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October 2018

CIRRELT-2018-44

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A Strategic Markovian Traffic Equilibrium Model for Capacitated Networks

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Abstract. In the realm of traffic assignment over a network involving rigid arc capacities, the aim of the present work is to generalize the model of Marcotte et al. (2004) by casting it within a stochastic user equilibrium framework. The strength of the proposed model is to incorporate two sources of stochasticity, stemming respectively from the users' imperfect knowledge regarding link costs (represented as a discrete choice model) as well as the probability of not accessing overcrowded links. Moreover, the arc-based formulation extends the Markovian traffic equilibrium model of Baillon and Cominetti (2008) through the explicit consideration of capacities.

Keywords: Traffic equilibrium, Markovian assignment, capacities, strategic behavior.

Acknowledgements. This work was partly funded by the Natural Sciences and Engineering Research Council of Canada (NSERC) discovery grant 435678-2013 and A5789.

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Dépôt légal – Bibliothèque et Archives nationales du Québec
Bibliothèque et Archives Canada, 2018

1 Introduction

Traffic equilibrium models are fundamental tools for the analysis of transportation networks performance as well as their design and planning. The traffic assignment problem consists in predicting arc flows over a network, given the known travel demand for each origin-destination (OD) pair. Flows are determined by the interaction of two mechanisms, users' travel decisions and congestion (Sheffi, 1985). Users' route choice preferences are incorporated in a generalized travel cost function which individual travelers aim to minimize, the primary component of which being travel time. Congestion is generally modeled by letting travel impedance functions depend on the usage of the network. As path costs increase with the amount of flow, travelers are induced to reroute on cheaper, less congested paths. The equilibrium assignment of travelers to routes is thus the result of a fixed point problem which is usually solved in an iterative manner. However, the classical equilibrium principles do not hold any more when side constraints, such as arc capacities, are entered into the model. A solution to that issue, proposed in Marcotte et al. (2004) is to embed within the users' objective function the probability that a link be unavailable, thus introducing a stochastic element that induces the strategic behaviour of users.

The main contribution of this paper is to generalize this model by including another source of stochasticity, namely the randomness of travel time estimation. By adopting the framework of Markovian equilibrium introduced by Baillon and Cominetti (2008), our model then generalizes the latter by incorporating arc capacities. More specifically, we embed the concept of strategies governing travelers' movements under capacity constraints in a Markovian traffic equilibrium setting. The key paradigm is to view strategies with recourse, according to which travelers readjust their path when reaching a saturated arc, as route choice behavior under imperfect information, similarly to Polychronopoulos and Tsitsiklis (1996). In order to deal with partial information, we expand the state space of the Markov Chain in Baillon and Cominetti (2008), such that a state encompasses two variables, an arc and an information set. User path choice behavior is then characterized by sequences of local arc choices depending on the current state and the destination. The network loading gives rise to availability probabilities, which are akin to access probabilities in Marcotte et al. (2004) and at the same time characterize action-state transition probabilities in the context of Markovian decision processes. The strength of the approach is to encompass two sources of stochasticity in the model by incorporating both unobserved elements and the risk of failure to access an arc in the cost of travel.

The rest of the paper is structured as follows. Section 2 presents a review of traffic assignments models and their underlying assumptions, helping to situate the two models on which this work is based, which we describe in detail in Section 3. We then introduce the proposed strategic Markovian traffic equilibrium model in Section 4. In Section 5, we describe algorithms related to those found in Marcotte et al. (2004) to compute availability probabilities from choice probabilities, to compute best response choice probability functions, and to determine an equilibrium. The Markovian traffic equilibrium model is then illustrated on a small network in Section 6, where we observe that the resulting flow patterns in the deterministic case are equivalent to those obtained with the model proposed in Marcotte et al. (2004). We then show in Section 7 the amenability of our approach to medium and large size networks, respectively corresponding to a simplified version of the Sioux Falls network, and the time-expanded Springfield transit network. Finally, in the concluding Section 8, we discuss extensions of the model that deserve further study.

2 Review on traffic assignment models

Traffic assignment models aim at predicting flow patterns in a network, under the assumption that travelers minimize some generalized cost, which itself may (or not) depend on flow volumes along the links (or paths) of the network. The equilibrium is thus the result of the interaction between demand and congestion. The first and simplest traffic assignment model formulated under these hypotheses is credited to Wardrop (1952), who posed the so-called user equilibrium principle. This states that, at equilibrium, all users are assigned to paths with minimum current cost, which implies that the cost of any unused path is greater or equal to the common cost of paths with positive flow. Beckmann et al. (1956) were the first to translate Wardrop's first principle of optimality into a convex mathematical program in order to obtain fast solution algorithms. The condition for this reformulation is that the function describing arc costs as a function of the total flow be separable. When this is not the case, the equilibrium problem is usually formulated as a variational inequality or a nonlinear complementarity problem (Dafermos, 1980), which are both a restatement of Wardrop's user equilibrium principle. This basic model has been extensively studied, with proofs of uniqueness and existence of the solution being developed, as well as efficient algorithms to reach it (Patriksson, 2004).

Several traffic equilibrium models extending Beckmann et al. (1956) were developed based on different assumptions regarding user behavior and congestion. In general, hypotheses can be formulated concerning (i) the knowledge that users have of the network and (ii) the effect of congestion on network's performance. We explain below how relaxing the basic assumptions in each direction led to different model developments.

The basic user equilibrium framework implies that users are able to minimize costs based on perfect knowledge and thus behave perfectly identically. This assumption is however counter-intuitive and assignment models based on it are known to exhibit unrealistic sensitivity to small changes in the network, as asserted by Dial (1971). Distinguishing between perceived and actual travel cost allows to account for users' lack of awareness, preference heterogeneity in the population, or the modeler's failure to identify all attributes of the cost function, and offers a more realistic modeling of route choice behavior. This spurred the development of another class of models based on stochastic user equilibrium conditions, which generalizes the previous (deterministic) user equilibrium condition by introducing a source of uncertainty in the model through random perceived costs. The equilibrium conditions for this class of models is that no user can unilaterally improve his/her *perceived* travel time by changing routes (Daganzo and Sheffi, 1977). This implies that travelers are distributed among several paths, according to the probability that each path is perceived to be the shortest, and the travel cost on all used paths is no longer equal. As with the deterministic case, a characterization of the equilibrium as the solution of a minimization problem has been proposed (Sheffi, 1985), provided that costs are a separable function of flows.

Link performance functions must be defined specifically by the modeler, but under Beckmann et al. (1956)'s formulation, they are assumed to be positive, increasing, and separable, meaning that a link cost depends on the amount of flow on that link only. A lot of research has however dealt with extensions of the traffic equilibrium model's travel cost function (e.g., Larsson and Patriksson, 1999). Such modifications allow to describe more realistic traffic conditions, such as interaction between flows or traffic flow restrictions, the consequence being that the classical Wardrop characterization as an optimization problem usually does not hold in part because the required cost functions are then non-separable, asymmetric and typically non integrable. For instance, Nagurney (2013) dedicated a large amount of work on more general model formulations, often involving variational inequalities, more adapted to characterize real-world congestion effects. A typical extension consists in relaxing the hypothesis that links may carry an unlimited amount of flow and associating a finite capacity to links. The problem of finite arc capacities has especially been studied in the context of transit assignment, where networks generally include links representing public transport lines between consecutive stops, which are assigned a capacity and travel cost. The effect of congestion is then different than that in a vehicular road network, as in-vehicle travel times are not affected by the number of users. Instead, crowded transit line vehicles may no longer be boarded once they are full, creating inherent uncertainty due to the potential unavailability of some network arcs. Incidentally, transit is not the only setting where studying restricted capacity on arcs may be helpful, see, e.g., the context of freight flows (Guélat et al., 1990).

In this context the classical Wardrop principle, which does not hold any more, must be adapted. One approach is through the use of asymptotic travel cost functions, meaning that as flow reaches capacity the cost goes to infinity. This solution allows to keep the convex optimization model structure, but has been criticized for entailing numerical difficulties as well as yielding unrealistic travel costs at equilibrium (Boyce et al., 1981, Larsson and Patriksson, 1995). Another solution is to add a well-defined extra cost interpreted as a queuing delay to saturated arcs, leading to a so-called generalized Wardrop equilibrium, as in, e.g., Larsson and Patriksson (1995). In both cases, the mechanism that increases travel costs as a result of capacity limits is somewhat implicit and not based on mathematical foundations. Flow constraints are respected but the equilibrium does not make behavioral sense, since users do not account for the risk to fail to access an arc in their path choice. Therefore, a third approach to capacities was proposed by Marcotte et al. (2004), which we describe below.

The fundamental notion in Marcotte et al. (2004) is the concept of strategy. Originally, this concept was introduced in transit assignment modeling to describe user behavior under uncertain outcomes. A strategy specifies for each node in the network a set of desired outgoing links, but the exact physical itinerary on which the user following the strategy travels depends on the realization of the random variables contained in the problem. In Spiess and Florian (1989), strategies are used to characterize user itinerary choice with respect to random arrivals of vehicles from several attractive lines. Marcotte et al. (2004) adapted the concept of strategy to relate it to the uncertainty induced by limits on available capacity, as we further explain in Section 3. This led to a theoretically appealing equilibrium model where user behavior is characterized by strategies with recourse. The model does not yield flows that may exceed arc capacities, in contrast to, e.g., De Cea and Fernández (1993), and may be applied not only to transit but generally to any acyclic network with capacities.

