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A Stochastic Programming Approach for Production Planning in a Manufacturing Environment with Random Yield

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Abstract. This paper investigates a multi-period, multi-product production planning problem in a manufacturing environment with random yield. It is concentrated on the case where the random yield originates from non-homogeneous characteristics of raw materials. A two-stage stochastic linear programming approach is proposed to address this problem. The random yields are modeled as scenarios with discrete probability distribution. The solution methodology is based on the sample average approximation method. The proposed approach is applied for sawmill production planning. The computational results for a real sawmill highlight the significance of using the stochastic model for production planning instead of the mean-value deterministic model.

Keywords. Production planning, random yield, stochastic programming, sample average approximation, sawmill.

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1. Introduction

Most production environments are characterized by multiple types of uncertainties. When planned production quantities are released, the outputs are often variable. These uncertainties affect and complicate the production plan and control. Random characteristic of raw materials is a common issue in manufacturing environments that process the natural resources, namely refineries, sawmills, etc. This randomness, as a consequent, can cause random yields of the production processes. The presence of random yield causes the fraction of the quantity actually processed, which turns out to be usable, to be uncertain.

The goal of this work is to address a multi-period, multi-product (MPMP) production planning problem in a manufacturing environment where alternative processes can produce simultaneously multiple products with random yields. In other words, the quantities of products that can be produced by each process are random variables. Besides, the randomness in process yields arises from random quality of raw materials which are supposed to be classified based on some attributes. In this production planning problem, we are looking for the number of times each process should be run as well as the quantity of each class of raw material that should be consumed by each process in each period in the planning horizon. The objective is to minimize products inventory/backorder and raw material costs, regarding fulfillment of products demands, machine capacities, and raw material inventory. This work is motivated by production planning for sawing units in sawmills where non-homogeneous characteristics of logs result in random yield.

This production planning problem can be considered as the combination of several classical production planning problems in the literature which have been modeled by linear programming (LP). Product mix problem and a special case of process selection problem (Johnson and Montgomery, 1974; Sipper and Bulfin, 1997) are the two main building blocks of this problem. On the other hand, the LP models include the assumption of deterministic parameters. One possible way to deal with uncertainties in an optimization model might be considering the expected value of the random parameters in the LP model and solve the mean-value deterministic model. Nevertheless, it has been shown in the literature (see for example Birge, 1997) that failure to include uncertainty in optimization models can cause expensive, even disastrous consequences if the anticipated situation is not realized. Most of the work in the literature for including uncertainty in production planning models is focused on considering random

product demand in the models. In (Bakir and Byrune, 1998), demand uncertainty in a MPMP production planning model has been studied. They have developed a demand stochastic LP model based on the two-stage deterministic equivalent problem. In (Escudero et al., 1993) a multi-stage stochastic programming approach has been used for solving a MPMP production planning model with random demand. In (Leung et al., 2006) the uncertain data for almost all the model parameters have been considered in an aggregate production planning problem. They have developed a robust optimization model to introduce production plans which are less sensitive to the change in the uncertain data. In (Kazemi et al., 2007) three approaches have been proposed as the potential methodologies to address MPMP production planning in a manufacturing environment with random yield. These approaches include stochastic programming, robust optimization and fuzzy linear programming.

In this paper, a two-stage stochastic program with recourse (Kall et al., 1994, 2005; Birge and Louveux, 1997) is proposed for MPMP production planning while considering random characteristics of raw materials and consequently random process yields. The random yields are modeled as scenarios with discrete probability distribution. Due to astronomic number of scenarios for random yields in the two-stage stochastic model, a Monte-Carlo sampling strategy, the sample average approximation (SAA) method (cf. Shapiro et al., 1998; Mak et al., 1999; Shapiro et al., 2000), is implemented to solve the stochastic model. The confidence intervals on the optimality gap for the candidate solutions are constructed based on common random number (CRN) streams (Mak et al., 1999). The proposed approach is applied for sawmill production planning. Our computational results involving one real sawmill indicate that the proposed approach can be served as a viable tool for production planning in manufacturing environments with random yield.

The remainder of this paper is organized as follows. In the next section, a theoretical framework for two-stage stochastic linear programming (LP) is provided; in section 3 we describe a two-stage stochastic linear program for MPMP production planning under uncertainty of process yields. In section 4, a scenario generation approach for process yields in the two-stage stochastic model is proposed. In section 5, we discuss about some challenges involved in developing a solution strategy for the model and we provide the solution methodology; we also explain the SAA scheme with the sampling technique based on the common random numbers. In section 6, the implementation results of the stochastic model and solution strategy for a sawmill production planning are presented. The solutions resulted from the stochastic model with those of the mean-value deterministic LP model are also compared in this section. Our concluding remarks are given in section 7.

2. A theoretical framework for two-stage stochastic LP

This section gives a brief review on two-stage stochastic LP; for more details, the reader is referred for example to (Kall et al., 1994, 2005; Birge and Louveux, 1997). When one or more of the parameters in a linear program is represented by a random variable, a stochastic linear program (SLP) is resulted. Model (1)-(3) is an example of a SLP.

$$\text{Minimize} \quad c^T x \tag{1}$$
$$\text{Subject to}$$
$$Ax = b, \tag{2}$$
$$T(\tilde{\xi})x \geq h^T(\tilde{\xi}), \tag{3}$$
$$x \geq 0.$$

where, $\tilde{\xi}$ is the vector of random parameters, $T(\tilde{\xi})$ and $h(\tilde{\xi})$ are random technological coefficient matrix and right-hand side vector, respectively. In the above model, constraints (2) and (3) represent the set of deterministic and stochastic constraints, respectively.

