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December 2009

CIRRELT-2009-56

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Abstract. We want to find a location of capacitated transshipment facilities which minimizes the expected total cost when the generalized transportation costs are random variables. Each generalized transportation cost is given by the sum of a deterministic transportation cost and a random transshipment facility throughput cost, whose probability distribution is unknown. A two-stage stochastic program with recourse for this problem and its deterministic nonlinear approximation are already available in literature. Unfortunately, both of them are unable to solve real-life instances in a reasonable computing time. In this paper a heuristic for solving the deterministic nonlinear approximation, which combines a Lagrangian-based method for the flows calculation with a Open&Close procedure for the facility location, is given. Computational results show a high performance of this heuristic.

Keywords.Transshipment location, random costs, Lagrangian heuristic.

Acknowledgements. This project has been partially supported by the Italian Ministry of University and Research, under the 2007 PRIN project "Optimization of Distribution Logistics".

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Dépôt légal – Bibliothèque nationale du Québec,
Bibliothèque nationale du Canada, 2009

1 Introduction

Given a set of origins with given supply, a set of destinations with given demand, a set of potential transshipment locations with deterministic fixed cost of location, a throughput capacity of the transshipment facilities, and random generalized transportation costs from origins to destinations through transshipment facilities, the Capacitated Transshipment Location Problem under Uncertainty consists in finding a transshipment location which minimizes the total cost, given by the sum of the total fixed cost plus the expected minimum total flow cost, subject to supply, demand, and facility capacity constraints.
Each generalized transportation cost is a random variable given by the sum of a deterministic transportation cost from an origin to a destination through a transshipment facility plus a random term, with unknown probability distribution, which represents the throughput cost of the transshipment facility.
The literature on location with stochastic costs is quite limited. Ricciardi et al. [7] consider a p-median problem where the throughput costs are random variables with given probability distribution, and develop a deterministic approximation model which is then solved heuristically. Daskin et al. [4] introduce a location-inventory model that minimizes the expected cost of locating facilities, transporting material, and holding inventory under stochastic daily demand. The stochastic model is then approximated by a deterministic model whith the objective to minimize mean and variance of the random parameters. Snyder et al. [8] consider a scenario-based stochastic version of the joint location-inventory model of [4], allowing mean and variance of the demand to be stochastic, as well as costs, lead times, and some other parameters. In Yanga et al. [11] the authors investigate the location problem of the logistics distribution centers under the condition that setup costs, turnover costs and customer demands are fuzzy variables. In Tadei et al. [10] the authors consider a stochastic p-median problem where the cost for using a facility is a stochastic variable with unknown probability distribution.
In Tadei et al. [9] the Capacitated Transshipment Location Problem under Uncertainty ($CTLPu$), addressed in this paper, is introduced. There the authors model the problem as a two-stage stochastic program with recourse and give a deterministic mixed-integer nonlinear approximation of it, named $CTLP_d$. Unfortunately, both the stochastic model and its approximation cannot solve real-life instances in a reasonable computing time. Then, an efficient heuristic for solving this kind of instances becomes necessary.
In this paper we address such an efficient heuristic. It is based on three interactive procedures. The first one, given a facility location, is a Lagrangian-based procedure for calculating optimal expected flows from origins to destinations through the given capacitated transshipment facilities. The second and the third ones are a opening and a closing procedure, respectively, for improving the given transshipment facility location. They both call the previous Lagrangian-based procedure for the optimal expected flows as a subroutine.
It will be shown that the overall heuristic is able to solve real-life instances in a reasonable computing time with a mean gap less than 0.2% with respect to the optimum of $CTLP_d$.

The remainder of the paper is organized as follows. In Sections 2 and 3 the Capacitated Transshipment Location Problem under Uncertainty as a two-stage stochastic programming model with recourse and its deterministic mixed-integer nonlinear approximation given in [9] are, respectively, reminded. In Section 4 a Lagrangian-based procedure for calculating the optimal expected flows is presented. In Sections 5 and 6 the criteria for opening and closing facilities, in order to improve the current optimum are, respectively, given. In Section 7 the overall heuristic is presented. In Section 8 the performance of the proposed heuristic is computationally tested. Finally, the conclusions of our work are reported in Section 9.

2 The Capacitated Transshipment Location Problem under Uncertainty

In [9] the Capacitated Transshipment Location Problem under Uncertainty ($CTLPu$) is introduced as follows.