Strategies exist in an exponential number for each OD pair, as do paths in a network. The optimization problem in Marcotte et al. (2004) is thus formulated in a high dimensional space, which can lead to practical implementation issues. The drawbacks of relying on path-based variables have been abundantly emphasized in other works (Dial, 2006, Fosgerau et al., 2013, Wie et al., 2002) of the user equilibrium and route choice modeling literature, noting that the issue remains associated with most existing models. A different approach was first provided by Akamatsu (1996) in the context of stochastic user equilibrium, as an alternative to Dial (1971)'s well known logit assignment model, which assigns travelers to paths under certain choice probability assumptions. The primary insight of the work is to consider path choice probabilities as products of sequential link choice probabilities, obviating explicit path variables. The link choice probability matrix is equivalent to the state transition probabilities of a Markov chain on the network's arcs with an absorbing state corresponding to the destination, yielding the denomination Markovian traffic equilibrium (MTE) by Baillon and Cominetti (2008). Their work established the existence and uniqueness of an equilibrium in the case of flow dependent arc costs, and showed that the approach conveniently circumvents traditional path enumeration issues and facilitates the operationalization of the model to large-scale networks. While this avenue is promising, it has nevertheless not been formally extended to the case of networks with rigid arc capacities.

3 Two subsumed models

In this section, we introduce the models of Marcotte et al. (2004) and Baillon and Cominetti (2008), on which we build in Section 4. Both models deal with traffic equilibrium under entirely different assumptions regarding user behavior and congestion. To describe each work, we assume some standard notation, i.e., the network is represented by a graph $G = (\mathcal{V}, \mathcal{A})$ with node set $\mathcal{V}$ and arc set $\mathcal{A}$, and each arc possesses a cost c_a and possibly a capacity u_a. For reasons explained later, it is convenient to denote arcs either by the letter k or a depending on the context. We denote by $\mathcal{A}(k)$ the set of outgoing arcs from the tail node of arc k.

3.1 A strategic flow model of traffic assignment

In the model of Marcotte et al. (2004), it is assumed that users have a perfect knowledge of arc costs, which casts the model within the deterministic user equilibrium framework. Regarding network performance, the model assumes that there exist strict capacity constraints on some of the network's arcs. Thus each arc $a \in \mathcal{A}$ is associated to a cost c_a and possibly a capacity u_a.

The model proposed by Marcotte et al. (2004) provides an entirely different approach to capacities than previous related works. Their solution consists in adopting strategies to describe user behavior, expanding a concept which was first denominated by Spiess and Florian (1989) for transit networks. In this case, users do not aim at minimizing path costs given by the sum of arc costs, but rather strategic costs.

The general idea of a strategy is to model complex decision making under uncertainty in the network service, providing travelers with the opportunity to readjust or refine their path choice as information on the network is gained. In this model, a strategy defines for each node a set of outgoing links ranked by order of preference, thus providing a recourse in case the preferred options have reached capacity. Users choose a strategy in advance, but do not know on which path they will eventually travel.

The inherent uncertainty induced by limited arc availability is encompassed into so-called *access probabilities*, which are conceptually similar to diversion probabilities or failure to board probabilities in some transit assignment models (Kurauchi et al., 2003). They allow the model to strictly enforce capacity constraints. Individuals' travel decisions take into account the stochasticity embedded into access probabilities, and consequently users are assumed to minimize the *expected* cost of each strategy s denoted C_s. This cost can be defined as the weighted sum of path costs by path access probabilities. Extending Wardop's principle to capacitated networks, Marcotte et al. (2004) state that a *strategic equilibrium* occurs when all users are assigned to strategies of minimum expected cost.

The complexity lies in the fact that the cost mapping C is not available in closed form as a function of strategic flows x. Pricing out strategies requires to obtain first the access probabilities $\pi(x)$ corresponding to the current distribution of users into strategies. It also depends on additional assumptions of the model, namely on the queuing mechanism at each node. Marcotte et al. (2004) rely on two algorithms to compute the expected price of strategies. The shape of access probabilities naturally induces nonlinearity and asymmetry in the cost mapping C, and Marcotte et al. (2004) show that it is not integrable, which prevents it from being reformulated as a standard optimization problem. Thus the equilibrium problem is expressed by the variational inequality

$$\langle C(x), x - y \rangle \leq 0, \qquad \forall x \in X,$$

where X is the set of feasible strategic flows.

3.2 A Markovian traffic assignment model

The underlying assumption in the model of Baillon and Cominetti (2008) is that travelers do not have perfect knowledge of arc costs, which are thus modeled as random variables representing how individuals perceive cost. In addition to being random variables, costs are also assumed to be flow-dependent to account for congestion.

Under these assumptions, the model of falls within the scope of stochastic user equilibrium. Perceived cost is defined as $\tilde{c}_a = c_a + \epsilon_a$, where c_a is the real arc cost and ϵ_a is an error term with zero mean. Congestion is accounted for by letting the mean cost c_a be a function of the flow f_a on the arc through known volume-delay functions $s_a : [0, \infty) \to [0, \infty)$, such that $c_a = s_a(f_a)$.

What distinguishes Baillon and Cominetti (2008) from other stochastic equilibrium models is that the approach is formulated in terms of arc-based variables, as it is embedded within a dynamic programming framework. Travelers' choice of path obeys a sequential process in which a discrete choice model at each arc k describes the choice probabilities $P^d_{k,a}$ of outgoing links a depending on the desired destination d. The arc-based formulation requires to define the notion of perceived cost to destination d from the source node of a given arc k, denoted $\tilde{w}^d_k = w^d_k + \epsilon_k$. The cost to destination w^d_k is the sum of the arc cost c_k and a destination specific value function defined recursively following Bellman equation of dynamic programming, i.e.

$$w^d_k = c_k + \varphi^d_k(w^d),$$

where

$$\varphi^d_k(w^d) = E\left(\min_{a \in \mathcal{A}(k)} w^d_a + \epsilon_a\right).$$

Thus the value function $\varphi^d_k(w^d)$ represents the expected minimum cost to go to destination d from the tail node of a given arc k in the network.

The model assumes that at the tail node of a link k, individuals traveling towards d observe $\tilde{w}^d_a$ for all outgoing arcs $a \in \mathcal{A}(k)$ and choose the link with the smallest perceived cost to destination. When the variance of error terms is null, individuals choose identically, while they are distributed according to link choice probabilities $P^d_{k,a}$ otherwise. Thus the Markovian traffic equilibrium model is also a generalization of a deterministic case and both can be formulated under the same framework. Although the traffic equilibrium model could be expressed as a variational inequality, it admits a characterization as a convex minimization problem, assuming that congestion functions s_a are integrable.

It is worth mentioning that the model's formulation in terms of link variables induces interesting properties. In particular, the link choice probability matrix P^d may be regarded as the transition probability matrix of an underlying Markov chain where states are network links, meaning that expected arc flows can be easily computed by matrix operations as the expected state visitation frequencies.

Finally, we note that Baillon and Cominetti (2008) mention the possibility of extending the model to networks with arc capacities u_a by considering bounded volume-delay functions, i.e. $s_a : [0, u_a) \to [0, \infty)$. However, doing so simply heuristically bounds predicted flows without providing a realistic model of how the risk that an arc becomes inaccessible affects behavior strategically. Moreover, the solution obtained does not satisfy Wardrop's equilibrium conditions.

4 Strategic Markovian traffic equilibrium model

In this section, we propose a strategic Markovian traffic equilibrium model for capacitated networks that subsumes the advantages of both previously described models. It incorporates two sources of stochasticity in user route choice behavior, induced by variations in cost perception and the risk associated with the failure to access an arc. We propose a model formulation in which the deterministic user equilibrium (i.e., arc cost is identical across users) is a specific case of the stochastic user equilibrium and for the sake of clarity we first describe the former in Subsection 4.2 before deriving the more general model in Subsection 4.3.

4.1 Notation and assumptions

We consider a directed connected graph $G = (\mathcal{A}, \mathcal{V})$, where $\mathcal{A}$ is the set of arcs, or links, and $\mathcal{V}$ is the set of nodes. Links are denoted k, a and $\mathcal{A}(k)$ is the set of outgoing links from the tail node of k. We assume that every link a has a strict capacity u_a and an associated generalized cost c_a. We add absorbing links without successors to each destination node and call $\mathcal{D}$ this set of destination links. We consider the demand to originate from each network link, and let g^d characterize the vector of demand from each link given a destination $d \in \mathcal{D}$. Throughout the paper, we assume that the network has sufficient capacity to accommodate the whole demand. In particular, we assume that for each node, there is at least one outgoing link with unlimited capacity. This can be trivially achieved by supplementing artificial arcs with large cost.

Users traveling in this network aim at finding the shortest path to their destination $d \in \mathcal{D}$. However, because of limited network capacity, some arcs may be saturated and thus inaccessible depending on route choices made by other travelers. Similarly to Marcotte et al. (2004), we assume a realistic modeling of user behavior in this context, dictating that travel decisions be strategic and include recourse actions, should a link in the preferred itinerary turn out to be unavailable. In addition, we make the hypothesis that travelers do not know in advance what arc will prove to be available, and only observe the outcome when reaching the source node of an arc, which is a realistic assumption for a transit network. Under these assumptions, the problem bears similarities to the stochastic shortest path problem in a probabilistic network studied in, e.g., Andreatta and Romeo (1988). As observed in Polychronopoulos and Tsitsiklis (1996), stochastic programming with recourse can be viewed as a stochastic control problem with imperfect information, and may be solved with dynamic programming methodology. Namely, instead of defining recourse actions, user behavior may equivalently be characterized by an optimal policy given the current state, where the state indicates

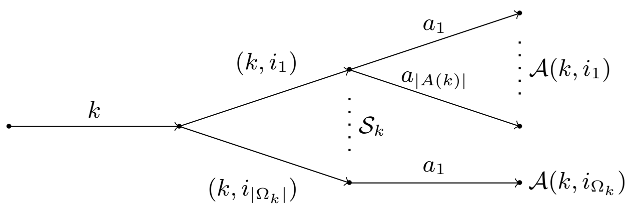

Figure 1: Illustration of state expansion

the realization of the random variables. Below, we explain how we formulate the model following this paradigm.