In two-stage stochastic models, we explicitly classify the decision variables according to whether they are implemented before or after an outcome of the random variable is observed. In other words, we have a set of decisions to be taken without full information on the random parameters. These decisions are called first-stage decisions, and are usually represented by a vector (x). Later, full information is received on realizations (scenarios) of some random vector $\tilde{\xi}$. Then, second-stage or recourse actions (y) are taken. These second-stage decisions allow us to model a response to each of the observed outcomes (scenarios) of the random variable, which constitutes our recourse. In general, this response will also depend upon the first-stage decisions. In mathematical programming terms, this defines the so-called two-stage stochastic program with recourse of the form:

$$\text{Minimize} \quad c^T x + E_{\tilde{\xi}}[Q(x, \tilde{\xi})] \tag{4}$$
$$\text{Subject to}$$
$$Ax = b, \tag{5}$$
$$x \geq 0.$$

where, $Q(x,\underset{\sim}{\xi}) = \min\left\{q^T(\underset{\sim}{\xi})y \mid Wy = h^T(\underset{\sim}{\xi}) - T(\underset{\sim}{\xi})x\right\}$, W is the recourse matrix, $q^T(\underset{\sim}{\xi})$ is the vector of penalty cost of second-stage (recourse) variables, $\underset{\sim}{\xi}$ is the random vector formed by the components of $q^T(\underset{\sim}{\xi})$, $h^T(\underset{\sim}{\xi})$, $T(\underset{\sim}{\xi})$, and $E_{\underset{\sim}{\xi}}$ denotes mathematical expectation with respect to $\underset{\sim}{\xi}$.

In the case of continuous distribution for random variables in model (4)-(5), the calculation of the expected value $E_{\underset{\sim}{\xi}}[Q(x,\xi)]$ requires the calculation of multiple integrals with respect to the measure describing the distribution of $\underset{\sim}{\xi}$. The computational effort increases with the dimension of the stochastic variables vector and this leads to tremendous amount of work. On the other hand, if $\underset{\sim}{\xi}$ has a finite discrete distribution $\left\{(\underset{\sim}{\xi}^i, p^i), i = 1,\ldots, n\right\}$, then (4)-(5) can be transformed into its *deterministic equivalent* which is an ordinary linear program as follows.

$$\text{Minimize} \quad c^T x + \sum_{i=1}^{n} p^i Q(x,\underset{\sim}{\xi}^i)$$

Subject to

$$Ax = b,$$
$$x \geq 0.$$

where, $Q(x,\underset{\sim}{\xi}^i) = \min\left\{q^{iT}(\underset{\sim}{\xi})y^i \mid Wy^i = h^{iT}(\underset{\sim}{\xi}) - T^i(\underset{\sim}{\xi})x\right\}$; y^i, $q^{iT}(\underset{\sim}{\xi})$, $h^i(\underset{\sim}{\xi})$ and $T^i(\underset{\sim}{\xi})$ represent the ith scenarios for y, $q^T(\underset{\sim}{\xi})$, $h^T(\underset{\sim}{\xi})$ and $T(\underset{\sim}{\xi})$, respectively. Finally the above model results in the following model (6)-(7):

$$\text{Minimize} \quad c^T x + \sum_{i=1}^{n} p^i q^{iT}(\underset{\sim}{\xi})y^i \tag{6}$$

Subject to

$$Ax = b,$$
$$Wy^i = h^{iT}(\underset{\sim}{\xi}) - T^i(\underset{\sim}{\xi})x, \qquad i = 1,\ldots,n, \tag{7}$$
$$x \geq 0.$$

Model (6)-(7) can be solved by the LP solvers. Although this model can become (very) large in scale, its particular block structure is amenable to specially designed algorithms. Solution methods for large-scale two-stage stochastic programs can be divided into two main categories: 1- exact methods including decomposition methods, namely L-shaped method (Kall et al., 1994 and 2005; Birge, 1997), and regularized decomposition method (Ruszczyński and Świetanowski, 1996) 2- approximate methods based on Monte-Carlo sampling: sample average approximation (SAA) (cf. Shapiro et al.,

1998; Mak et al., 1999; Shapiro et al., 2000), and stochastic decomposition method (Higle and Sen, 1996).

3. Problem formulation by mathematical programming

In this section we first describe the deterministic linear program (LP) formulation for MPMP production planning considered in this paper. Then we develop the proposed stochastic model to address the problem by considering the uncertainty of process yields.

3.1. The deterministic LP model for MPMP production planning

Consider a production unit with a set of products P, a set of classes of raw materials C, a set of production processes A, a set of resources (machines) R, and a planning horizon consisting of T periods. To state the deterministic linear programming model for this production planning problem, the following notations are used:

3.1.1. Notations

Indices

p product

t period

c raw material class

a production process

r resource (machine)

Parameters

i_{pt} Inventory holding cost per unit of product *p* in period *t*

b_{pt} Backorder cost (lost opportunity and goodwill) per unit of product *p* in period *t*

m_{ct} Raw material cost per unit of class *c* in period *t*

I_{c0} The inventory of raw material class *c* at the beginning of planning horizon

I_{p0} The inventory of product *p* at the beginning of planning horizon

s_{ct} The quantity of material of class *c* supplied at the beginning of period *t*

d_{pt} Demand of product *p* by the end of period *t*

ϕ_{ac} The units of class *c* raw material consumed by process *a* (consumption factor)

ρ_{ap} The units of product *p* produced by process *a* (yield of process *a*)

δ_{ar} The capacity consumption of resource *r* by process *a*

M_{rt} The capacity of resource *r* in period *t*

Decision variables

X_{at} The number of times each process *a* should be run in period *t*

I_{ct} Inventory level of raw material of class *c* by the end of period *t*

I_{pt} Inventory level of product *p* by the end of period *t*

B_{pt} Backorder level of product *p* by the end of period *t*

3.1.2. The LP model

$$\text{Minimize } Z = \sum_{p\in \mathrm{P}} \sum_{t=1}^{T} (i_{pt} I_{pt} + b_{pt} B_{pt}) + \sum_{c\in \mathrm{C}} \sum_{t=1}^{T} \sum_{a\in A} m_{ct} \phi_{ac} X_{at} \tag{8}$$