Let be

- I: set of origins
- J: set of destinations
- K: set of potential transshipment locations
- L_k: set of throughput operation scenarios at transshipment facility $k \in K$
- P_i: supply at origin $i \in I$
- Q_j: demand at destination $j \in J$
- U_k: throughput capacity of transshipment facility $k \in K$
- f_k: fixed cost of locating a transshipment facility $k \in K$
- y_k: binary variable which takes value 1 if transshipment facility $k \in K$ is located, 0 otherwise
- c_{ij}^k: unit transportation cost from origin $i \in I$ to destination $j \in J$ through transshipment facility $k \in K$
- θ_{kl}: unit throughput cost of transshipment facility $k \in K$ in throughput operation scenario $l \in L_k$
- s_{ij}^k: flow from origin $i \in I$ to destination $j \in J$ through transshipment facility $k \in K$.

Let us assume

i) the system is balanced, i.e. $\sum_{i\in I} P_i = \sum_{j\in J} Q_j = T$

ii) the unit throughput costs $\{\theta_{kl}\}$ are independent and identically distributed (i.i.d.) random variables with a common and unknown probability distribution (for details see [9])

$$Pr\{\theta_{kl} \geq x\} = F(x). \tag{1}$$

Let $r_{ij}^{kl}(\theta)$ be the stochastic generalized unit transportation cost from origin i to destination j through transshipment facility k in throughput operation scenario l given by

$$r_{ij}^{kl}(\theta) = c_{ij}^k + \theta_{kl}, \quad i \in I, j \in J, k \in K, l \in L_k \tag{2}$$

with unknown probability distribution

$$\Pr\left\{r_{ij}^{kl}(\theta) \geq x\right\} = \Pr\left\{c_{ij}^k + \theta_{kl} \geq x\right\} = \Pr\left\{\theta_{kl} \geq x - c_{ij}^k\right\} = F(x - c_{ij}^k). \tag{3}$$

Let us define

$$\overline{\theta}_k = \min_{l \in L_k} \theta_{kl}, \quad k \in K \tag{4}$$

with unknown probability distribution

$$H(x) = \Pr\left\{\overline{\theta}_k \geq x\right\} \tag{5}$$

The stochastic generalized unit transportation cost from origin i to destination j through transshipment facility k is the minimum among the costs for the different throughput operation scenarios at facility k and becomes

$$\overline{r}_{ij}^k(\theta) = \min_{l \in L_k} r_{ij}^{kl}(\theta) = c_{ij}^k + \min_{l \in L_k} \theta_{kl} = c_{ij}^k + \overline{\theta}_k, \quad i \in I, j \in J, k \in K. \tag{6}$$

In order to write $CTLPu$ as a two-stage stochastic program with fixed recourse [1], the authors consider the variables $\{y_k\}$, $\{s_{ij}^k\}$ as the first-stage decision variables and introduce the second-stage decision variables $\{x_{ij}^k(\theta)\}$, such that $x_{ij}^k(\theta) = s_{ij}^k, \ \forall \theta_{kl}, \ k \in K, l \in L_k$.

The $CTLPu$ is then formulated as follows

$$\min_y \sum_{k \in K} f_k y_k + \mathbb{E}_\theta \left[\min_s \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \overline{r}_{ij}^k(\theta) s_{ij}^k \right] \tag{7}$$

subject to

$$\sum_{j \in J} \sum_{k \in K} s_{ij}^k = P_i, \quad i \in I \tag{8}$$

$$\sum_{i\in I}\sum_{k\in K} s_{ij}^k = Q_j, \quad j \in J \tag{9}$$

$$\sum_{i\in I}\sum_{j\in J} s_{ij}^k \leq U_k y_k, \quad k \in K \tag{10}$$

$$x_{ij}^k(\theta) = s_{ij}^k, \ \forall \theta_{kl}, \ i \in I, j \in J, k \in K, l \in L_k \tag{11}$$

$$x_{ij}^k(\theta) \geq 0, \ \forall \theta_{kl}, \ i \in I, j \in J, k \in K, l \in L_k \tag{12}$$

$$s_{ij}^k \geq 0, \quad i \in I, j \in J, k \in K \tag{13}$$

$$y_k \in \{0, 1\}, \quad k \in K \tag{14}$$

where $\mathbb{E}_\theta$ denotes the expected value with respect to θ; the objective function (7) expresses the minimization of the total cost given by the sum of the minimum total fixed cost plus the expected minimum total flow cost; constraints (8) and (9) ensure that supply at each origin i and demand at each destination j are satisfied; constraints (10) ensure the capacity restriction at each transshipment facility k; constraints (11) tie the first-stage decision variables s_{ij}^k (revealed flows) to the second-stage decision variables $x_{ij}^k(\theta)$ for any occurrence of the random variables θ_{kl}; (12), (13) are the non-negativity constraints, and (14) are the integrality constraints.