We assume that the set of available outgoing arcs from link k is a random subset of $\mathcal{A}(k)$, and define the random vector I_k, which indicates whether each outgoing arc is accessible and may take values in $\Omega_k = \{0, 1\}^{|\mathcal{A}(k)|}$. Consequently, we define a state $s = (k, i_k)$ as a set of two variables, i.e., a link k and a realization i_k of random vector I_k. The set of states at link k is denoted $\mathcal{S}_k$, while the set of all possible states is denoted $\mathcal{S}$. A policy, or action, is then a choice of outgoing arc among the set $\mathcal{A}(s)$ of available links depending on the current state $s = (k, i_k)$, as illustrated by Figure 1. For an unvisited arc k, the random vector I_k follows availability probability distribution π_k, with support on $\{0, 1\}^{|\mathcal{A}(k)|}$. Upon arrival at the tail node of arc k, the user learns the realization of I_k. Therefore, travelers choose their paths sequentially in a dynamic fashion, choosing in each state an action that leads stochastically to a new state. Note that we choose to include an arc instead of a node variable in the state space, as it allows to extend the model to more complex queuing dynamics, where travelers from a certain incoming arc may have priority over another, an issue we discuss in Section 5.1.

Travelers' route choice behavior is characterized by the destination specific choice probability matrix $P^d = \{P^d_{s,a}\}_{s \in \mathcal{S}, a \in \mathcal{A}}$, which describes in what proportion individuals choose each action conditionally on the state and the destination. The role of availability probabilities π is analog to that of state transition probabilities conditional on choices in a Markov Decision Process. Given a state $s_t = (k, i_k)$ and an action $a \in \mathcal{A}(s_t)$, the probability $\Pr(s_{t+1}|s_t, a)$ of reaching the new state $s_{t+1} = (a, i_a)$ is given by the distribution π_a of random vector I_a. In other words, the new state consists of the chosen available link and a realization of the availability random vector at that link. We can here draw a parallel with the model of Baillon and Cominetti (2008), where the choice of outgoing link may also be viewed as a choice of action leading to a new state. While in Baillon and Cominetti (2008) the new state is given with certainty once the action is selected, and is equal to the chosen link, we obtain a more complex model with non degenerate action-state transition probabilities. Therefore, in a capacitated network, passengers' motions are directed by an underlying Markov chain dependent on both choice and availability probabilities.

The probability of accessing an arc naturally depends on the choices of all other users of the network. Hence, as in Marcotte et al. (2004), availability probabilities π actually depend on both capacities and choice probabilities P through a loading process, which we further explain in Section 5.

We observe that the probability matrix P has a close tie to the strategic flow vector x in the model of Marcotte et al. (2004), since both specify the distribution of travelers between different travel strategies or policies. The major difference in this work is that we model behavior using local choices at each node instead of a choice of strategy for the entire itinerary. Also, in Marcotte et al. (2004), the model requires one strategic flow vector x per OD pair, while the matrix P in our arc-based model works implicitly with all the strategies but is only destination specific. In addition, the framework we propose lends itself to model both deterministic and stochastic equilibrium. Indeed, although P is dubbed a probability matrix, it may be degenerate as exemplified in Section 4.2.

We summarize below the notation used throughout the paper:

$\mathcal{A}$	set of arcs
$\mathcal{V}$	set of nodes
$\mathcal{A}(k)$	set of outgoing arcs from arc k
$\mathcal{S}$	set of states
I_k	random vector indicating available outgoing arcs from k
$\mathcal{A}(s)$	set of available outgoing arcs in state $s = (k, i_k)$
g^d	demand vector to destination d
c_a	cost on arc a
$V^d(a, i_a)$	minimum expected cost to destination from state (a, i_a) to destination d
w_a^d	expected cost of arc a with regard to destination d
u_a	capacity on arc a
f_a^d	expected arc flow on a to destination d
π_k	availability distribution of random vector I_k
P^d	matrix of link choice probabilities to destination d

4.2 Deterministic user equilibrium

In this section, we propose a deterministic user equilibrium model assuming that individuals have a perfect knowledge of arc costs c_a. We emphasize that perfect knowledge does not refer to availability of outgoing arcs, which we still assume to be unknown for downstream parts of the network.

As in Baillon and Cominetti (2008), in each state $s = (k, i_k)$ individuals minimize the expected cost to destination of actions $a \in \mathcal{A}(s)$ corresponding to available outgoing links, where the stochasticity is induced by availability probabilities π. This quantity w_a^d is the sum of two terms, the link cost c_a associated to the action, and the minimum expected cost to destination $V^d(a, i_a)$ from the future state (a, i_a), weighted by the probability distribution π_a of reaching each possible state conditional on the action:

$$w_a^d = c_a + E_{i_a \sim \pi_a} V^d(a, i_a). \tag{1}$$

The minimum expected cost of traveling to destination d from state (k, i_k) is denoted the value function and defined recursively by the Bellman equation:

$$V^d(k, i_k) = \min_{a \in \mathcal{A}_i(k)} \left\{ c_a + E_{i_a \sim \pi_a} V^d(a, i_a) \right\}. \tag{2}$$

An equilibrium is reached when, in each possible state, no user can reduce its expected cost to destination by modifying its action choice. Hence, for each state $s = (k, i_k) \in \mathcal{S}$ and destination $d \in D$, all available actions $a \in \mathcal{A}(s)$ which have a non null choice probability $P^d_{s,a}$ must have the same expected cost w^d_a. For this reason, $P^d_{s,a}$ are not choice probabilities obtained by a discrete choice model, and are degenerate if a single action possesses the minimum cost. More simply, $P^d_{s,a}$ express the proportion of individuals choosing each action.

As for the strategic cost function in the model of Marcotte et al. (2004), the cost w^d explicitly depends on access probabilities π, which themselves depend on users' choices P^d through a loading mechanism which mirrors the queuing taking place to access each capacitated link. As a result, non integrability of the costs arises and the equilibrium problem must be expressed as a variational inequality. Let us denote $C^d_{s,a}$ as equal to w^d_a if $a \in \mathcal{A}(s)$, and ∞ otherwise. Then C^d_s is the vector of expected costs to destination d of all actions from state s. Omitting for simplicity the destination index d, the equilibrium probabilities $P^*_{s,a}$ for each destination must satisfy the variational inequality

$$\langle C_s(P^*), P^*_s - P_s \rangle \leq 0 \quad \forall P \in \mathcal{P},\ \forall s \in \mathcal{S}, \tag{3}$$

where the set $\mathcal{P}$ includes all feasible $|\mathcal{S}| \times |\mathcal{A}|$ probability matrices.

Alternatively, the problem may be formulated as the nonlinear complementarity problem:

$$P^*_{s,a} \left[C_{s,a} - V_s \right] = 0 \quad \forall s \in \mathcal{S}, a \in \mathcal{A}(s), \tag{4}$$

$$P^*_{s,a} \geq 0 \quad \forall s \in \mathcal{S}, a \in \mathcal{A}(s). \tag{5}$$

Intuitively, V^d_s thus represents the minimum expected cost to travel between a state s in the network and the desired destination d.

4.3 Stochastic user equilibrium

In this section, we propose an extension where perception of travel costs c_a varies across the user population. We model *perceived* arc costs as random variables $\widetilde{c}_a = c_a + \mu\epsilon_a$, letting the measured arc cost be disrupted by an error term with $E(\epsilon_a) = 0$. This can be interpreted as users not having perfect information, or the modeler failing to properly identify and measure the cost function.

Under these assumptions, the expected cost of an action a to reach destination d also becomes a random variable $\widetilde{w}^d_a$, which is the sum of both the error term ϵ_a and the term w^d_a:

$$\widetilde{w}^d_a = w^d_a + \mu\epsilon_a. \tag{6}$$

In this case, individuals observe in each state s the realization of $\widetilde{w}^d_a$ for all available actions $a \in \mathcal{A}(s)$ and choose the one associated with the minimum cost. Probabilities

$P^d_{s,a}$ represent the probability that each action has the minimum cost, i.e.,

$$P^d_{s,a} = \Pr\left(\widetilde{w}^d_a \leq \widetilde{w}^d_{a'}, \forall a' \in \mathcal{A}(s)\right). \tag{7}$$

On the other hand, the cost w^d_a becomes

$$w^d_a = c_a + E_{i_a \sim \pi_a} V^d(a, i_a), \tag{8}$$

where $V^d(a, i_a)$ is now the *expected* value function. Therefore, according to the Bellman equation, we have

$$V^d(k, i_k) = E_{\epsilon_a}\left[\min_{a \in \mathcal{A}_i(k)} \left\{c_a + E_{i_a \sim \pi_a} V^d(a, i_a) + \mu\epsilon_a\right\}\right]. \tag{9}$$

In particular, if ϵ_a is an extreme value type I error term, (9) can be rewritten as the following log sum:

$$V^d(k, i_k) = \mu \ln\left(\sum_{a \in \mathcal{A}_i(k)} e^{\frac{1}{\mu}\left(c_a + E_{i_a \sim \pi_a} V^d(a, i_a)\right)}\right). \tag{10}$$

Following the notation introduced in the previous section, we can formulate the equilibrium problem as a similar variational inequality. We define $\widetilde{C}^d_{s,a}$ as the sum $w^d_{s,a} + \mu \ln(P^d_{s,a})$, where $w^d_{s,a}$ is equal to w^d_a if $a \in \mathcal{A}(s)$ and ∞ otherwise. Then for each destination the equilibrium choice probabilities $P^*_{s,a}$ are the solution of

$$\langle \widetilde{C}_s(P^*), P^*_s - P_s \rangle \leq 0 \quad \forall P \in \mathcal{P},\ \forall s \in \mathcal{S}, \tag{11}$$

where the destination index is omitted. Equivalently, in the stochastic user equilibrium framework, the nonlinear complementarity problem becomes

$$P^*_{s,a}\left[\widetilde{C}_{s,a} - V_s\right] = 0 \quad \forall s \in \mathcal{S}, a \in \mathcal{A}(s), \tag{12}$$

$$P^*_{s,a} \geq 0 \quad \forall s \in \mathcal{S}, a \in \mathcal{A}(s). \tag{13}$$

In this case all available outgoing arcs have some positive flow since probabilities $P^d_{s,a}$ are non null for all available actions $a \in \mathcal{A}(s)$. Thus we note that at equilibrium V^d_s is equal to $\widetilde{C}^d_{s,a}$ for all arcs $a \in \mathcal{A}(s)$.