Subject to

Material inventory constraint

$$I_{ct} = I_{ct-1} + s_{ct} - \sum_{a\in A} \phi_{ac} X_{at}, \qquad t = 1,\ldots,T;\ c \in \mathrm{C}, \tag{9}$$

Product inventory constraint

$$I_{p1} - B_{p1} = I_{p0} + \sum_{a\in \mathrm{A}} \rho_{ap} X_{a1} - d_{p1},$$

$$I_{pt} - B_{pt} = I_{pt-1} - B_{pt-1} + \sum_{a\in \mathrm{A}} \rho_{ap} X_{at} - d_{pt}, \qquad t = 2,\ldots,T;\ p \in \mathrm{P}, \tag{10}$$

Production capacity constraint

$$\sum_{a\in \mathrm{A}} \delta_{ar} X_{at} \le M_{rt}, \qquad t = 1,\ldots,T;\ r \in \mathrm{R}, \tag{11}$$

Non-negativity of all variables

$$X_{at} \ge 0, I_{ct} \ge 0, I_{pt} \ge 0, B_{pt} \ge 0, \qquad t = 1,\ldots,T;\ p \in \mathrm{P};\ c \in \mathrm{C};\ a \in A. \tag{12}$$

The objective function (8) is a linear cost minimization equation. It consists of total inventory and backorder cost for all products and raw material cost for all classes in the planning horizon. Constraint (9) ensures that the total inventory of raw material of class *c* at the end of period *t* is equal to its inventory in the previous period plus the quantity of material of class *c* supplied at the beginning of that

period (s_{ct}) minus its total consumption in that period. It should be noted that the total consumption of each class of raw material in each period is calculated by multiplying material consumption factor of each process (ϕ_{ac}) by the number of times that process is executed in that period. Constraint (10) ensures that the sum of inventory (or backorder) of product *p* at the end of period *t* is equal to its inventory (or backorder) in the previous period plus the total production of that product in that period, minus the product demand for that period. Total quantity of production for each product in each period is calculated as the sum of the quantities yielded by each of the corresponding processes regarding the yield (ρ_{ap}) of each process. Finally, constraint (11) requires that the total production do not exceed the available production capacity. In other words, the sum of capacity consumption of a machine *r* by corresponding processes in each period should not be greater than the capacity of that machine in that period.

3.2. The Two-stage stochastic model for MPMP production planning

To include the random nature of process yields in MPMP production planning we expand the model (8)-(12) to a two-stage stochastic linear program with recourse. It is assumed that the probability distribution of random yields is known. We represent the random yield vector by $\underset{\sim}{\xi}$, where $\underset{\sim}{\xi} = \{\rho_{ap} \mid a \in \mathrm{A}, p \in \mathrm{P}\}$. We also represent each realization of random process yields by $\rho_{ap}(\underset{\sim}{\xi})$. It should be emphasized that the stages of the two-stage recourse problem do not refer to time units. They correspond to steps in the decision making. In other words, in the first-stage, the decision maker does not have any information about the process yields due to lack of complete information on the characteristic of raw materials. However, the production plan should be determined before the complete information is available. In the second stage when the realized yields are available, based on the first-stage decision, the recourse actions (inventory or backorder levels) can be computed. The objective of the second-stage problem is to minimize the inventory and backorder costs (recourse action costs). The resulting formulation is as follows:

First-stage model

$$\text{Minimize } Z = \sum_{c \in \mathrm{C}} \sum_{t=1}^{T} \sum_{a \in A} m_{ct} \phi_{ac} X_{at} + E_{\underset{\sim}{\xi}}[Q(X_{at}, \underset{\sim}{\xi})] \tag{13}$$

Subject to

$$I_{ct} = I_{ct-1} + s_{ct} - \sum_{a \in A} \phi_{ac} X_{at}, \qquad t = 1,\ldots,T;\ c \in \mathrm{C}, \tag{14}$$

$$\sum_{a \in \mathrm{A}} \delta_{ar} X_{at} \le M_{rt}, \qquad t = 1,\ldots,T;\ r \in \mathrm{R}, \tag{15}$$

$$X_{at} \ge 0,\ I_{ct} \ge 0, \qquad a \in \mathrm{A};\ c \in C;\ t = 1,\ldots,T. \tag{16}$$

where, $Q(X_{at}, \underset{\sim}{\xi})$ is the optimal value of the following problem:

Second-stage model

$$\text{Minimize } Q(X_{at}, \underset{\sim}{\xi}) = \sum_{p \in \mathrm{P}} \sum_{t=1}^{T} (i_{pt} I_{pt} + b_{pt} B_{pt}) \tag{17}$$

Subject to

$$I_{p1} - B_{p1} = I_{p0} + \sum_{a \in \mathrm{A}} \rho_{ap}(\underset{\sim}{\xi}) X_{a1} - d_{p1},$$

$$I_{pt} - B_{pt} = I_{pt-1} - B_{pt-1} + \sum_{a \in \mathrm{A}} \rho_{ap}(\underset{\sim}{\xi}) X_{at} - d_{pt}, \quad t = 2,\ldots,T;\ p \in \mathrm{P}, \tag{18}$$

$$I_{pt} \ge 0, B_{pt} \ge 0, \qquad p \in \mathrm{P}; t = 1,\ldots,T. \tag{19}$$

Note again that $\underset{\sim}{\xi}$ is a random vector corresponding to different scenarios for the uncertain process yields, and the optimal value $Q(X_{at}, \underset{\sim}{\xi})$ of the second-stage problem (17)-(19) is a function of the first-stage decision variable X_{at} and a realization (or a scenario) of the uncertain yield ($\rho_{ap}(\underset{\sim}{\xi})$). The expectation in (13) is taken with respect to the probability distribution of $\underset{\sim}{\xi}$ which is supposed to be known.