3 A deterministic nonlinear approximation of the Capacitated Transshipment Location Problem under Uncertainty

In [9] the authors show that $CTLPu$ can be approximated, using the asymptotic approximation method derived from the extreme value theory [5], by a deterministic nonlinear model, called $CTLP_d$. Given a positive parameter β, $CTLP_d$ is formulated as follows

$$\max_{y,s}\left[-\frac{1}{\beta}\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} s_{ij}^k \ln s_{ij}^k - \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} s_{ij}^k (c_{ij}^k - \frac{1}{\beta}) - \sum_{k\in K} f_k y_k\right] \tag{15}$$

subject to

$$\sum_{j\in J}\sum_{k\in K} s_{ij}^k = P_i, \quad i \in I \tag{16}$$

$$\sum_{i\in I}\sum_{k\in K} s_{ij}^k = Q_j, \quad j \in J \tag{17}$$

$$\sum_{i\in I}\sum_{j\in J} s_{ij}^k \leq U_k y_k, \quad k \in K \tag{18}$$

$$s_{ij}^k \geq 0, \quad i \in I, j \in J, k \in K \tag{19}$$

$$y_k \in \{0,1\}, \quad k \in K \tag{20}$$

The authors also observe that the nonlinearity affects only the objective function through the "Entropy" term
$-\frac{1}{\beta}\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} s_{ij}^k \ln s_{ij}^k$, while all constraints are linear.

As far as the performance of $CTLP_d$ is concerned, the authors show that the mean gap between its optimum and that of $CTLPu$ is around 2%, and small size instances (up to 3 origins, 40 destinations, and 20 potential transshipment locations) run in a computing time of 1.000 seconds.

We tried to run larger real-life instances (e.g., up to 5 origins, 100 destinations, and 50 potential transshipment locations) but, even by using the best state-of-the-art solvers, the necessary computing time turns to be a huge amount of hours. Then, some efficient heuristics for solving instances of that, or even larger, size become necessary. One of these heuristics is addressed in Section 7. The proposed heuristic, given a transshipment location, calculates the optimal expected flows of $CTLP_d$, then it updates the given location by opening and closing transshipment facilities until a local optimum is found.

4 Finding the optimal expected flows

Let us assume a given transshipment location $\{y_k\}$. Then $CTLP_d$ is now in the unknowns $\{s_{ij}^k\}$ only and, neglecting the constant $F = -\sum_{k\in K} f_k y_k$ in the objective function (15), it becomes

$$\max_s \left[-\frac{1}{\beta}\sum_{i\in I}\sum_{j\in J}\sum_{k\in K/y_k=1} s_{ij}^k \ln s_{ij}^k - \sum_{i\in I}\sum_{j\in J}\sum_{k\in K/y_k=1} s_{ij}^k (c_{ij}^k - \frac{1}{\beta})\right] \tag{21}$$

subject to

$$\sum_{j\in J}\sum_{k\in K/y_k=1} s_{ij}^k = P_i, \quad i \in I \tag{22}$$

$$\sum_{i\in I}\sum_{k\in K/y_k=1} s_{ij}^k = Q_j, \quad j \in J \tag{23}$$

$$\sum_{i\in I}\sum_{j\in J} s_{ij}^k \leq U_k y_k, \quad k \in K \tag{24}$$

$$s_{ij}^k \geq 0, \quad i \in I, j \in J, k \in K \tag{25}$$

4.1 Lagrangian relaxation

Let us consider (21), multiplied by β, and the Lagrangian multipliers $\nu_i, i \in I$, $\mu_j, j \in J$, and $\lambda_k \geq 0, k \in K$, associated to constraints (22), (23), and (24), respectively.
By building the Lagrangian of problem (21)-(25) one gets

$$\begin{aligned} L_1 = - \quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K/y_k=1} s_{ij}^k \ln s_{ij}^k - \beta \sum_{i\in I}\sum_{j\in J}\sum_{k\in K/y_k=1} s_{ij}^k (c_{ij}^k - \frac{1}{\beta}) + \sum_{i\in I} \nu_i (P_i - \sum_{j\in J}\sum_{k\in K/y_k=1} s_{ij}^k) + \\ & \sum_{j\in J} \mu_j (Q_j - \sum_{i\in I}\sum_{k\in K/y_k=1} s_{ij}^k) + \sum_{k\in K} \lambda_k (U_k y_k - \sum_{i\in I}\sum_{j\in J} s_{ij}^k) \end{aligned} \tag{26}$$

