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Cost Allocations in Combinatorial Auctions for Bilateral Procurement Markets

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Abstract. We propose a trading mechanism for bilateral markets that is based on combinatorial auctions. Unlike previous reported works, the market bilateral aspect is handled by considering one-sided reverse auctions rather than exchanges during the trading process. The bilateral component is then reconsidered at the end of the auction to decide on the price to be paid or received by each participant. The proposed approach takes advantage of the simplicity of one-sided auctions compared to exchanges, while exploiting combinatorial bidding as a mean to enforce collaboration between buyers. We propose, evaluate and compare different procedures for allocating costs to buyers during this post-auction pricing phase. These procedures are inspired by two key concepts in cooperative game theory: the nucleolus and the Shapley value.

Keywords. Combinatorial auctions, bilateral markets, linear prices, cooperative games, cost sharing.

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1 Introduction

Auctions have received a lot of attention in the last two decades in economics, operations research and computer science literature. This is in part due to the growing popularity of electronic markets and to the efficiency and fairness characterizing these mechanisms (ANANDALINGAN *et al.*, 2005). Auction design requires the market maker and participants to address several important issues: Who should be the final winners? What should every participant pay or receive? What is the best bidding strategy? Is it attractive for bidders to reveal their true values? Etc. Some of these problems become more complex when the so-called *combinatorial auctions* are used. Combinatorial auctions refer to auction mechanisms in which bids on combinations of items are permitted. Such package bids are beneficial when synergy exists between the traded items (e.g. RASSENTI *et al.*, 1982; LEDYARD *et al.*, 2002; DE VRIES and VOHRA, 2003; PEKEČ and ROTHKOPF, 2003). Participants are thus offered the possibility to express their valuations for any collection of items they want to trade and the well-known *exposure* problem is avoided (ROTHKOPF *et al.*, 1998).

The majority of research on eMarkets and auctions design considers one-sided markets, i.e., markets including either one seller and multiple buyers (a *one-to-many* or *forward* auction), or one buyer and multiple sellers (a *many-to-one* or *reverse* auction). Determining winning bids is often done with respect to an objective function that, depending on the type of market considered, maximizes the seller's revenue (forward auction) or minimizes the buyer's cost (reverse auction). When only simple bids are permitted (i.e. bids on single items), winners are straightforwardly determined (PEKEČ and ROTHKOPF, 2003). When bids on combinations of items are allowed, the winner determination problem (WDP) however must be formulated as a set packing, a set covering or a set partitioning problem and is much harder to solve (ABRACHE *et al.*, 2007).

The *pricing problem* is another important issue that must be tackled when designing auctions. One must indeed specify the price that must be paid or received by each participant at the end of the auction process. Pricing rules have to be consistent in regards to winning and losing bids and explain why some bids win and others lose (XIA *et al.*, 2004). For iterative auctions, i.e., auctions made up of several rounds, the pricing problem is generally addressed at each round to disclose price information to bidders and help them elaborate efficient bidding strategies.

Both winner determination and pricing problems were intensively studied in the last two decades for unilateral markets. A few studies dealt with bilateral markets, i.e., markets including multiple buyers and multiple sellers. Many-to-many auctions, also called exchanges, were proposed in which both buyers and sellers submit package bids. Applications of combinatorial exchanges have been suggested for trading assets in financial markets (e.g., Fan *et al.*, 2000, 2002), supply chain formation and coordination (Walsh *et al.*, 2000), and market clearing in process industries (Kalagnanam *et al.*, 2001). Few papers addressed the problem of designing combinatorial exchanges (Parkes *et al.*, 2001; Kothari *et al.*, 2002; Smith *et al.*, 2002). Some others focused on modeling and solving the winner determination problem (Xia *et al.*, 2005) or considered only non-combinatorial bids (Bourbeau *et al.*, 2005). Abrache *et al.* (2004) and Cavallo *et al.* (2005) suggested bidding languages for combinatorial exchanges that allow participants to express preferences for both buying and selling goods in the same structure. Recently, Chu (2009) considered a bilateral *bundle/multiunit market* in which each buyer requests some bundles of heterogeneous items and each seller provides multiple units of a single item. The proposed mechanism is proved to be *strategy-proof* (i.e., bidding truthfully is the best strategy for each bidder), *individual rational* (each bidder has a nonnegative utility function), budget-balanced (the payoff of the auctioneer is nonnegative) and *asymptotically efficient*.

