac{d}{v_{con}}$. The total cost as a function of $\mu \in \left[h_0 + \frac{d}{v_{con}}, \eta\right]$ is

$$TC(\mu) \quad = \quad (B + C{v_{con}}^3)\frac{d}{v_{con}} + D\eta,$$

which does not depend on μ. Therefore any value of $\mu \in \left[h_0 + \frac{d}{v_{con}}, \eta\right]$ is optimal.

Next we assume that $\eta > a$. We consider two cases: (i) $a < h_0$ and (ii) $a > h_0$. In each case, we use the optimal values of z_0 and v defined in the proof of Proposition 5 to calculate the optimal cost as a function of $\mu \leq \eta$.

In Case (i), we get for $\mu \leq \eta$:

$$TC(\mu) = Ad + D\eta + \begin{cases} B(\mu - h_0) + Cd\left(\frac{d}{\mu - h_0}\right)^2 & \text{if } \mu < h_0 + \frac{d}{\underline{v}} \\ B\frac{d}{\underline{v}} + Cd\underline{v}^2 & \text{if } \mu > h_0 + \frac{d}{\underline{v}}. \end{cases}$$

The expression in the first piece is convex and minimized at $\mu = h_0 + \frac{d}{\underline{v}}$, which is its upper limit, therefore the first piece is decreasing in μ. The second one is constant. Hence if $\eta > h_0 + \frac{d}{\underline{v}}$ any value between $h_0 + \frac{d}{\underline{v}}$ and η is optimal, otherwise η is the optimal value.

In Case (ii), we get for $\mu \leq \eta$:

$$TC(\mu) = Ad + D\eta +$$

$$\begin{cases} B(\mu - h_0) + C\left((a - h_0){v_{con}}^3 + \frac{(d-(a-h_0)v_{con})^3}{(\mu-a)^2}\right) & \text{if } \mu < a + \frac{d-(a-h_0)v_{con}}{\hat{v}} \\ B\left(\frac{d-(\mu-a)\hat{v}}{v_{con}} + (\mu - a)\right) + C\left([d - (\mu - a)\hat{v}]{v_{con}}^2 + (\mu - a)\hat{v}^3\right) & \text{if } a + \frac{d-(a-h_0)v_{con}}{\hat{v}} \leq \mu < a + \frac{d}{\hat{v}} \\ B(\mu - a) + C\frac{d^3}{(\mu-a)^2} & \text{if } a + \frac{d}{\hat{v}} \leq \mu < a + \frac{d}{\underline{v}} \\ B\frac{d}{\underline{v}} + Cd\underline{v}^2 & \text{if } \mu \geq a + \frac{d}{\underline{v}} \end{cases}$$

For $\mu \leq \eta$, the expression in the first piece is convex and minimized at $a + \frac{d-(a-h_0)v_{con}}{\underline{v}}$ but since $a + \frac{d-(a-h_0)v_{con}}{\underline{v}} \geq a + \frac{d-(a-h_0)v_{con}}{\hat{v}}$, the first piece is decreasing in μ. The second piece is linear decreasing in μ. The expression in the third piece is convex and minimized at $a + \frac{d}{\underline{v}}$ which is its upper limit, therefore the third piece is decreasing in μ. The last piece is constant. Therefore, as in case (i), any value between $h_0 + \frac{d}{\underline{v}}$ if $\eta > h_0 + \frac{d}{\underline{v}}$, otherwise, $\mu = \eta$ is optimal.

Finally we prove the result under the late policy. If $\eta \leq a$, then the vehicle drives the entire arc in congestion, so the free-flow speed v is irrelevant and the arrival time at location 1 is $\mu = h_0 + y_0 + \frac{d}{v_{con}}$, in other words, $y_0 = \mu - h_0 - \frac{d}{v_{con}}$. The total cost as a function of $\mu \in \left[h_0 + \frac{d}{v_{con}}, \eta\right]$ is

$$TC(\mu) = (B + C{v_{con}}^3)\frac{d}{v_{con}} + D(\eta - \mu),$$

which is decreasing in μ therefore setting $\mu = \eta$ is optimal.

Next we assume that $\eta > a$ and as with the early policy, we consider two cases: (i) $a < h_0$ and (ii) $a > h_0$. In each case, we use the optimal values of

y_0 and v defined in the proof of Proposition 5 to calculate the optimal cost as a function of $\mu \leq \eta$.

In Case (i), we get for $\mu \leq \eta$:

$$TC(\mu) = Ad + \begin{cases} B(\mu - h_0) + Cd\left(\frac{d}{\mu - h_0}\right)^2 + D(\mu - h_0) & \text{if } \mu < h_0 + \frac{d}{\overline{v}} \\ B\frac{d}{\overline{v}} + Cd\overline{v}^2 + D(\frac{d}{\overline{v}} + \eta - \mu) & \text{if } \mu > h_0 + \frac{d}{\overline{v}}. \end{cases}$$

The expression in the first piece is convex and minimized at $\mu = h_0 + \frac{d}{\overline{v}}$, which is its upper limit, therefore the first piece is decreasing in μ. The second one is decreasing in μ. Therefore the optimal value is $\mu = \eta$.

In Case (ii), we get for $\mu \leq \eta$:

$$TC(\mu) = Ad +$$

$$\begin{cases} B(\mu - h_0) + C\left((a - h_0){v_{con}}^3 + \frac{(d-(a-h_0)v_{con})^3}{(\mu-a)^2}\right) & \text{if } \mu < a + \frac{d-(a-h_0)v_{con}}{\tilde{v}} \\ +D(\mu - h_0) & \\ (B + D)\left(\frac{d-(\mu-a)\tilde{v}}{v_{con}} + (\mu - a)\right) & \text{if } a + \frac{d-(a-h_0)v_{con}}{\tilde{v}} \leq \mu < a + \frac{d}{\tilde{v}} \\ +C\left([d - (\mu - a)\tilde{v}]{v_{con}}^2 + (\mu - a)\tilde{v}^3\right) & \\ B(\mu - a) + C\frac{d^3}{(\mu-a)^2} + D(\eta - a) & \text{if } a + \frac{d}{\tilde{v}} \leq \mu < a + \frac{d}{\overline{v}} \\ B\frac{d}{\overline{v}} + Cd\overline{v}^2 + D(\frac{d}{\overline{v}} + \eta - \mu) & \text{if } \mu \geq a + \frac{d}{\overline{v}} \end{cases}$$

For $\mu \leq \eta$, the expression in the first piece is convex and minimized at $a + \frac{d-(a-h_0)v_c}{\overline{v}}$ but since $a + \frac{d-(a-h_0)v_c}{\overline{v}} \geq a + \frac{d-(a-h_0)v_{con}}{\tilde{v}}$, the first piece is actually decreasing in μ. The second piece is linear decreasing in μ. The third piece is convex and minimized at $a + \frac{d}{\overline{v}}$, which is its upper limit, therefore the third piece is decreasing in μ. The last piece is decreasing in μ. Therefore the optimal value is $\mu = \eta$. □

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