5 Algorithmic framework

This section presents algorithms for solving the proposed model. First, we give two inner algorithms related to the ones found in Marcotte et al. (2004). The first one is a loading mechanism to obtain availability probabilities from choice probabilities, and the second computes the best response choice probabilities to a given assignment. Then, we propose an iterative outer algorithm for determining an equilibrium solution.

5.1 Capload

Similarly to Marcotte et al. (2004), the model requires a procedure for recovering availability probabilities $\pi = \{\pi_{a,s}\}_{(a,s)\in\mathcal{A}\times\mathcal{S}}$ from choice probabilities $P^d = \{P_{s,a}\}^d_{(s,a)\in\mathcal{S}\times\mathcal{A}}$. In order to compute the availability probability distribution of outgoing arcs at the tail node of a given arc k, it is necessary to obtain the total arc flow $f_k = \sum_{d\in D} f_k^d$, itself dependent on upstream availability probabilities. Thus the components of π must be computed for each arc in topological order from origin to destination.

In a Markovian traffic equilibrium perspective, expected arc flows can be computed as the expected state visitation frequencies of the implicit Markov Chain on the network's arcs, where the destination is the absorbing state. If state transition probabilities to a destination d are characterized by the matrix P^d and the demand by the vector g^d, Baillon and Cominetti (2008) proved that we obtain

$$f^d = (I - {(P^d)}^T)^{-1} g^d. \tag{14}$$

In the context of this work, travelers' movements are characterized by both choice and availability probabilities. If capacities were unlimited, the choice probability matrix P^d would fully capture transitions between links. However, probabilities π modify the flow distribution given by P^d. Thus the underlying Markov chain in a network with rigid capacities is characterized by the transition probability matrix Q^d, where we have

$$Q^d_{k,a} = \sum_{s\in\mathcal{S}} \pi^d_{k,s} P_{s,a}. \tag{15}$$

Therefore arc flows f^d are given by applying (14) with the transition matrix Q in (15).

The algorithm to compute availability probabilities $\pi = \{\pi_{a,s}\}_{(a,s)\in\mathcal{A}\times\mathcal{S}}$ makes use both of (15) and a loading mechanism at each node. Initially π in (15) is set such that the probability that all outgoing arcs are accessible is 1 for all links. The network nodes with capacitated outgoing links are then processed in topological order. At each arc k terminating at the current node, a loading mechanism emulates the queuing process taking place when users attempt to access outgoing arcs and updates the availability distribution π_k. In essence, the total flow arriving at the node must be loaded on outgoing arcs with respect to user preferences, potential access priorities and limited capacities. In this work, we make the following assumptions regarding the loading process:

- The individuals on each arc k which terminates at the current node have equal access priority.
- The queuing discipline implemented is the single queue processing (SQP) described in Marcotte et al. (2004), corresponding to letting users be randomly and uniformly distributed in a single queue.

Both assumptions may be relaxed. We could consider, for instance, that users coming from certain arcs have priority over others, resulting in a more complex loading process. This extension is worthwhile when considering a time-expanded transit network, where

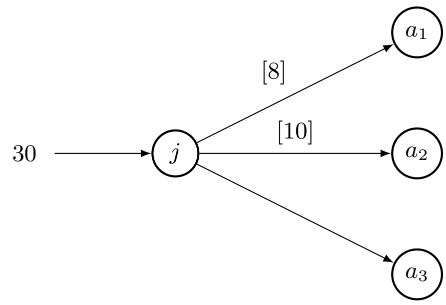

Figure 2: Loading example

users coming from a transit line and are already on board have priority over those seeking to board. For instance, Hamdouch and Lawphongpanich (2008) extend the model to accommodate this specific case. It is also possible to implement more complex queuing disciplines, such as parallel queue processing (PQP), as described in Marcotte et al. (2004).

We exemplify the loading process for a given node j with capacitated outgoing links, as illustrated by Figure 2. As the nodes are visited in topological order, the total flow f_k on all arcs k with j as tail node is known. It consists of the demand originating from k plus the flow arriving to k from previous arcs. In Figure 2, the incoming flow arriving at node j amounts to 30. At the first iteration, users are assigned to each link, which fill up at a rate proportional to the ratio between capacity and the number of individuals who want to access the link. In this case, since 10 and 20 users wish to access a_1 and a_2 respectively according to the choice probabilities given in Table 1, the ratios are 8/10 and 10/20. Therefore, having the smallest ratio, the arc leading to a_2 is the one to reach capacity first. At this point, half of the users have been assigned, therefore the probability of a user reaching the state $s_1 = (1, 1, 1)$, corresponding to all links available, is 1/2. The 15 users that have not been assigned and are in the remaining of the queue follow the behavior in state $s_2 = (1, 0, 1)$ given by Table 1. Before performing the next iteration, the capacity of remaining arcs is replaced by the residual capacity, which is obtained by removing the fraction of users who have successfully accessed the arc. Now, all 15 users want to access a_1. Since the residual capacity is 3, the ratio is 1/5. Therefore, the probability that a user reaching the tail node ends up in state $s_2 = (1, 0, 1)$ is equal to $1/2 \cdot 1/5 = 1/10$. The remaining users behave as in state $s_3 = (0, 0, 1)$ and are all able to access the arc leading to a_3. We conclude that the probability of state $s_3 = (0, 0, 1)$ is 4/10.

Finally, Algorithm 1 formally describes the procedure Capload. Note that the procedure is identical for both the deterministic and stochastic assignment.

5.2 Capshort

The algorithm Capshort computes a best response choice probability matrix $\bar{P}^d$ (not to be mistaken for an equilibrium solution) after the loading of choice probability P^d. To

Algorithm 1 CAPLOAD

```
procedure CAPLOAD(u, P)
    Initialization:
    for k ∈ \mathcal{A} do
        \pi_{k,s} = 1 if s = (k, \{1\}^{|\mathcal{A}(k)|}), 0 else
    end for
    k = first arc in topological order with capacitated outgoing link
    while not all arcs visited do
        for d ∈ D do
            Q^d = \pi \cdot P^d
            f^d = (I - Q^d)^{-1} g^d
        end for
        f = \sum_{d \in D} f^d                                  ▷ Computing flow
        I = \{1\}^{|\mathcal{A}(k)|}
        s = (k, I)                                   ▷ Current state: all arcs available
        while not stop do
            \beta = \min\{1, \min_{a \in \mathcal{A}(k)}\{u_a/f_a\}\}
            if \beta \leq 1 then
                p = \pi_{k,s}                                ▷ Probability of current state
                \pi_{k,s} = \beta p                          ▷ Updating probability of current state
                \tau_{k,s} = 0                               ▷ Updating residual probability of current state
                i = \arg\min_{a \in \mathcal{A}(k)}\{u_a/f_a\}    ▷ New saturated arc
                I_i = 0
                s' = (k, I)                                  ▷ New state
                \pi_{k,s'} = (1 - \beta)p                    ▷ Updating probability of new state
                \tau_{k,s'} = \pi_{k,s'}                     ▷ Updating residual probability of new state
                s = s'                                       ▷ Current state = new state
                for a ∈ \mathcal{A}(k) do
                    u_a = u_a - \beta f_a                    ▷ Updating residual capacities
                end for
                for d ∈ D do
                    Q^d = \tau \cdot P^d
                    f^d = (I - Q^d)^{-1} g^d
                end for
                f = \sum_{d \in D} f^d                       ▷ Updating residual flow
            else
                stop
            end if
        end while
        k = next arc in topological order with capacitated outgoing link
    end while
end procedure
```

State	a_1	a_2	a_3
$(1,1,1)$	$1/3$	$2/3$	0
$(0,1,1)$	0	1	0
$(1,0,1)$	1	0	0
$(0,0,1)$	0	0	1

Table 1: Probability of users choosing each outgoing link in each possible state for the loading example

that end, the algorithm starts by computing recursively the minimum expected cost to destination from each state, as defined in equations (2) and (9), for the deterministic and stochastic case respectively.