In this paper, we propose a trading mechanism for bilateral markets based on combinatorial auctions. The proposed approach deals with the market bilateral aspect differently by considering one-sided reverse auctions rather than exchanges during the trading process. The bilateral component is then reconsidered at the end of the auction to decide on the price to be paid or received by each participant. The proposed approach takes advantage of the simplicity of one-sided auctions compared to exchanges, while exploiting combinatorial bidding as a mean to enforce collaboration between buyers. More specifically, we consider a particular trading context in which a set of buyers requiring given services or goods, or more generally, *contracts*, decide to participate in the same procurement market together with a number of competing sellers. Buyers' requests are private information that are submitted at the beginning of the trade and are not permitted to change during the whole process. Participating sellers try then to win some of these requests by submitting combinatorial bids. A seller may combine in its combinatorial bid contracts submitted by different buyers. At this stage, only the WDP and the bidder pricing problems need to be solved. When the auction ends, given the results output of the one-sided auction, one knows exactly the winning sellers, the price they should receive, and consequently the total price to be paid by all buyers. The next step consists in determining the price to allocate to each buyer individually. The main object of this paper is to propose, evaluate and compare different procedures for allocating costs to buyers at this post-auction pricing phase. In fact, by participating together in the same combinatorial auction, buyers give sellers more possibilities for combining contracts and offering interesting sell prices in their package bids which, in turn, should result in a gain on the total payment to be handled out by buyers when compared to the situation where buyers participate independently in the same auction. This gain must be perceived by buyers through the payment they are allocated.

When defining the buyer-cost allocation procedures, we looked for satisfying three properties: *fairness*, *proposal neutral* , and *budget balance*. The proposal-neutral property ensures that the cost allocated to a given buyer is independent of its identity. Budget balance means that the total amount paid by buyers as a result of the cost allocation procedures must be equal to the total payment to be received by the winning bidders as output by the auction process. To satisfy the proposal-neutral property, cost allocation procedures are defined with respect to individual contracts rather than individual buyers. Once these prices are determined, each buyer pays an amount equal to the sum of the prices of the contracts it requested. This *contract-pricing approach* assumes that one can determine the price of each contract separately at the end of the auction process. Such an approach is in fact closely related to the problem of determining linear prices in iterative combinatorial auctions that has been treated for one-sided markets (e.g. Xia *et al.*, 2004; Kwasnica *et al.*, 2005; Bichler *et al.*, 2009). In the reported studies, exact or approximate, and possibly multiple, linear prices are derived with respect to winning and losing bids. Some of these studies limited the search to determining single-item prices although they were non-unique (O'Neill *et al.*, 2005). Others proposed procedures to ensure unicity in a way that guided bidders solve the threshold problem in multi-round auctions (Kwasnica *et al.*, 2005). The procedures that we propose in this paper to determine exact or approximate (and possibly multiple) contract prices recall the main ideas already exposed in the literature and adapt them to reverse auctions. For contracts price unicity, we propose a new approach directed toward auctioneer rather than bidders' interests. In fact, the unicity procedures reported in the literature were elaborated for price-feedback information purposes in iterative one-sided auctions to help bidders construct interesting and potentially winning bids for the next rounds. Prices unicity was thus met with respect to some arbitrary or bidder-specific criteria. In this paper, linear prices are used to determine the price that should be paid by each buyer individually, once the whole auction process terminates. Hence, unlike previous work, fixing contract prices is done with respect to auctioneers (i.e., buyers) so as to define fair payments for them. Moreover, based on the idea that combinatorial bidding is a mean to incitate buyers to collaborate and participate in the same auction, the proposed procedures are inspired by a number of cooperative game theory concepts, namely the *nucleolus* and the *Shapley value*. We define two procedures that are based on the concept of the nucleolus applied to a particular cooperative game with restrictions. We define two other procedures based on the Shapley value concept adapted to a series of sub-games. We prove that these two procedures are equivalent and will always yield the same price allocation.