Model (13)-(19) is a two-stage stochastic program. The first-stage consists of deciding the number of times each process should be run in each period (X_{at}), respecting raw material inventory and machine capacity constraints, and the second-stage consists of finding the optimal recourse action (i.e. inventory or backorder level of different products in each period) based on the first-stage decision and the realized uncertain yields. The objective is to minimize the raw material consumption cost and the expected future inventory and backorder costs. The second-stage program is a complete recourse

program, since the backorder and inventory costs guarantee that $Q(X_{at},\underset{\sim}{\xi})<\infty$ for all X_{at} and $\underset{\sim}{\xi}$. Furthermore, we assume that inventory and backorder costs are sufficiently high such that $Q(X_{at},\underset{\sim}{\xi})>-\infty$ for all X_{at} and $\underset{\sim}{\xi}$, and hence $Q(X_{at},\underset{\sim}{\xi})$ is finite valued for all X_{at} and possible realizations of random yields. We assume further that the expected value $E_{\underset{\sim}{\xi}}[Q(X_{at},\underset{\sim}{\xi})]$ is well defined and finite valued for the considered distribution of $\underset{\sim}{\xi}$. Consequently problem (13)-(16) has a well defined objective function and, since its feasible region is non-empty and finite, possesses an optimal solution.

Before explaining how the stochastic model should be solved, in the next section we will first illustrate how the random yields should be modeled to be incorporated in the stochastic model.

4. Scenario generation

In this section we explain how different scenarios for random yields ($\rho_{ap}(\underset{\sim}{\xi})$) can be generated in the stochastic model. We define a global scenario in the two-stage model as the combinations of scenarios for yields of individual processes. We suppose that the yields of different processes are independent. Therefore as the first step, all possible scenarios for yields of each process should be determined and then these scenarios should be aggregated to generate the global scenarios for the stochastic model. Figure 1 illustrates better this approach for scenario generation.

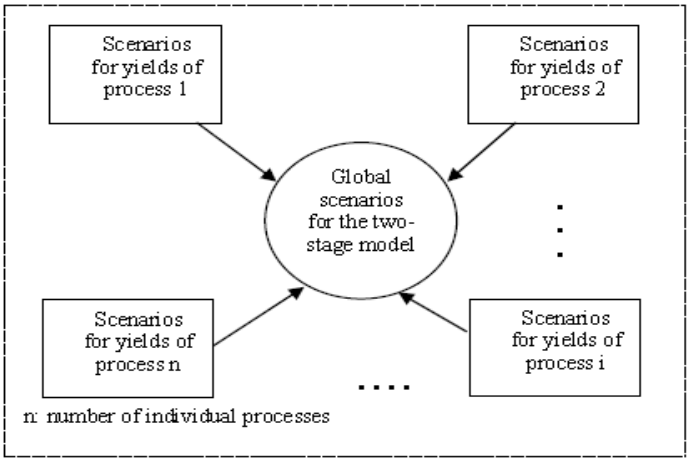

Figure 1. Scenario creation approach

In the following we explain how the scenarios for individual processes are defined and how they can be generated. As we have mentioned before, we are studying a manufacturing environment where each process produces simultaneously all its corresponding products after processing each unit of raw material (co-production). Thus, the quantities of products that are yielded by the same process ($\rho_{ap}(\underset{\sim}{\xi})$) are not independent. Such scenarios can be determined simultaneously for all the corresponding products as different quantities of each product that can be yielded by that process. For example, consider a process that can produce potentially 4 products (P1, P2, P3, P4). Table 1 represents two scenarios among all possible scenarios for yields of this process.

Table 1. Two scenarios for yields of a process

Scenario number	Products	Quantity (yield)
1	P1	2
	P2	3
	P3	1
	P4	0
2	P1	1
	P2	0
	P3	3
	P4	2

On the other hand, respecting the limited volume of raw materials and dimensions of different products, it is evident to consider a discrete distribution for random yields of processes. According to the above scenario definition approach for process yields, the only questions that remain to be answered are how the real scenarios in industry can be determined and also how their probability distribution can be estimated.

Such scenarios and their probability distribution can be determined as follows.

1) Take a sample of raw materials in each class (e.g. 300 parts in each class) and let them to be processed by each process.

2) Register the yield of the process (the corresponding products with their quantity) for each individual part and consider the result as a scenario.

3) After finding all the resulted scenarios, calculate their probabilities as their proportion in the population of scenarios.

5. Solution strategy

In this section, we give the details of the proposed methodology to solve the two-stage stochastic production planning model. We use the sample average approximation (SAA) scheme to solve this problem. In the first part, the deterministic equivalent of the stochastic model is presented and the challenges to solve this model are discussed. The SAA scheme is explained in the second part.

5.1. The deterministic equivalent model

As we have mentioned in the previous section, the probability distribution of process yields has a discrete distribution and the yields of different processes are independent. Consequently, the probability distribution of global scenarios for the two-stage model (13)-(19) has also a discrete distribution with a known probability for each scenario. Therefore, the expected value $E_{\tilde{\xi}}[Q(X_{at}, \tilde{\xi})]$ in (13) can be written as $\sum_{i=1}^{n} p^i Q(X_{at}, \tilde{\xi}^i)$, where *n* denotes the total number of scenarios, $\tilde{\xi}^i$ denotes the *i*th scenario, and p^i denotes the probability of scenario *i*. Finally, the first and second-stage problems (13)-(19) can be summed in a single large LP model, which is also called in the literature the "deterministic equivalent model". This model is presented as follows.

First-stage model

$$\text{Minimize } Z = \sum_{c\in C}\sum_{t=1}^{T}\sum_{a\in A} m_{ct}\phi_{ac}X_{at} + \sum_{i=1}^{n} p^i Q(X_{at}, \tilde{\xi}^i) \tag{20}$$

Subject to

Constraints (14)-(16).

Second-stage model

Model (17)-(19).