In the deterministic case, the optimal action $\alpha^d(s)$ for an individual in state $s = (k, i_k)$ going to destination d consists in choosing arc $a \in \mathcal{A}(s)$ such that

$$\alpha^d(s) = \arg \min_{a \in \mathcal{A}(s)} \left\{ w_a^d \right\}. \tag{16}$$

and the best response choice probability matrix $\bar{P}^d$ is simply given by letting all users choose the best action in each state:

$$\bar{P}^d_{s,a} = \frac{I\{\alpha^d(s) = a\}}{\sum_{a' \in \mathcal{A}_i(k)} I\{\alpha^d(s) = a'\}}. \tag{17}$$

In the stochastic case, the optimal action $\alpha^d(s)$ for an individual in state s traveling to d is

$$\alpha^d(s) = \arg \min_{a \in \mathcal{A}(s)} \left\{ w_a^d + \mu\epsilon(a) \right\}. \tag{18}$$

Thus each arc a is associated to a probability of being the best action in each state, and the best response choice probability matrix $\bar{P}^d$ distributes the demand on available outgoing arcs according to this probability function, such that

$$\bar{P}^d_{s,a} = E_{\epsilon_a} \left[I\{\alpha^d(s) = a\} \right], \tag{19}$$

which, in the case of extreme value type I error terms, is equivalent to a multinomial logit

$$\bar{P}^d_{s,a} = \frac{e^{\frac{1}{\mu}(w_a^d)}}{\sum_{a' \in A(s)} e^{\frac{1}{\mu}(w_{a'}^d)}}. \tag{20}$$

Algorithms 2 and 3 implement the procedure Capshort for the deterministic and stochastic equilibrium respectively.

5.3 Solution algorithm

We aim to find, for each destination, the equilibrium choice probabilities P^* corresponding to the solution of the variational inequality described in the previous section, where

Algorithm 2 Capshort deterministic

```
procedure CAPSHORT(π, P)
    for all destinations d ∈ D do
        Initialization:
        V_s^d = 0 if s is a destination state
        for all arcs k in inverse topological order do
            for all realizations i_k of I_k do
                V^d(k, i_k) = min_{a∈A_i(k)} {c_a + E_{I_a∼π_a} V^d(a, I_a)}
                α^d(k, i_k) = arg min_{a∈A_i(k)} {c_a + E_{I∼π_a} V^d(a, I)}
                for all outgoing arcs a ∈ A_i(k) do
                    P̄_{s,a}^d = I{α^d(s)=a} / Σ_{a'} I{α^d(s)=a'}
                end for
            end for
        end for
    end for
end procedure
```

Algorithm 3 Capshort stochastic

```
procedure CAPSHORT(π, P, μ)
    for all destinations d ∈ D do
        Initialization:
        V_s^d = 0 if s is a destination state
        for all arcs k in inverse topological order do
            for all realizations i_k of I_k do
                V^d(k, i_k) = μ ln( Σ_{a∈A_i(k)} e^{(1/μ)(c_a + E_{I_a∼π_a} V^d(a, I_a))} )
                for all outgoing arcs a ∈ A_i(k) do
                    P̄_{s,a}^d = e^{(1/μ)(c_a + E_{I_a∼π_a} V^d(a, I_a))} / Σ_{a'∈A_i(k)} e^{(1/μ)(c_{a'} + E_{I_{a'}∼π_{a'}} V^d(a', I_{a'}))}
                end for
            end for
        end for
    end for
end procedure
```

the destination index d is omitted. Algorithm 4 describes the method of successive averages (MSA) using a relevant stopping criterion.

Algorithm 4 Method of successive averages

procedure MSA(P, u, μ, ϵ)
    Initialization:
    $n = 1$
    while $g_R(P) > \epsilon$ **do**
        $\pi \leftarrow \text{Capload}(u, P)$
        $\bar{P}, w \leftarrow \text{Capshort}(\pi, P, \mu)$
        for all destinations $d \in D$ **do**
            $g_R(P^d) = \frac{\sum_{s \in \mathcal{S}} \langle w_s^d, P_s^d - \bar{P}_s^d \rangle}{\sum_{s \in \mathcal{S}} \langle w_s^d, P_s^d \rangle}$
            $P^d \leftarrow P^d + \theta_n(\bar{P}^d - P^d)$
        end for
        $n \leftarrow n + 1$
    end while
end procedure

At each iteration n, the destination specific choice probability matrix P_n^d is updated by taking a step in the direction of the best solution $\bar{P}_n^d$. The size of the step at each iteration n depends on the value θ_n, which can be defined in the following manners. The classic method is to use a common step size based on the sequence $\theta_n = 1/(n+1)$ to update all components of P^d, which gives

$$P_{n+1}^d = P_n^d + \theta_n(\bar{P}_n^d - P_n^d).$$

An alternative approach is to update the choice probabilities $P_{s,a}^d$ for each state s with a different step size θ_s^d, defined as

$$\theta_s^d = 1 - \frac{V_s^d}{\bar{w}_s^d},$$

where $\bar{w}_s^d$ is the average of $w_{s,a}^d$ over all arcs $a \in \mathcal{A}(s)$ weighted according to the flow on each arc.

Algorithm 4 resorts to a gap function to evaluate the proximity of the iterate with the equilibrium solution. Defining an appropriate aggregate gap measure for the entire network is not trivial. It is however straightforward to define the gap associated to a specific state s and destination d as

$$g(P_s^d) = \max_{R \in \mathcal{P}} \langle w_s^d, P_s^d - R_s^d \rangle,$$

and its scaled version as

$$g_R(P_s^d) = \frac{g(P_s^d)}{\langle w_s^d, P_s^d \rangle}.$$

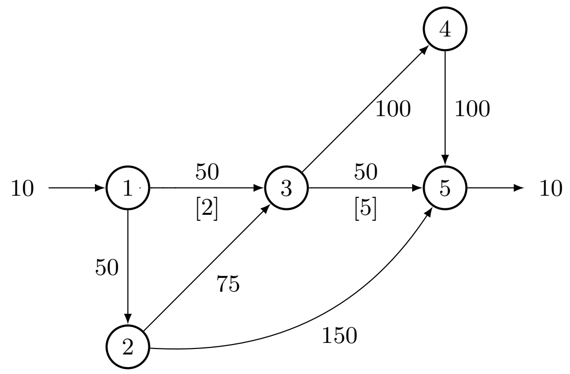

Figure 3: Small capacitated network

We define the aggregate relative gap $g_R(P^d)$ for a destination $d \in \mathcal{D}$ as a weighted average of the state specific relative gaps by the flow on each state, i.e.

$$g_R(P^d) = \sum_{s \in \mathcal{S}} p_s^d g_R(P_s^d),$$

where the weights p_s^d are given by $\frac{f_s^d}{\sum_{s \in \mathcal{S}} f_s^d}$. Note in addition that we exclude from the sum all states where only one outgoing arc is available, since the gap in such states is trivially null. Finally the aggregate relative gap $g(P)$ for all destinations is given by

$$g_R(P) = \sum_{d \in \mathcal{D}} q^d g_R(P^d),$$

where the weights q^d are given by the proportion of the total demand associated to destination d.

While the gap measure $g_R(P)$ is used as a stopping criterion for Algorithm 4, it remains interesting to analyze the gap at a more disaggregate level, since there may be considerable variance in the state specific gaps.

6 An illustrative example

In this example, we consider the small network in Figure 3, in which each link is associated with a length L and possibly a capacity u (bracketed number) as illustrated. Links are numbered from 1 to 9, including an artificial origin and destination link. The link length is the only component of the travel cost function for this network, such that the deterministic cost of an arc a is given by $c_a = L_a$. For origin and destination links, this cost is 0. The demand between origin link 1 and destination link 9 is set to 10 units. Since we only consider one destination, we omit the destination index d in the following.

Since there is at most one outgoing arc with limited capacity, each network link corresponds to at most two possible states. In total, there are 12 possible states for a

State ID	Link	Link ID	Availability of outgoing links
1	o-1	1	(1,1)
2	o-1	1	(1,0)
3	1-2	2	(1,1)
4	1-3	3	(1,1)
5	1-3	3	(1,0)
6	2-3	4	(1,1)
7	2-3	4	(1,0)
8	2-5	5	(1)
9	3-4	6	(1)
10	3-5	7	(1)
11	4-5	8	(1)
12	5-d	9	-

Table 2: Possible states in the example

user traveling in this network, listed in Table 2. In addition, since the maximum number of outgoing arcs per link is two, in any state where an outgoing link has reached capacity, the choice automatically reduces to the other only available link.

To apply the algorithms described in Section 5.3 it is necessary to initialize the choice probability matrix P. For this experiment, we assume that initially choice probabilities are given by

$$P = \begin{pmatrix} 0 & 0.50 & 0.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \end{pmatrix}.$$

Each row of the matrix corresponds to a state (destination excluded) and each column to a link, as numbered in Table 2. Entries of the matrix P specify in which proportion individuals choose outgoing links available in each state. For example, in this case, the initial flow on arc 1 is split in half between outgoing arcs 2 and 3 when both are available, and directed on arc 2 only otherwise.

In the following, we compare the deterministic and stochastic assignment algorithms described in Section 5.3 to the deterministic assignment model in Marcotte et al. (2004) and applied on the same network. In their work, Marcotte et al. (2004) consider strategies represented as vectors of size equal to the number of network nodes, prescribing for each an ordered list of successor nodes. Examples of such strategies are displayed in Table

Nodes	1	2	3	4	5
s_1	$[3, 2]$	$[3]$	$[5, 4]$	$[5]$	-
s_2	$[3, 2]$	$[5]$	$[5, 4]$	$[5]$	-
s_3	$[2]$	$[3]$	$[5, 4]$	$[5]$	-

Table 3: A set of strategies (Marcotte et al., 2004) for the small network

3. For instance, a user following strategy s_1 would choose node 3 from node 1 if the link is available, and node 2 otherwise. From node 2, the user would pick node 3, and in this case no recourse action is needed. Finally, at node 3, the preferred choice would be node 5, then node 4, and trivially from node 4 the user would go to node 5. There exists many such strategies, and their number grows exponentially with the size of the network.