Then by imposing the necessary first order conditions for the expected flows $\{s_{ij}^k\}$ one gets

$$\partial L_1 / \partial s_{ij}^k = -\ln s_{ij}^k - \beta c_{ij}^k - \nu_i - \mu_j - \lambda_k = 0 \tag{27}$$

and the optimal expected flows become

$$s_{ij}^k(\nu, \mu, \lambda) = e^{-\nu_i} e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k} \tag{28}$$

By substituting (28) into (22)

$$e^{-\nu_i} = P_i / \sum_{j\in J}\sum_{k\in K/y_k=1} e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k} = P_i / \sum_{j\in J}\sum_{k\in K} y_k e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k}, \quad i \in I \tag{29}$$

and (28) becomes

$$s_{ij}^k(\mu, \lambda) = P_i \frac{e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k}}{\sum_{j\in J}\sum_{k\in K} y_k e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k}}, \quad i \in I, j \in J, k \in K \tag{30}$$

It is interested to observe that, although eq. (30) has been derived for open facilities, it holds for all k, provided the possibility to calculate the multipliers $\{\lambda_k\}$ also for closed facilities. Then, for an open transshipment facility k, s_{ij}^k represents the actual flow from i to j through that open facility. Vice versa, for a closed transshipment facility k, it represents the "potential" flow from i to j through that closed facility.

In order to calculate the optimal expected flows s_{ij}^k (both actual and potential) by (30), we must first calculate the value of the Lagrangian multipliers $\{\mu_j\}$ and $\{\lambda_k\}$.

A procedure to do that is given in the next section.

4.2 Lagrangian multipliers calculation

We remind we assumed that a transshipment location $\{y_k\}$ is already given. We calculate the Lagrangian multipliers $\{\mu_j\}$ and $\{\lambda_k\}$ by an efficient iterative method as follows.

Let us start with $\lambda_k = 0, k \in K$.

Calculate $\mu_j, j \in J$ such that the demand satisfaction constraints (23), where s_{ij}^k are given by (30), are satisfied.
To do that solve iteratively in $e^{-\mu_j}$ the following system of equations, starting with any value for $\{e^{-\mu_j}\}$ (e.g., $e^{-\mu_j} = 1, j \in J$)

$$e^{-\mu_j} = Q_j / \sum_{i \in I} \sum_{k \in K / y_k = 1} P_i \frac{e^{-\lambda_k} e^{-\beta c_{ij}^k}}{\sum_{j \in J} \sum_{k \in K} y_k e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k}}, \quad j \in J \tag{31}$$

Once $\{e^{-\mu_j}\}$ are calculated, the multipliers $\{\lambda_k\}$ are updated. Note that these multipliers can be calculated also for closed facilities.

Let $D_k(\lambda)$ be the throughput of facility k

$$D_k(\lambda) = \sum_{i \in I} \sum_{j \in J} s_{ij}^k(\lambda), \quad k \in K \tag{32}$$

$\{s_{ij}^k\}$ are expressed only in the unknowns $\{\lambda_k\}$ because $\{\mu_j\}$ are already known.

Because of $\{s_{ij}^k\}$, also $\{D_k\}$ are defined for all k, i.e. for both open and closed facilities. When k is open, D_k represents its actual throughput, while it represents its "potential" throughput, when k is closed.

By (30), eq. (32) becomes

$$D_k(\lambda) = e^{-\lambda_k} \sum_{i \in I} P_i \frac{\sum_{j \in J} e^{-\mu_j} e^{-\beta c_{ij}^k}}{\sum_{j \in J} \sum_{k \in K} y_k e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k}} = e^{-\lambda_k} \rho_k, \quad k \in K \tag{33}$$

where

$$\rho_k = \sum_{i \in I} P_i \frac{\sum_{j \in J} e^{-\mu_j} e^{-\beta c_{ij}^k}}{\sum_{j \in J} \sum_{k \in K} y_k e^{-\mu_j} e^{-\lambda_k} e^{-\beta c_{ij}^k}}, \quad k \in K \tag{34}$$

is the current size of facility k (actual, if k is open or "potential", if k is closed).

The updating of the multipliers $\{\lambda_k\}$ is made as follows

- if $\rho_k \leq U_k$ (where U_k is the capacity of the facility k), leave $\lambda_k = 0$, then $e^{-\lambda_k} = 1$
- otherwise set $e^{-\lambda_k} = U_k/\rho_k$.

The rationale of the above updating mechanism is the following. If the current size ρ_k of facility k is less than or equal to the capacity U_k, then it is kept like it is, otherwise, in order to satisfy the capacity constraint, it will be reduced by multiplying it by a proper coefficient $e^{-\lambda_k} < 1$.
Given the updated $\{\lambda_k\}$, the multipliers $\{\mu_j\}$ are then recalculated by (31) and the iterative procedure goes on until the capacity constraints (24) are satisfied.
With the final values of $\{\mu_j\}$ and $\{\lambda_k\}$ one can calculate the optimal expected flows $\{s_{ij}^k\}$ for the given transshipment location $\{y_k\}$ by (30), then the corresponding optimum by (15).