Furthermore, this model results in model (21)-(25) as follows.

$$\text{Minimize } Z = \sum_{c\in C}\sum_{t=1}^{T}\sum_{a\in A} m_{ct}\phi_{ac}X_{at} + \sum_{i=1}^{n}\sum_{p\in P}\sum_{t=1}^{T} p^i[i_{pt}I^i_{pt} + b_{pt}B^i_{pt}] \tag{21}$$

Subject to

$$I_{ct} = I_{ct-1} + s_{ct} - \sum_{a \in A} \phi_{ac} X_{at}, \qquad t = 1,\dots,T;\ c \in \mathrm{C}, \tag{22}$$

$$\sum_{a \in \mathrm{A}} \delta_{ar} X_{at} \le M_{rt}, \qquad t = 1,2,\dots,T;\ r \in \mathrm{R}, \tag{23}$$

$$I^i_{p1} - B^i_{p1} = I_{p0} + \sum_{a \in \mathrm{A}} \rho_{ap}(\underset{\sim}{\xi}^i) X_{a1} - d_{p1},$$

$$I^i_{pt} - B^i_{pt} = I^i_{pt-1} - B^i_{pt-1} + \sum_{a \in \mathrm{A}} \rho_{ap}(\underset{\sim}{\xi}^i) X_{at} - d_{pt},\ t = 2,\dots,T;\ p \in \mathrm{P};\ i = 1,\dots,n, \tag{24}$$

$$X_{at} \ge 0,\ I_{ct} \ge 0,\ I^i_{pt} \ge 0,\ B^i_{pt} \ge 0, \qquad c \in C;\ p \in P;\ t = 1,\dots,T;\ a \in A;\ i = 1,\dots,n. \tag{25}$$

where, I^i_{pt} and B^i_{pt} denote the inventory and backorder levels of product *p* in period *t* under scenario *i*, respectively. In this model there are $|\mathrm{A}| \times T$ first-stage decisions, whereas there are $2 \times |\mathrm{P}| \times T \times n$ second-stage decisions. Where $|\mathrm{A}|$ and $|\mathrm{P}|$ denote the sizes of process and product sets, respectively. The first-stage decisions X_{at} cannot anticipate the yield scenarios and must be feasible for all of the scenarios and should have the minimum expected inventory/backorder costs.

It is evident that the LP model (21)-(25) can be solved by the linear programming solvers. However, in the case of a huge number of scenarios, solving this model would be far beyond the present computational capacities. In such situations, it is not practical to solve the two-stage model or its deterministic equivalent, directly. In fact, it is not also possible to evaluate the function $\sum_{i=1}^{n} p^i Q(X_{at}, \underset{\sim}{\xi}^i)$, for a given X_{at}, in a reasonable amount of time, since the number of second-stage linear programs (17)-(19) to be solved is just too large.

We can however use Monte Carlo sampling techniques, which consider only randomly selected subsets of the set $\left\{\underset{\sim}{\xi}^1, \underset{\sim}{\xi}^2, \dots, \underset{\sim}{\xi}^n\right\}$ to obtain approximate solutions. Monte Carlo solution procedures for solving stochastic programs can use "internal sampling" or "external sampling". The "internal sampling" procedures include sampling-based cutting plane methods (e.g. Higle and Sen, 1996) and stochastic quasi-gradient algorithms (e.g. Ermoliev, 1998). In the "external sampling" procedures, sampling is performed external to (prior to) the solution procedure. The sample average approximation (SAA)

scheme (cf. Shapiro et al., 1998; Mak et al, 1999; Shapiro et al., 2000) which is selected as the solution approach in this work is an "external sampling" procedure.

5.2. Sample average approximation (SAA) scheme

In the SAA scheme, a random sample of *n* realizations (scenarios) of the random vector $\underset{\sim}{\xi}$ is generated and the expectation $E_{\underset{\sim}{\xi}}[Q(X_{at}, \underset{\sim}{\xi})]$ is approximated by the sample average function $\frac{1}{n}\sum_{i=1}^{n} Q(X_{at}, \underset{\sim}{\xi}^i)$. In other words, the "true" problem (20) is approximated by the sample average approximation (SAA) problem (26).

First-stage model

$$\text{Minimize } \hat{Z} = \sum_{c\in C}\sum_{t=1}^{T}\sum_{a\in A} m_{ct}\phi_{ac}X_{at} + \frac{1}{n}\sum_{i=1}^{n} Q(X_{at}, \underset{\sim}{\xi}^i) \qquad (26)$$

Subject to

Constraints (14)-(16).

Second-stage model

Model (17)-(19).

It can be shown that under mild regularity conditions, as the sample size *n* increases, the optimal solution vector $\hat{X}_n$ and optimal value $\hat{Z}_n$ of the SAA problem (26) converge with probability one to their true counterparts, and moreover $\hat{X}_n$ converges to an optimal solution of the true problem with probability approaching one exponentially fast (Shapiro and Hommem-de-Mello., 1998 and 2000). This convergence analysis suggests that a fairly good approximate solution to the true problem (20) can be obtained by solving an SAA problem (26) with a modest sample size. The mentioned regularity conditions include: 1) the objective function of the stochastic model has finite mean and variance, 2) the independent identically distributed (i.i.d.) observations of vector $\underset{\sim}{\xi}$ can be generated, 3) instances of SAA problem can be solved for sufficiently large *n* to result "good" bounding information, and 4) the objective function of the stochastic model can be evaluated exactly for specific values of X_{at} and realizations of vector $\underset{\sim}{\xi}$. These regularity conditions are satisfied for our problem, especially regarding the discrete distribution of random yields.