There exists a correspondence between choice probabilities P in this work and strategic flows x as defined in Marcotte et al. (2004). For instance, the link choice probabilities characterized by the initial P are equivalent to a flow on strategies given by $x = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, when restricting the number of possible strategies to the three ones displayed in Table 3. Marcotte et al. (2004) state that at equilibrium, demand is equally split between strategies s_1 and s_2, of equal expected cost 185, and receiving each 5 units of flow. In other words, the optimal flow on strategies is $x^* = (\frac{1}{2}, \frac{1}{2}, 0)$. The first strategy consists in selecting node 3 from nodes 1 and 2, and node 5 from nodes 3 and 4. The second strategy differs only by selecting node 5 from node 2. We can find an equivalent deterministic equilibrium in the space of choice probabilities, given by

$$P^* = \begin{pmatrix} 0 & 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.50 & 0.50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \end{pmatrix},$$

as the demand splits in equal proportion between both outgoing arcs in state 3, corresponding to node 2.

6.1 Deterministic assignment

We apply the deterministic assignment algorithm described in Section 5.3, with common and disaggregate step sizes respectively. We display in Table 4 and 5 the relevant values

	Choice probabilities				Costs					Gap (%)		
# Iter	$P_{1,2}$	$P_{1,3}$	$P_{3,4}$	$P_{3,5}$	$C_{1,2}$	$C_{1,3}$	$C_{3,4}$	$C_{3,5}$	V_1	$g_R(P_1)$	$g_R(P_3)$	$g_R(P)$
0	0.5000	0.5000	0.7500	0.2500	200.00	156.25	181.25	150.00	182.50	12.28	13.51	9.25
1	0.2500	0.7500	0.3750	0.6250	175.00	100.00	125.00	150.00	155.00	15.79	11.11	8.36
2	0.1667	0.8333	0.5833	0.4167	200.00	137.50	162.50	150.00	185.00	7.04	4.64	3.51
3	0.1250	0.8750	0.4375	0.5625	188.64	113.64	138.64	150.00	171.49	7.62	4.41	3.45
4	0.1000	0.9000	0.5500	0.4500	200.00	132.81	157.81	150.00	185.07	4.82	2.78	2.17
5	0.0833	0.9167	0.4583	0.5417	192.65	117.65	142.65	150.00	176.28	5.04	2.72	2.16
10	0.0455	0.9545	0.5227	0.4773	200.00	128.68	153.68	150.00	185.06	2.46	1.26	1.01
20	0.0238	0.9762	0.5119	0.4881	200.00	126.95	151.95	150.00	185.03	1.35	0.66	0.54
50	0.0098	0.9902	0.5049	0.4951	200.00	125.81	150.81	150.00	185.02	0.57	0.27	0.22
100	0.0050	0.9950	0.5025	0.4975	200.00	125.41	150.41	150.00	185.01	0.29	0.14	0.11
200	0.0025	0.9975	0.5012	0.4988	200.00	125.21	150.21	150.00	185.00	0.15	0.07	0.06
500	0.0010	0.9990	0.5005	0.4995	200.00	125.08	150.08	150.00	185.00	0.06	0.03	0.02
1000	0.0005	0.9995	0.5002	0.4998	200.00	125.04	150.04	150.00	185.00	0.03	0.01	0.01

Table 4: Iterations of the deterministic assignment algorithm (common step size)

of P for successive iterations of the algorithm. In particular we look at the choice probabilities from states 1 and 3, since we do not expect the other components of P to be updated. We also show the corresponding cost C of choosing the respective actions in each state, and the corresponding state-specific relative gap. Finally, the last column displays the aggregate relative gap value.

We observe that both methods converge slowly towards the solution P^* given above, which is equivalent to the equilibrium solution found in Marcotte et al. (2004). The gap at state 3 is smaller using the disaggregate step size, however the opposite is true for state 1. When comparing the aggregate gap values, we observe that it is smaller with the common step size. Therefore the disaggregate step size does not improve convergence in this example. In general, we also observe that the gap at specific states may be higher than the aggregate gap, since the latter is lowered by taking into account some states where the gap is zero.

While in Marcotte et al. (2004), all used strategies have the same expected cost of 185 at equilibrium, similarly we observe here that all chosen actions at a given state have the same expected cost. When an equilibrium is reached, both outgoing links in state 3 have a cost of 150, while in state 1 the only chosen outgoing link has a cost of 185, which is less than the cost of the other link. Note that the expected cost of the best strategy for the OD pair is equivalent to the value function at the origin state in this work, as it represents the minimum expected cost to reach destination. From Table 4 we observe that V in (2) at the origin state given by $s = 1$ indeed converges to $\min\{185, 200\} = 185$.

6.2 Stochastic assignment

In a second phase, we apply the stochastic version of the algorithm described in Section 5.3 to obtain equilibrium flows in this network. In this setting, arc costs are assumed to be stochastic and given by $c_a + \mu\epsilon_a$, and we do not expect the choice probabilities to converge to the same solution as in Section 6.1. Therefore, we examine the impact of

	Choice probabilities				Costs					Gap (%)		
# Iter	$P_{1,2}$	$P_{1,3}$	$P_{3,4}$	$P_{3,5}$	$C_{1,2}$	$C_{1,3}$	$C_{3,4}$	$C_{3,5}$	V_1	$g_R(P_1)$	$g_R(P_3)$	$g_R(P)$
0	0.5000	0.5000	0.7500	0.2500	200.00	156.25	181.25	150.00	182.50	12.28	13.51	9.25
1	0.4386	0.5614	0.6486	0.3514	200.00	145.68	170.68	150.00	180.65	14.06	8.21	6.99
2	0.3769	0.6231	0.5954	0.4046	200.00	139.11	164.11	150.00	180.45	14.16	5.30	5.42
3	0.3236	0.6764	0.5638	0.4362	200.00	134.81	159.81	150.00	180.72	13.53	3.55	4.29
4	0.2798	0.7202	0.5438	0.4562	200.00	131.90	156.90	150.00	181.09	12.62	2.44	3.46
5	0.2445	0.7555	0.5305	0.4695	200.00	129.89	154.89	150.00	181.44	11.16	1.70	2.84
10	0.1446	0.8554	0.5056	0.4944	200.00	125.93	150.93	150.00	182.68	7.84	0.31	1.36
20	0.0777	0.9233	0.5002	0.4998	200.00	125.03	150.04	150.00	183.74	4.45	0.02	0.64
50	0.0324	0.9676	0.5000	0.5000	200.00	125.00	150.00	150.00	184.50	1.91	0.00	0.26
100	0.0164	0.9836	0.5000	0.5000	200.00	125.00	150.00	150.00	184.75	0.98	0.00	0.13
200	0.0083	0.9917	0.5000	0.5000	200.00	125.00	150.00	150.00	184.87	0.49	0.00	0.06
500	0.0033	0.9967	0.5000	0.5000	200.00	125.00	150.00	150.00	184.95	0.20	0.00	0.03
1000	0.0017	0.9983	0.5000	0.5000	200.00	125.00	150.00	150.00	184.98	0.09	0.00	0.01

Table 5: Iterations of the deterministic assignment algorithm (disaggregate step size)

the scale μ of the random term on the results. Table 6 gives the choice probabilities P and value function V at origin state after 1000 iterations for different values of μ.

As expected, when μ is small, the assignment is close to a deterministic one and the equilibrium choice probabilities are close to the values in Table 4. On the other hand, when μ becomes very large, we observe that the choice of arc is close to random. From arc 2, the flow splits between arcs 4 and arcs 5 in proportion $\frac{2}{3}$ and $\frac{1}{3}$ respectively. This is because there are two paths to the destination from arc 4, and only one from arc 5. Similarly from arc 1, we notice that arc 2 contains three feasible paths to the destination, while arc 3 only contains two. Thus the choice probabilities at the origin state converge towards $\frac{3}{5}$ and $\frac{2}{5}$ respectively. The expected minimum cost given by V at the origin state is close to 185 when the value of μ is small, and decreases as μ tends to infinity and the magnitude of the error term becomes large. Intuitively, the large variance among perceived costs decreases the expected value of the minimum cost.

In Table 7, we look in detail at the iterations of the algorithm for $\mu = 0.5$. We compare once more the algorithms with common and disaggregate step size and note that with the former the aggregate gap converges faster to zero.