5 Opening a transshipment facility

The following theorem does hold

Theorem 1 *Problem $CTLP_d$ given by (15)-(20) is equivalent to*

$$\max_{y}\min_{\mu,\lambda}\left[\sum_{i\in I}P_i\ln\sum_{j\in J}\sum_{k\in K}y_k e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}+\sum_{j\in J}Q_j\mu_j+\sum_{k\in K}U_k y_k\lambda_k-\sum_{k\in K}f_k y_k\right] \quad (35)$$

Proof. Let us consider problem (21)-(25), which is problem (15)-(20) when a transshipment location $\{y_k\}$ is already known, and its Lagrangian L_1 given by (26).
Substituting (30) into (26), after some manipulations, one gets

$$L_1(\mu,\lambda)=\sum_{i\in I}P_i\ln\sum_{j\in J}\sum_{k\in K}y_k e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}+\sum_{j\in J}Q_j\mu_j+\sum_{k\in K}U_k y_k\lambda_k$$

Then problem (21)-(25) becomes equivalent to

$$\min_{\mu,\lambda}L_1(\mu,\lambda)=\min_{\mu,\lambda}\left[\sum_{i\in I}P_i\ln\sum_{j\in J}\sum_{k\in K}y_k e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}+\sum_{j\in J}Q_j\mu_j+\sum_{k\in K}U_k y_k\lambda_k\right] \quad (36)$$

Let us consider now problem (15)-(20) which differs from (21)-(25) only by the presence of the unknowns $\{y_k\}$.
Because of (36), problem (15)-(20) then becomes equivalent to

$$\max_{y}\min_{\mu,\lambda}\left[\sum_{i\in I}P_i\ln\sum_{j\in J}\sum_{k\in K}y_k e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}+\sum_{j\in J}Q_j\mu_j+\sum_{k\in K}U_k y_k\lambda_k-\sum_{k\in K}f_k y_k\right] \quad (37)$$

□

By imposing to (35) the necessary first order conditions for $\{y_k\}$ one gets

$$\begin{aligned}
&\partial\left\{\sum_{i\in I}P_i\ln\sum_{j\in J}\sum_{k\in K}y_k e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}+\sum_{j\in J}Q_j\mu_j+\sum_{k\in K}U_k y_k\lambda_k-\sum_{k\in K}f_k y_k\right\}/\partial y_k=\\
&=\sum_{i\in I}P_i\frac{\sum_{j\in J}e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}}{\sum_{j\in J}\sum_{k\in K}y_k e^{-\mu_j}e^{-\lambda_k}e^{-\beta c_{ij}^k}}+U_k\lambda_k-f_k=\\
&=e^{-\lambda_k}\rho_k+U_k\lambda_k-f_k \qquad (38)
\end{aligned}$$

(38) represents the impact on the optimum of a variation of location y_k. Then, if one wants to open a transshipment facility by improving as much as possible the optimum, the

transshipment facility, among the closed ones, with the highest term $(e^{-\lambda_k}\rho_k + U_k\lambda_k - f_k)$ will be candidate to be opened, i.e. the facility r such that

$$r = argmax_{k/y_k=0}\left[e^{-\lambda_k}\rho_k + U_k\lambda_k - f_k\right]$$

If we define as *revenue* of facility k the difference between its *profit* $(e^{-\lambda_k}\rho_k + U_k\lambda_k)$ and its fixed cost f_k, the facility with the highest revenue will be candidate for opening. Let us consider a closed facility k. Because of the complementary slackness conditions of Linear Programming, we know that if the facility potential current size ρ_k is less than or equal to its capacity U_k, then the associated multiplier $\lambda_k = 0$ and the facility revenue becomes $(\rho_k - f_k)$. Vice versa, if $\lambda_k > 0$ the potential current size of facility k is over capacity (then its opening should be extremely recommended) and the term $U_k\lambda_k$ will encourage the choice of this facility to be candidate for opening. In such a case $e^{-\lambda_k}\rho_k = U_k$ and the facility revenue becomes $[U_k(1 + \lambda_k) - f_k]$.
Note that the above criterion for opening facilities requires the possibility to calculate the size ρ_k also for closed facilities, but, as previously seen, this can be easily done by (34), which holds for all k.