In practice, the SAA scheme involves repeated solutions of the SAA problem (26) with independent samples. Statistical confidence intervals are then derived on the quality of the approximate solutions (Mak et al., 1999). According to the work of Mak et al. (1999), an obvious approach to test solution quality for a candidate solution ($\bar{X}$) is to bound the optimality gap, defined as $E_{\underset{\sim}{\xi}}[f(\bar{X},\underset{\sim}{\xi})]-z^*$ using standard statistical procedures, where $f(\bar{X},\underset{\sim}{\xi})$ and z^* are the true objective value for $\bar{X}$ and the true optimal solution to the problem (20), respectively, and $E_{\underset{\sim}{\xi}}[f(\bar{X},\underset{\sim}{\xi})]$ is the expected value of $f(\bar{X},\underset{\sim}{\xi})$. In our work a sampling procedure based on common random numbers (CRN) is used to construct the optimality gap confidence interval which provides significance variance reduction over naive sampling as shown in (Mak et al., 1999). This approach is described next.

The SAA algorithm (with common random number streams)

Step 1- Generate n_g independent identically distributed (i.i.d.) batches of samples each of size *n* from the distribution of $\underset{\sim}{\xi}$, i.e., $\left\{\underset{\sim}{\xi}_j^{\,1},\underset{\sim}{\xi}_j^{\,2},\ldots,\underset{\sim}{\xi}_j^{\,n}\right\}$ for $j=1,\ldots,n_g$. For each sample solve the corresponding SAA problem (26). Let $\hat{Z}_n^j$ and $\hat{X}_n^j$, $j=1,\ldots,n_g$, be the corresponding optimal objective value and an optimal solution, respectively.

Step 2- Compute

$$\bar{Z}_{n,n_g}=\frac{1}{n_g}\sum_{j=1}^{n_g}\hat{Z}_n^j, \text{ and} \tag{27}$$

$$s^2_{\bar{Z}_{n,n_g}}=\frac{1}{n_g(n_g-1)}\sum_{j=1}^{n_g}(\hat{Z}_{n_g}^j-\bar{Z}_{n,n_g})^2. \tag{28}$$

It is well known that the expected value of $\hat{Z}_n$ is less than or equal to the optimal value z^* of the true problem (see e.g., Mak et al., 1999). Since $\bar{Z}_{n,n_g}$ is an unbiased estimator of $E[\hat{Z}_n]$, we obtain that $E[\bar{Z}_{n,n_g}]\le z^*$. Thus $\bar{Z}_{n,n_g}$ provides a lower statistical bound for the optimal value z^* of the true problem and $s^2_{\bar{Z}_{n,n_g}}$ is an estimate of the variance of this estimator.

Step 3- Choose a candidate feasible solution $\bar{X}$ of the true problem, for example, a computed $\hat{X}^{j}_{n'}$ by using a sample size (n') larger than used for lower bound estimation (n). Estimate the true objective function value $f(\bar{X})$ for all batches of samples ($j = 1, \ldots, n_g$) as follows.

$$\tilde{f}_n^j(\bar{X}) = \sum_{c\in C}\sum_{t=1}^{T}\sum_{a\in A} m_{ct}\phi_{ac}\bar{X} + \frac{1}{n}\sum_{i=1}^{n} Q(\bar{X}, \underset{\sim}{\xi}^i_j). \tag{29}$$

Step 4- Compute the observations of the optimality gap G_n^j for the candidate solution $\bar{X}$ for all $j = 1, \ldots, n_g$ as follows.

$$G_n^j = \tilde{f}_n^j(\bar{X}) - \hat{Z}_n^j. \tag{30}$$

It has been shown in (Mak et al., 1999) that:

$$\underbrace{E\left[\tilde{f}_n(\bar{X}) - \hat{Z}_n\right]}_{G_n} \geq E_{\underset{\sim}{\xi}}[f(\bar{X}, \underset{\sim}{\xi})] - z^*. \tag{31}$$

where, $f(\bar{X}, \underset{\sim}{\xi})$ and z^* are the true objective value for $\bar{X}$ and the true optimal solution to the problem (20), respectively, and ($E_{\underset{\sim}{\xi}}[f(\bar{X}, \underset{\sim}{\xi})] - z^*$) is the true optimality gap for the candidate solution $\bar{X}$. We also have:

$$\sqrt{n_g}\left[\bar{G}_{n_g} - EG_n\right] \Rightarrow \mathrm{N}(0, \sigma_g^2) \quad \text{as } n_g \to \infty$$

where $\sigma_g^2 = \mathrm{var}\, G_n$.

Step 5- Compute the sample mean and sample variance for the optimality gap G_n^j as follows.

$$\bar{G}_{n_g} = \frac{1}{n_g}\sum_{j=1}^{n_g} G_n^j, \text{ and} \tag{32}$$

$$s^2_{G_n^j} = \frac{1}{n_g(n_g-1)}\sum_{j=1}^{n_g}(G_n^j - \bar{G}_{n_g})^2. \tag{33}$$

Step 6- Compute the approximate $(1-\alpha)$-level confidence interval for the optimality gap of $\bar{X}$ as $\left[0, \bar{G}_{n_g} + \tilde{\varepsilon}_g\right]$, where $\tilde{\varepsilon}_g = \frac{t_{n_g-1,\alpha} s_{G_n^j}}{\sqrt{n_g}}$.

For solving the SAA problem (26) for each of the n_g batches of n randomly sampled scenarios, we propose either to solve directly its deterministic equivalent, or in the cases where the number of scenarios is very large and the deterministic equivalent model cannot be solved in a reasonable amount of time, to implement the regularized decomposition method (Ruszczyński and Świetanowski, 1996) with l_∞ trust region as it is used in (Linderoth and Wright, 2003; Linderoth and Shapiro, 2006).