μ	$P_{1,2}$	$P_{1,3}$	$P_{3,4}$	$P_{3,5}$	V_1
0.5	0.0005	0.9995	0.5000	0.5000	184.72
1	0.0005	0.9995	0.5000	0.5000	184.44
5	0.0005	0.9995	0.5000	0.5000	182.22
10	0.0016	0.9984	0.5000	0.5000	179.43
20	0.0454	0.9546	0.5003	0.4997	173.12
30	0.1406	0.8594	0.5035	0.4965	165.28
50	0.2996	0.7004	0.5224	0.4776	145.83
100	0.4495	0.5505	0.5789	0.4211	84.41
1000	0.5857	0.4143	0.6561	0.3439	−1125.60
10000	0.5985	0.4015	0.6656	0.3344	−13309.00

Table 6: Choice probabilities P and expected minimum cost at origin state V_1 for different values of μ after 1000 iterations (common step size)

	Choice probabilities				Costs					Gap (%)		
# Iter	$P_{1,2}$	$P_{1,3}$	$P_{3,4}$	$P_{3,5}$	$\tilde{C}_{1,2}$	$\tilde{C}_{1,3}$	$\tilde{C}_{3,4}$	$\tilde{C}_{3,5}$	V_1	$g_R(P_1)$	$g_R(P_3)$	g_R
0	0.5000	0.5000	0.7500	0.2500	199.65	155.90	181.11	149.31	182.50	12.30	13.77	9.38
1	0.2500	0.7500	0.3750	0.6250	174.31	99.86	124.51	149.77	155.00	15.71	11.25	8.42
2	0.1667	0.8333	0.5833	0.4167	199.10	137.41	162.23	149.56	185.00	6.96	4.71	3.53
3	0.1250	0.8750	0.4375	0.5625	187.60	113.57	138.22	149.71	171.49	7.53	4.47	3.46
4	0.1000	0.9000	0.5500	0.4500	198.85	132.76	157.51	149.60	185.07	4.74	2.83	2.18
5	0.0833	0.9167	0.4583	0.5417	191.40	117.60	142.26	149.69	176.28	4.97	2.75	2.17
10	0.0455	0.9545	0.5227	0.4773	198.45	128.65	153.35	149.63	185.06	2.41	1.28	1.01
20	0.0238	0.9762	0.5116	0.4884	198.12	126.89	151.57	149.64	185.02	1.32	0.63	0.51
50	0.0098	0.9902	0.5002	0.4998	197.36	125.03	149.69	149.65	184.60	0.56	0.00	0.08
100	0.0050	0.9950	0.5000	0.5000	197.00	125.00	149.65	149.65	184.65	0.28	0.00	0.04
200	0.0025	0.9975	0.5000	0.5000	196.66	125.00	149.65	149.65	184.69	0.14	0.00	0.02
500	0.0010	0.9990	0.5000	0.5000	196.20	125.00	149.65	149.65	184.71	0.00	0.00	0.01
1000	0.0005	0.9995	0.5000	0.5000	195.85	125.00	149.65	149.65	184.72	0.00	0.00	0.00

Table 7: Iterations of the stochastic assignment algorithm for $\mu = 0.5$ (common step size)

	Choice probabilities				Costs					Gap (%)		
# Iter	$P_{1,2}$	$P_{1,3}$	$P_{3,4}$	$P_{3,5}$	$\tilde{C}_{1,2}$	$\tilde{C}_{1,3}$	$\tilde{C}_{3,4}$	$\tilde{C}_{3,5}$	V_1	$g_R(P_1)$	$g_R(P_3)$	g_R
0	0.5000	0.5000	0.7500	0.2500	199.65	155.90	181.11	149.31	182.50	12.30	13.77	9.38
1	0.4395	0.5605	0.6497	0.3503	199.59	145.51	170.58	149.48	180.66	14.04	8.40	7.08
2	0.3785	0.6215	0.5972	0.4028	199.51	139.10	164.08	149.55	180.48	14.12	5.49	5.51
3	0.3256	0.6744	0.5661	0.4339	199.44	134.93	159.85	149.58	180.76	13.47	3.74	4.38
4	0.2822	0.7178	0.5465	0.4535	199.36	132.13	156.99	149.60	181.14	12.56	2.63	3.55
5	0.2471	0.7529	0.5335	0.4665	199.30	130.21	155.04	149.62	181.50	11.59	1.90	2.93
10	0.1472	0.8528	0.5095	0.4905	199.02	126.48	151.23	149.64	182.76	7.79	0.49	1.45
20	0.0798	0.9202	0.5037	0.4963	198.61	125.57	150.27	149.65	183.73	4.43	0.12	0.70
50	0.0335	0.9665	0.5018	0.4982	198.08	125.29	149.96	149.65	184.37	1.91	0.03	0.28
100	0.0170	0.9830	0.5012	0.4988	197.71	125.19	149.85	149.65	184.58	0.98	0.01	0.14
200	0.0086	0.9914	0.5008	0.4992	197.34	125.13	149.79	149.65	184.67	0.49	0.01	0.07
500	0.0035	0.9965	0.5005	0.4995	196.86	125.08	149.73	149.65	184.72	0.20	0.00	0.03
1000	0.0017	0.9983	0.5004	0.4996	195.51	125.06	149.71	149.65	184.73	0.09	0.00	0.01

Table 8: Iterations of the stochastic assignment algorithm for $\mu = 0.5$ (disaggregate step size)

OD pair	Notation	Demand	Destination index
(1,24)	**OD1**	35	d_1
(1,22)	**OD2**	25	d_2
(7,24)	**OD3**	20	d_1
(7,22)	**OD4**	20	d_2

Table 9: OD pairs for Sioux Falls network

7 Applications

In the following we present two applications of the model. The first one is a simplified and acyclic version of the Sioux Falls network, also used as a numerical example in Marcotte et al. (2004). The network is more complex than the illustrative example, but small enough to study in detail the solution of the assignment. The second one is a larger scale experiment with a time-expanded transit network of over 2000 links, where we assume that users are loaded in a random manner.

7.1 Sioux Falls network

The network is depicted in Figure 4 and contains 24 nodes and 41 links. It has up to 4 outgoing arcs per node, up to three of which may have a limited capacity. In total there are 14 capacitated arcs. The state space is thus more complex than for the previous illustrative example, and there are 75 states in total. We consider four OD pairs with demand described in Table 9. Accordingly, dummy origin and destination links are added to the network, to nodes 1 and 7, and from nodes 22 and 24 respectively.

We compute the deterministic equilibrium using both the common and the disaggregate stepsize rule, and display the value function V^d at origin for each OD pair in Table

	Expected minimum cost V_o^d				Gap (%)
Heuristic	**OD1**	**OD2**	**OD3**	**OD4**	$g_R(P)$
Common step size	120.00	139.94	112.97	99.97	$2.97 \cdot 10^{-2}$
Disaggregate step size	120.00	139.90	114.18	100.00	$2.67 \cdot 10^{-2}$

Table 10: Expected minimum cost of OD pairs after 1000 iterations of the deterministic assignment algorithm

Destination	Node	Tail node of outgoing links	Strategic costs $C_{s,a}^d$		Choice probabilities $P_{s,a}^d$		Gap (%) $g_R(P_s^d)$
24	3	4,12	110.0122	110.0000	0.3263	0.6737	$3.61 \cdot 10^{-3}$
	19	20,22	54.6733	55.0000	0.9222	0.0778	$9.92 \cdot 10^{-2}$
22	1	2,3	139.9361	139.9748	0.9980	0.0020	$5.52 \cdot 10^{-5}$
	7	8,18	99.9748	100.0000	0.7176	0.2824	$7.12 \cdot 10^{-3}$
	3	4,12	129.9361	130.0000	0.8812	0.1188	$5.84 \cdot 10^{-3}$
	4	5,11	119.9392	119.9322	0.4880	0.5120	$2.87 \cdot 10^{-3}$
	10	11,17	119.9322	119.4173	0.2834	0.7166	$1.20 \cdot 10^{-1}$

Table 11: Outgoing links with equal strategic cost for each destination after 1000 iterations with common step size

10. The value can be interpreted as the expected minimum cost to travel between each OD pair, and the values are close to the minimum strategic costs found in Marcotte et al. (2004). The aggregate relative gap is well below 1%, at around 0.03%.

In contrast with Marcotte et al. (2004), it is not possible to analyze the number of different strategies used at deterministic equilibrium, since we cannot recover strategic flows from arc flows. Instead, we may observe for how many couples (s, d) there exist two different outgoing arcs in $\mathcal{A}(s)$ with non null choice probabilities $P_{s,a}^d$. Therefore in Table 11, we display the nodes for which there exists outgoing links with equal expected minimum cost, and display the value of corresponding choice probabilities in the state where both links are available. We also analyze the specific relative gap at the corresponding states. In all cases the value is small, illustrating that the aggregate gap value does not conceal important variance.

We then apply the stochastic user equilibrium algorithm on the network for several values of μ. We display the results in Table 12. We observe that for $\mu = 0.5$, the expected minimum costs obtained are close to the deterministic solution, while they decrease as μ increases. The algorithm with common step size converges slightly faster for all values of μ.

		Expected minimum cost V_o^d				Gap (%)
Heuristic	μ	**OD1**	**OD2**	**OD3**	**OD4**	$g_R(P)$
Common step size	0.5	119.74	138.92	114.72	99.61	$2.15 \cdot 10^{-2}$
Disaggregate step size		119.69	138.84	114.15	99.60	$3.21 \cdot 10^{-2}$
Common step size	5	116.83	131.42	113.00	96.23	$6.26 \cdot 10^{-3}$
Disaggregate step size		117.05	131.34	112.61	96.16	$4.13 \cdot 10^{-2}$
Common step size	10	112.00	119.25	107.38	88.50	$7.99 \cdot 10^{-2}$
Disaggregate step size		112.59	119.80	107.21	88.57	$1.10 \cdot 10^{-1}$
Common step size	20	95.57	106.02	94.11	80.25	$9.10 \cdot 10^{-3}$
Disaggregate step size		95.18	105.59	93.38	79.88	$7.62 \cdot 10^{-2}$

Table 12: Expected minimum cost of OD pairs after 1000 iterations of the stochastic assignment algorithm with different values of μ