6 Closing a transshipment facility

We observe that a mechanism similar to that of Section 5 can be adopted for finding, among the open facilities, that one candidate to be closed down. In such a case, the transshipment facility q such that

$$q = argmin_{k/y_k=1}\left[(e^{-\lambda_k}\rho_k + U_k\lambda_k) - f_k\right]$$

will be the candidate facility to be closed down. Of course, facility r will be really closed if some criteria are satisfied, among them the fact that the total capacity of the remaining open facilities is greater than or equal to the total flow T.

We are now ready to build the overall heuristic for solving problem $CTLP_d$.

7 A Lagrangian heuristic for solving $CTLP_d$

It is based on three procedures which interact each other: the first one is the procedure for finding the optimal expected flows when a facility location is already given (see Sec-

tion 4), the second one is that for opening a closed facility (see Section 5), and the third one is that for closing down an open facility (see Section 6).
In particular, we assume that a transshipment location $\{y_k\}$ is already given and derive, by the procedure developed in Section 4, the optimal expected flows $\{s_{ij}^k\}$ and the corresponding optimum. Then, we try to improve such a given transshipment location by opening and closing facilities. This process calls the above procedure for the optimal expected flows as a subroutine. We reiterate until no more improvements for the optimum are found.

More in detail, the heuristic for solving problem $CTLP_d$ acts as follows

- Problem Feasibility check.
  If the total capacity is less than the total flow, $\sum_{k \in K} U_k < T$, then STOP, the problem is infeasible.
- While the number of iterations is no more than MAXITER (maximum number of iterations) and the overall computational time is not greater than MAXTIME (maximum computational time), apply the Core heuristic (see Subsection 7.1).
- Keep the best solution as the optimal one.

7.1 Core heuristic

The Core heuristic builds a solution according to the following steps

1. Open all the facilities, i.e. $y_k = 1, k \in K$.
2. Compute the optimal expected flows as in Section 4 and the corresponding optimum by (15). Set the best solution $BestSol$ to the optimal expected flows.
3. Repeat the following steps
   (a) Decide if just closing down a transshipment facility or simultaneously closing down and opening two different facilities. The decision is taken according to a rule based on a randomized process and a short term search memory. This rule, called *Operation Choosing Rule*, is described in depth in Subsection 7.1.1.
   (b) Let be q the facility to be closed and r the facility to be opened (if any). Close q and open r.
   (c) Compute the optimal expected flows as in Section 4 and the optimum by (15). Set the current solution $CurrSol$ to the optimal expected flows.
   (d) If no opening operation has been performed and the objective function of $BestSol$ is worse than the one of $CurrSol$, then exit from the heuristic and return $BestSol$. Otherwise, set $BestSol$ to $CurrSol$.

7.1.1 Operation Choosing Rule

In our heuristic, we can decide if just closing down a facility, or simultaneously closing down and opening two different facilities. The choosing rule uses both a dynamic random process guided by the search history and a short-term memory structure. The latter is a list FL which forbids, after opening a facility, its closing down for a fixed amount m of iterations.
The rule works as follows:

- Initialize the opening probability step $\delta_O = 0$ if we are at the first iteration of the overall heuristic, $\delta_O = 2/K$ otherwise, where K is the number of potential transshipment locations. Empty the list FL, set its size equal to 3 and put $p_O = 0$.
- While the solution is feasible
  - Get a random number $p \in (0, 1]$.
  - If $p \geq p_O$, find the facility q candidate to be closed down as in Section 6 and check that $q \notin FL$. Increment p_O by δ_O and decrement by one for all facilities in FL the number of iterations for which they cannot be closed. Remove from FL the facilities for which the number of iterations is 0.
  - Otherwise, set $p_O = \delta_O$, find the facility q candidate to be closed down and the facility r candidate to be opened, as in Sections 6 and 5, respectively. If it is possible and convenient (in terms of the optimum) close and open these two facilities, add r to FL and set the amount of iterations for which r cannot be closed to 3.

Notice that in the first iteration of the heuristic we apply only closing operations, while the opening operations will be considered in the next iterations of the overall heuristic.

8 Computational results

This section is devoted to test the performance of the proposed heuristic by comparing its results, in terms of optimum and computing time, versus those of $CTLP_d$. Following the paper by [6] and the instance sets introduced in [9], we consider 3 instance classes. The instances are generated randomly using uniform distribution with corresponding ranges. The parameters used to generate the instance classes are given in Table 1. In Table 1 the first column represents the problem class; the second, third and fourth columns represent the range for the number of origins, destinations and potential transshipment locations, respectively. For each class, 10 instances are generated, for a total of 30 instances.