6. Application to sawmill production planning

In this section, the proposed approach for MPMP production planning by considering random yields is implemented for a sawmill production planning as a special application. In sawmills, logs are classified based on some attributes namely: diameter class, species, length, taper, etc. Logs are broken down into different pieces of green lumbers (products) by means of different cutting patterns. We define a production process in a sawing unit as a combination of a log class and a cutting pattern. It should be mentioned that, possible combinations of log classes and cutting patterns can produce simultaneously different mix of lumbers. However, due to non homogeneity in quality of logs, each cutting pattern yields a random quantity of corresponding products after processing a known quantity of each log class. In the production line, whenever a log from a special class enters into a cutting pattern, after some preliminary activities, it passes though an X-ray scanner. The result of the scanning is transferred to a log sawing optimizer which decides about the optimal mix of lumbers with their quantity that should be yielded by that cutting pattern. The objective of the optimizer is to maximize value/volume of the yielded products for each log. Figure 2 is a schematic illustration of sawing process in a sawmill.

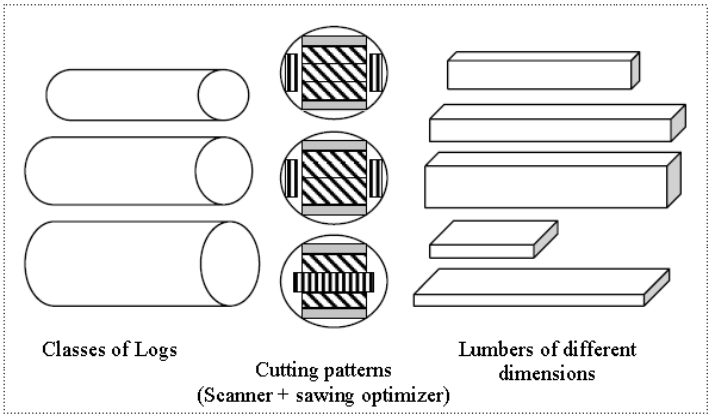

Figure 2. Sawing process in a sawmill

Production planning in a sawing unit is to decide about the optimal quantity of log consumption from different classes and selection of corresponding cutting patterns in each period of the planning horizon, in order to fulfill customer demand regarding machine capacities and logs inventory. The objective is to minimize log consumption, as well as products inventory/backorder costs.

Two different approaches have been already proposed in the literature to address sawmill production planning. In the first approach, the randomness of the process yields is ignored and their expected value is considered in a MPMP linear programming model (Gaudreault et al., 2004). However, the production plans issued by these models result usually in extra inventory of products with lower quality and price while backorder of products with higher quality and price. The second approach is focused on combined optimization type solutions linked to real-time simulation sub systems (Maness and Norton, 2002; Maness and Adams, 1991; Mendoza et al., 1991). In this approach, the stochastic characteristic of logs is taken into account by assuming that all the input logs are scanned through an X-ray scanner before planning. Maness and Norton, (2002) have developed an integrated multi-period production planning model which is the combination of an LP model and a log sawing optimizer (simulator). The LP model acts as a coordinating model that allocates limited resources. A series of dynamic programming sub-problems, titled in the literature as "log sawing optimization models" are used to generate activities (columns) for the coordinating LP based on the products' shadow prices. The log sawing optimization model is a sawing algorithm for lumber grade, based on data collected from the X-ray scanner. Although the stochastic characteristics of logs are considered in the second approach, they include the following drawbacks: logs, needed for the next planning horizon, are not always available in a sawmill to be scanned before planning. Furthermore, to implement this method, the logs should be processed in production line in the same order they have been simulated, which is not an easy practice.

In the following, we first explain how the scenarios are generated for random yields in a sawmill; afterwards, we provide the computational results of using the proposed approach to solve a real sawmill production planning problem.

6.1. Scenario generation for sawmills

The scenarios and their probability distribution for individual processes in a sawmill can be determined by the approach proposed in section 4. However, the implementation of this approach is very difficult in sawmills. In fact, the high production speed in the sawing unit makes it almost impossible to track the logs through the line and to observe the result of sawing individual logs. As a more feasible alternative for scenario definition for process yields in sawmills, we propose to use the historical data on the yields that are proposed by the sawing optimizer in the production line. In other words, we should take a sample of these data for each process (e.g. 300 samples), then by statistical analysis on this sample, different scenarios with their corresponding probabilities (as their proportion in the population) should be determined.

In the real sawmill example where we implemented the stochastic programming approach, regarding the fact that the SAA method is selected to solve the stochastic model, we used a log sawing simulator named as “Optitek” (Forintek Canada Corp.) to generate randomly different batches of samples for random yields. “Optitek” has been developed to simulate the sawing process in Quebec sawmills. The inputs to this simulator consist of log class, cutting pattern, and the number of logs to be processed. The simulator considers the logs in the requested class with random physical and internal characteristics and based on sawing rules which are similar to those of a sawmill, generates different yields for each log. Afterwards, the yields of each log can be considered as a scenario for the yields of corresponding process. Finally, the combinations of such scenarios for all processes construct the global scenarios for the stochastic model.

6.2. Computational results

In this section, we describe the numerical experiments using the proposed approach to solve a real sawmill production planning problem. We first describe the characteristics of the test industrial problem and some implementation details, and then comment on the quality of the stochastic model solution in comparison to those obtained using the mean-value deterministic model.

6.2.1. Data and implementation

Our test problem is that of the production planning for a sawing unit in a sawmill in Quebec (Canada) where 3 classes of logs with 10 feet length can be processed by 5 cutting patterns for producing 27 products (lumbers with different dimensions). Therefore, we have 15 processes all can produce 27 products with random yields. All the processes are run on two machines: Trimmer and Bull. The planning horizon consists of 30 periods (days). Products demands in each period are supposed to be deterministic and known parameters, which are calculated based on the received and forecasted orders. The number of scenarios for random yields in this example can be estimated as $5^{405} \approx 1.2 \times 10^{283}$!

Recall from section 5 that the SAA method calls for the solution of n_g instances of the approximating stochastic program (26), each involving n sampled scenarios. Statistical validation of a candidate solution is then carried out by evaluating the objective function value using the same n sampled scenarios in each batch. In our implementation, we used n=60, 100, and 150; and $n_g = 30$. Our candidate solutions are computed by solving the SAA problem (26) with $n' = 100, 150,$ and 250. To illustrate the complexity of solving (26) within the SAA scheme, we present the sizes of the deterministic equivalents of the SAA problems corresponding to the different values of n' in Table 2.