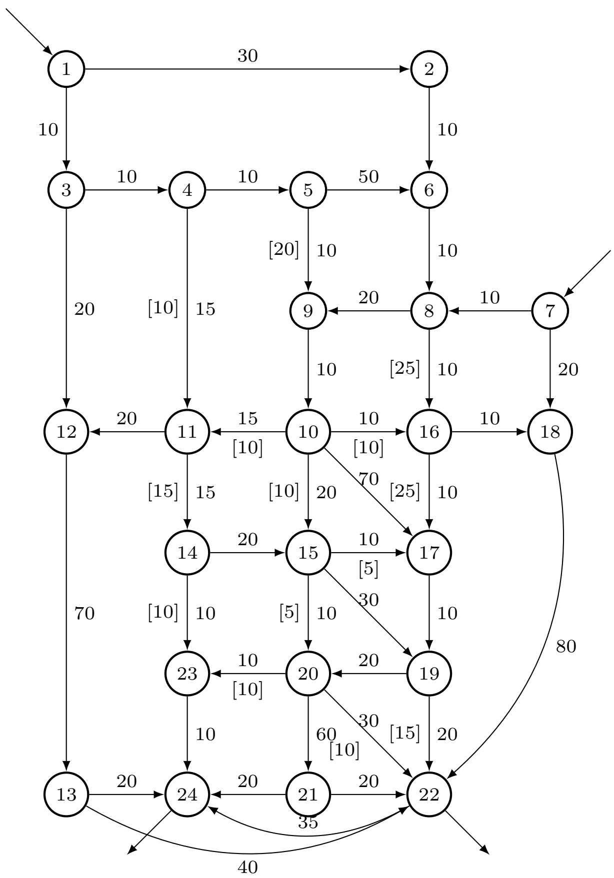

Figure 4: Sioux Falls network

7.2 Springfield network

The Springfield network is a 5 zones network that was developed as a generic example for the fast trips Dynamic Transit Passenger Assignment tool (Khani, 2013). The network has three transit lines, as displayed in Figure 5, several walking links and a transfer link between transit stops B_2 and R_2. The train line going through stops R_1, R_2 and R_3 has a capacity of 16 units, and the two bus lines have a capacity of 10.

In this example, we apply the strategic Markovian traffic equilibrium model to the time-expanded version of the Springfield network. The transit schedule is given between 3PM and 6PM and there are 152 runs of the transit lines. Demand starts at 3:15 PM, ends at 5:15 PM and is characterized by a trip every ten seconds between two of the five possible zones. Each trip has a latest desired arrival time of 30 minutes after departure time.

We create an acyclic time-space network based on the static bidirectional network in Figure 5 and the given schedule. To do so, we build four types of arcs: transit arcs, corresponding to each run of a transit line between two consecutive stops; transfer arcs, connecting two transit stops (here B_2 and R_2); walking arcs, between zones and accessible transit stops; waiting arcs, connecting the same zone or transit stop between two consecutive discrete points in time. Transfer and walking arcs are created not at regular time intervals but rather for each arrival or departure of a transit line at the stop. Thus time in this approach is discretized according to the transit schedule. We assume that the capacity of waiting, transfer and walking arcs is infinite.

Artificial origin and destination links are also created to match the dynamic OD information. For each trip in the OD table, an origin link is created at the origin zone, so as to be connected with the first walking arc to leave the zone from the stated departure time. Similarly, a destination link without successor is added at the arrival zone and is connected to the link arriving at the zone at a time closest to the latest desired arrival time. We ensure that the time interval between earliest possible departure and latest possible arrival is at least 30 minutes. Note that origin and destination links are also connected to waiting arcs. Therefore, the demand may leave and arrive at any time between the stated departure time and latest possible arrival time, and use waiting arcs in between. The total number of arcs in the time-expanded network is 2961, while the number of possible states is 4032.

Transit, transfer and walking arcs have a cost displayed in Figure 5, corresponding to the travel time in minutes between nodes. The cost of waiting arcs is equal to the waiting time, which can be inferred from the time index at the nodes of the time-expanded network. However, the cost of waiting arcs at origin and destination zones is upper bounded by a small value (20 seconds). Thus the cost provides individuals with an incentive to arrive earlier at destination if possible and spend less time travelling.

We assume that passengers are loaded randomly at each node. It is usual in dynamic transit assignment to make more complex assumptions, typically that passengers arriving first at a node are loaded before those arriving at a later time step. However, since boarding priorities and first-come first-serve loading is beyond the scope of this paper, we illustrate the model on this example with the assumptions described in Section 5.1.

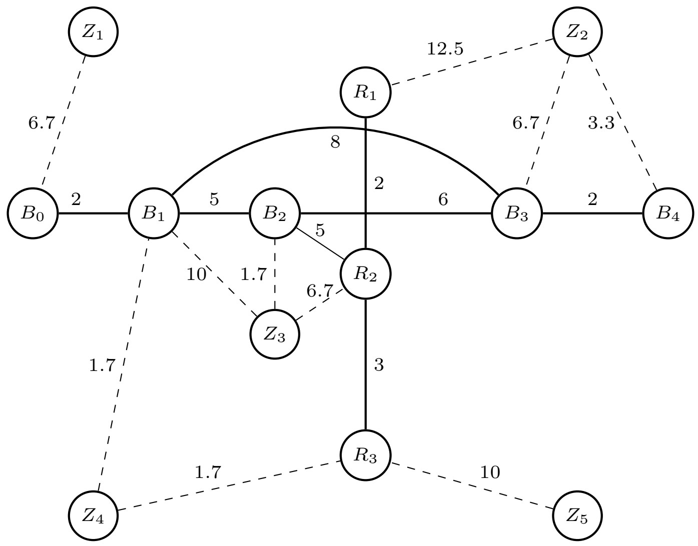

Figure 5: Springfield network

Since the disaggregate step size does not demonstrate a significant improvement in convergence, we use the common step size update to compute the deterministic and stochastic user equilibrium. For the stochastic case, we use an intermediate value of 5 for μ. Table 13 shows the value of the aggregate gap for iterations of the deterministic and stochastic algorithms. We observe that the algorithm follows the typical slow convergence rate where the gap is roughly divided by two when the number of iterations double.

While the aggregate gap shows that choice probabilities globally tend towards the equilibrium solution, it is not the only way to look at the gap. Therefore, Table 14 displays disaggregate values of the gap for specific destinations and states. In particular, for each destination d, we compare two different gap functions, i.e., the maximum relative gap across all states $\max_s g_R(P_s^d)$, and the average of $g_R(P_s^d)$ over all states. We then show the lowest, highest and average values of these measures across all destinations after 1000 iterations. For the worse destination and state, there is still an 7.60% and 11.34% relative gap for the deterministic and stochastic model respectively. Although it is unnecessarily demanding to require the gap to reach a very low value in all states, this shows that there may be significant variance in the gap across the network.

8 Conclusion

We presented a strategic Markovian traffic equilibrium model for capacitated networks, which provides a framework to compute both deterministic and stochastic user equilibrium. The model extends the work of Baillon and Cominetti (2008) on Markovian traffic equilibrium by considering travel cost functions which instead of bounding flows through

	Gap (%)	
# Iter	Deterministic	Stochastic
0	1.70	2524.00
1	1.77	31.61
2	1.62	9.54
3	1.42	$9.43 \cdot 10^{-1}$
4	1.29	$7.40 \cdot 10^{-1}$
5	1.14	$5.83 \cdot 10^{-1}$
10	$7.30 \cdot 10^{-1}$	$2.85 \cdot 10^{-1}$
20	$4.44 \cdot 10^{-1}$	$1.19 \cdot 10^{-1}$
50	$1.94 \cdot 10^{-1}$	$3.49 \cdot 10^{-2}$
100	$1.07 \cdot 10^{-1}$	$2.93 \cdot 10^{-2}$
200	$5.63 \cdot 10^{-2}$	$1.73 \cdot 10^{-2}$

Table 13: Values of aggregate gap at iterations of the deterministic and stochastic assignment algorithm

	Maximum state specific gap (%)		
Assignment	$\min_d \max_s g_R(P_s^d)$	$\max_d \max_s g_R(P_s^d)$	$\text{mean}_d \max_s g_R(P_s^d)$
Deterministic	$2.92 \cdot 10^{-2}$	7.60	0.33
Stochastic	$6.98 \cdot 10^{-4}$	11.34	0.44
	Average state specific gap (%)		
Assignment	$\min_d \text{mean}_s g_R(P_s^d)$	$\max_d \text{mean}_s g_R(P_s^d)$	$\text{mean}_d \text{mean}_s g_R(P_s^d)$
Deterministic	$6.93 \cdot 10^{-4}$	$1.69 \cdot 10^{-2}$	$8.38 \cdot 10^{-3}$
Stochastic	$7.45 \cdot 10^{-6}$	$7.83 \cdot 10^{-2}$	$3.28 \cdot 10^{-3}$

Table 14: Different gap values after 1000 iterations for both the deterministic and stochastic assignment algorithm

exogenous volume-delay functions incorporate the risk of failing to board an arc, thereby allowing users to behave strategically with respect to the stochasticity induced by limits on capacity. The model possesses a travel cost function which explicitly derives delay from an emulation of the queuing process to access capacitated arcs. In that respect, the model is also an extension of the work of Marcotte et al. (2004), who first proposed the concept of strategic equilibrium in the context of deterministic arc costs. Both approaches are relatively disconnected in the literature, and our contribution consists in merging both models. The resulting model has the advantage of incorporating two sources of stochasticity in user route choice behavior, induced by variations in cost perception and the risk associated with the failure to access an arc. The model formulation is arc-based, which also avoids path enumeration issues.

Future work could be dedicated to extending the proposed framework by incorporating the complex queuing dynamics that would allow for a more realistic transit assignment application, such as letting passengers be loaded on a first-come first-serve basis when the network represents a time-expanded graph. Another appealing prospect is to adapt this framework to cyclic networks, which would require to reshape the algorithms proposed in this work.

Acknowledgements

This work was partly funded by the National Sciences and Engineering Research Council of Canada, discovery grants 435678-2013 and A5789.

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