Data set	$\mid I \mid$	$\mid J \mid$	$\mid K \mid$
Class 1	$[2,3]$	$[30,40]$	$[10,20]$
Class 2	$[2,5]$	$[30,40]$	$[20,30]$
Class 3	$[2,5]$	$[40,100]$	$[30,50]$

Table 1: Test problem classes (10 instances solved for each set)

As in [9], we generate the instances randomly according to the following criteria

- supply P_i is drawn from $U[900, 1000]$
- demand Q_j is drawn from $U[1, \sum_{i \in I} P_i / \mid J \mid]$. The last demand is adjusted so that the total demand is equal to the total supply.
- capacity U_k is drawn from $U[0.5avU, 3avU]$, where $avU = \sum_{i \in I} P_i / \mid K \mid$
- unit transportation cost c_{ij}^k is drawn from $U[1, 10]$
- fixed cost $f_k = TC\ U_k / (\mid I \mid\mid J \mid)$ where TC is the total transportation cost over all possible arcs.

Our heuristic has been implemented in Matlab 2007, while the non-linear deterministic model $CTLP_d$, according to the tests reported in [9], has been solved by means of BonMIN release 1.1 [2, 3]. All the tests have been performed on a Pentium Quad Duo 2.4 GhZ workstation with 2 Gb of Ram.

After a preliminary testing phase on a subset of 20% of the instances, the parameters of the heuristic have been set to the following values

- MAXTIME= 10.000 seconds;
- MAXITER= 50;
- Size of FL= 3.

For BonMIN, we use its standard parameters with a time limit of 24 hours for each instance.
We summarize the results of the heuristic in Tables 2 and 3. More in details, Table 2 reports detailed statistics of each instance, where the meaning of each column is the following

- Columns 1 and 2: instance class and instance number;
- Columns 3 and 4: optimum and computing time (s) of the heuristic at the end of its first iteration, where only the closing operation is allowed;
- Columns 5 and 6: optimum and computing time (s) of the best solution found by the heuristic by means of the opening and closing operations;
- Column 7: total computing time (s) at the end of the heuristic;
- Columns 8 and 9: optimum and computing time (s) of the best solution found by

BonMIN;
- Column 10: total computing time (s) of BonMIN;
- Columns 11 and 12: percentage gap between the heuristic (first and best solution) and BonMIN.

The mean values obtained for each instance class are reported in Table 3, where the columns have the following meaning

- Column 1: instance class;
- Columns 2 and 3: percentage gap between the heuristic (first and best solution) and BonMIN;
- Columns 4, 5 and 6: computing time (s) of the heuristic (first and best solution) and BonMIN;
- Columns 7 and 8: total computing time (s) at the end of the heuristic and BonMIN.

For each value, the mean results of the 10 random generated instances in each class are reported in the first three rows, while the last row give the overall mean over the three classes.

Class	Inst	C		O&C			BonMIN			Gap	
		OPT	Time	OPT	Opt Time	Tot Time	OPT	Opt Time	Tot Time	C	O&C
1	1	144659.8	2.3	142856.2	2.3	201.2	142712.7	45.6	47.6	1.36	0.10
	2	210328.5	3.7	210328.5	3.7	266.7	209429.0	64.9	77.8	0.43	0.43
	3	152415.0	3.3	151213.8	62.4	222.8	150860.4	108.3	125.1	1.03	0.23
	4	167532.8	3.7	167532.8	3.7	245.8	167358.9	174.3	187.2	0.10	0.10
	5	157326.0	4.7	157326.0	4.7	316.6	157159.6	69.7	171.7	0.11	0.11
	6	214347.1	4.9	211891.7	106.0	294.4	211108.0	219.4	347.1	1.53	0.37
	7	246348.5	5.4	245983.0	16.3	326.4	244105.3	154.9	254.6	0.92	0.77
	8	248560.9	3.7	248359.7	141.7	295.3	248086.0	275.0	369.3	0.19	0.11
	9	249641.5	5.1	247229.3	8.6	428.6	247004.6	441.0	453.9	1.07	0.09
	10	188749.8	3.5	188749.8	3.5	253.4	188291.0	32.0	34.3	0.24	0.24
2	1	604886.6	18.5	595126.8	32.4	1081.1	594360.6	9656.7	17146.2	1.77	0.13
	2	471361.0	10.4	471361.0	10.4	635.8	471361.0	12057.5	86400.0	0.00	0.00
	3	442685.6	9.8	442280.5	28.2	563.4	441550.2	5586.0	11944.2	0.26	0.17
	4	583198.1	12.3	583198.1	12.3	714.9	582520.6	11309.9	11534.6	0.12	0.12
	5	229398.4	5.1	228540.4	87.9	351.6	227864.6	301.6	565.0	0.67	0.30
	6	443419.6	12.4	441867.0	228.5	692.4	441090.0	8718.4	25371.9	0.53	0.18
	7	264759.5	10.5	264664.9	412.2	564.6	264457.3	3917.3	4812.3	0.11	0.08
	8	357978.9	8.3	356801.1	22.9	457.7	355382.8	960.5	1812.0	0.73	0.40
	9	750515.9	18.4	746740.8	163.3	1088.5	745853.8	16306.9	23643.8	0.63	0.12
	10	625309.4	10.5	621629.8	72.1	601.0	621288.9	20276.6	47286.0	0.65	0.05
3	1	919082.2	81.6	919082.2	81.6	4224.4	918609.8	6468.8	86400.0	0.05	0.05
	2	470979.0	124.5	470979.0	124.5	6579.1	468864.8	57121.8	86400.0	0.45	0.45
	3	598003.3	30.8	596171.8	89.8	1497.3	595747.5	55795.5	86400.0	0.38	0.07
	4	992675.4	111.3	981112.9	2879.6	7578.0	980429.9	69090.3	86400.0	1.25	0.07
	5	889800.0	48.1	889673.5	172.6	2157.1	889326.4	30936.1	86400.0	0.05	0.04
	6	484328.4	79.0	479308.0	2269.2	3660.0	478182.8	77238.0	86400.0	1.29	0.24
	7	556649.4	59.1	556363.0	1061.7	2949.3	555947.9	59261.3	86400.0	0.13	0.07
	8	768839.8	103.4	768839.8	103.4	4842.6	767101.5	34394.1	86400.0	0.23	0.23
	9	1017825.8	41.2	1015630.2	566.3	1887.5	1015575.6	65542.0	86400.0	0.22	0.01
	10	652024.6	72.2	652024.6	72.2	3974.9	650924.9	48652.0	86400.0	0.17	0.17