Table 2. Size of the deterministic equivalent of the SAA problem

n'	Constraints	Variables
1	960	2160
100	81150	162540
150	121650	243540
250	202650	405540

The SAA scheme was implemented in OPL Studio 3.7.1. The OPL Script is used for solving the deterministic equivalents for different instances of SAA problems as well as for calculating the true objective function value for the candidate solutions. All computations were carried out on a Pentium(R) IV 1.8 GHz PC with 512 MB RAM running Windows XP.

6.2.2. Quality of stochastic solutions

In this section we first present the results of applying the SAA scheme for our test problem as well the evaluation of quality of several candidate solutions; afterwards, we compare the solution of the

stochastic programming model to that of the mean-value model involving the expected values of the uncertain yields. The point estimates of the lower statistical bound for the optimal value of the stochastic problem ((27)-(28)) are reported in table 3. They are computed based on 30 batches of sampled scenarios with 3 different batch sizes. Table 4 displays the quality of 3 candidate solutions and contains the 95% confidence intervals on their optimality gaps based on CRN method (see section 5). The candidate solutions $\bar{X}^{100}, \bar{X}^{150}, \bar{X}^{250}$ for the RCN strategy are computed by solving the approximating problem (26) that includes 100, 150 and 250 scenarios. The CPU times for computing each candidate solution are also reported in table 4.

Table 3. Lower bound estimation results for the optimal value (n_g = 30 batches)

Batch size (n)	60	100	150
Average ($\bar{Z}_{n,n_g}$)	515829	527981	519226
SD ($s_{\bar{Z}_{n,n_g}}$)	35582	25562	22590

As it can be observed from Table 4, by increasing the sample size, the quality of approximate solutions improves monotonically and the tighter confidence intervals for the optimality gaps of candidate solutions are constructed.

Table 4. Optimality gaps for candidate solutions

Candidate solution	$\bar{X}^{100}$	$\bar{X}^{150}$	$\bar{X}^{250}$
Batch size (n)	60	100	150
No. of batches (n_g)	30	30	30
Point estimate ($\bar{G}_{n_g}$)	13253	9284	4783
Error estimate ($\alpha = 95\%$) ($\tilde{\varepsilon}_g$)	1555	1268	393
Confidence interval (95%)	[0,14808]	[0,10552]	[0, 5176]
CPU time (sec.)	45	80	198

To compare the candidate stochastic model solution to the mean-value model solution, we calculated the value of the stochastic solution (VSS) (Birge, 1997) for three candidate solutions. The VSS indicates the difference between the expected cost of the mean-value model solution and the stochastic model one and is computed as follows.

Step 1- Solve the deterministic problem (mean-value problem) (8)-(12) by considering the expected value of process yields and find the optimal solution X^{MVP} .

Step 2 – Compute the real objective function value (the expected cost) for X^{MVP} ($\tilde{f}_n(X^{MVP})$) by (29) (see section 5).

Step 3- The value of the stochastic solution (VSS) for each candidate solution ($\bar{X}$) is calculated by:

VSS= $\tilde{f}_n(X^{MVP})$ - $\tilde{f}_n(\bar{X})$,

where, $\tilde{f}_n(\bar{X})$ is the objective value of the SAA problem for the solution $\bar{X}$.

In order to compute the VSS for the three mentioned candidate solutions, we selected one batch (among all 30 batches of samples) that has been used to construct the optimality gap for those solutions. Thus, the value of '*n*' for calculating the VSS for each solution is the one used for constructing the optimality gap for that solution. The comparison between three candidate solutions $\bar{X}^{100}$, $\bar{X}^{150}$, $\bar{X}^{250}$ and X^{MVP} is reported in Table 5.

Table 5. Comparison of the solutions of the stochastic model and mean-value deterministic model

Solution	X^{MVP}			$\bar{X}^{100}$	$\bar{X}^{150}$	$\bar{X}^{250}$
	n=60	*n*=100	*n*=150	(*n*=60)	(*n*=100)	(*n*=150)
Objective function value ($\tilde{f}_n(X)$)	1735702	17135702	1704186	509108	504536	502162
VSS				1226594	1215266	1202024

It is clear that the estimated total average cost for all three candidate stochastic programming solutions are significantly smaller than that of the mean-value problem solution which reveals that the stochastic model is a more reliable production planning tool in the presence of random yields. Finally we can conclude that, by considering a moderate number of scenarios (250 scenarios) among the potential enormous number of scenarios we obtain an approximate solution in a very short time with an optimality gap of [0, 5176] which is less than 1% of the lower bound of the real optimal value (see Tables 3 and 4). Thus, this solution can be accepted as a relatively good approximation to the optimal solution especially regarding the higher expected cost of mean-value problem solution compared to those of stochastic model solution (see table 5).

7. Conclusions

In this paper, we developed a two-stage stochastic programming model for MPMP production planning under the uncertainty of process yields. The SAA method was implemented to solve the stochastic model which provided us an efficient framework for identifying and statistically testing a variety of

candidate production plans. The proposed approach was applied for sawmill production planning by considering random characteristics of logs. We provided the empirical results for production planning in a real sawmill and we identified several candidate plans in a short time by solving the approximate SAA problem. Furthermore, the confidence intervals for the optimality gap of candidate solutions were constructed by common random number (CRN) streams. Our results reveal that the production plans resulted by the stochastic model are superior to those obtained by traditional mean-value deterministic model. Although these results are found for sawmill production planning, the proposed approach in this work can be applied for production planning in other manufacturing environments where non-homogeneous and random characteristics of raw materials result in random yield. Future research will consider in the stochastic model the decision maker's risk preferences towards the cost of different scenarios in addition to their expected cost. Furthermore, by considering also the products demands as random variables, more realistic production plans can be obtained.

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