Table 2: Detailed results of the heuristic

From Table 3 we can see that the mean gap between the initial solution and BonMIN (column C) is less than 0.6%, which is reduced to 0.18% after applying the opening and closing operations (column $O\&C$). Moreover, the worst mean gap is 0.7%. These results

Class	Gap		Opt Time			Tot Time	
	C	O&C	C	O&C	BonMin	O&C	BonMin
1	0.70	0.26	4.0	35.3	158.5	285.1	206.9
2	0.55	0.15	11.6	107.0	8909.1	675.1	23051.6
3	0.42	0.14	75.1	742.1	50450.0	3935.0	86400.0
Mean	0.56	0.18	30.25	294.80	19839.2	1631.7	36552.8

Table 3: Mean results of the heuristic

are more impressive if one thinks that they can be obtained in a reasonable computing time. In fact, the mean computing time of the heuristic is less than half an hour, while BonMIN needs a mean computing time of about 10 hours, with peaks of 24 hours. The efficiency of our heuristic is more evident if one considers the time at which the heuristic and BonMIN find their optimal solution. In this case the time gap between the two methods is almost two order of magnitude (almost 300 seconds versus almost 20.000). Moreover, the computing time of the heuristic could be further reduced by implementing an ad hoc fixed point method to compute the optimal flows in Section 4. In fact, the present implementation uses the standard Matlab $fsolve$ function, which is not efficient when the number of flows increases.

If we consider the impact of the opening operation on the final solution, we can see that a certain number of facilities which have been opened during the procedure are still open in the final solution. Then it would seem that the closing operation has made some mistake in closing down these facilities during the first iteration of the heuristic (where only the closing operation is applied). Actually, this may happen when the revenue term $(e^{-\lambda_k}\rho_k + U_k\lambda_k) - f_k$, used as the criterion for closing down a facility, is quite similar for two or more facilities. In this case, the heuristic could take the wrong decision for closing, but the mistake is then recovered by the opening operation.

9 Conclusions

In this paper the deterministic approximation of the Capacitated Transshipment Location Problem under Uncertainty $CTLP_d$ has been solved by a Lagrangian-based heuristic. The performance of this heuristic is extremely good both in terms of effectiveness and efficiency. In fact the mean gap for the optimum is 0.18% and the computing time is almost two order of magnitude less than that of $CTLP_d$.
Future research will be devoted to check whether this heuristic is able to cope with transshipment location problems where not only an upper capacity but also a lower capacity constraint for the facilities must be satisfied. This aspect is particularly important from an economic point of view, in order to avoid to locate facilities whose expected size is either under a minimum or over a maximum throughput threshold, both situations being economically inefficient.

10 Acknowledgments

This project has been partially supported by the Italian Ministry of University and Research, under the 2007 PRIN Project "Optimization of Distribution Logistics".

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