Regarding the budget-balance property, we prove that when a set partitioning formulation is used to model the winner determination problem, the proposed contract-pricing approach is always budget-balanced independently of the unicity procedure used. We also prove that a set covering formulation may yield an unbalanced budget, in some cases, and propose a simple way to recover the budget balance.

The proposed cost allocation procedures differ in the method used to fix contract prices when multiple solutions exist after the exact or approximate contract-pricing phase. Yet, even though defined on a contract level to satisfy the proposal-neutral property, these methods are used to determine the payment to be paid by a buyer (i.e., a subset of contracts). We thus propose performance-measure criteria, defined at buyer level, to compare the quality of the solutions output by these procedures.

We tested the proposed cost-allocation procedures on instances derived from the CATS generator elaborated by Leyton-Brown *et al.* (2000). All the proposed procedures need relatively short computing times even for large problems.

The remainder of the paper is organized as follows. The next section defines the problem addressed and the related assumptions. In Section 3, we adapt the exact and approximate procedures for determining linear prices in combinatorial auctions to procurement markets. In Section 4, we prove that the budget-balance property is not always satisfied when a contract-pricing approach is used and propose a simple idea to recover it. Section 5 describes the procedures proposed to ensure contracts price unicity. Section 6 presents the performance-measure criteria used to evaluate the quality of the allocations yielded by these procedures and reports the experimental results obtained for different sets of instances. We conclude in Section 7.

2 Problem Setting

We consider a bilateral procurement market including multiple buyers and multiple sellers trading heterogeneous contracts. A contract is specific to a buyer. It refers to either services or goods and may include other additional information such as trading conditions, buyer specifications, etc. As already mentioned, the trading process is modeled as a one-sided procurement auction in which all buyers are grouped and treated as a single "artificial" auctioneer and sellers are the bidders. It is assumed that contracts are not allowed to change during the auction process. Moreover, all contracts are *indivisible* implying that each must be served by a unique seller offer. In transportation markets, for example, buyers correspond to shippers who need to move commodities between specified locations and sellers are carriers that make offers to win shipping contracts. A contract, in this context, specifies the commodities to move, their volumes, and possibly other components such as pick-up and delivery time windows, carrying conditions, etc. Contract indivisibility implies that the commodities within a shipper contract have to be carried together by a single transportation service.

The “artificial” auctioneer, acting as representative of the buyers, communicates the buyer contracts to participating sellers. The latter make offers in form of package bids. A seller presents in its bid a set of contracts it is ready to perform and the price it asks for this service. Seller bids are assumed *indivisible* in the sense that the contracts submitted in a package must be all allocated or none. Hereafter, we illustrate the problem for the more general iterative auction process. In such contexts, bidders have the possibility to submit new bids in different rounds of the auction. In each round, the set of winning bids are determined by solving a winner determination problem. Prices information together with current winning bids are communicated to bidders to help them in their bidding tasks, and the process is repeated until some stopping criteria are met.

2.1 Winner Determination Problem

Let K denote the set of contracts submitted by the buyers. Set K is fixed at the beginning of the auction and is not permitted to change during the trading process. A combinatorial bid b submitted by a seller is described by a pair (K_b, P_b), where K_b is the set of contracts the seller offers to perform in bid b ($K_b \subseteq K$) and P_b is the price asked if bid b wins. Let B^r denote the set of bids considered in a round r of the auction. This set includes the new bids submitted in round r, as well as the winning bids of the previous round. The WDP corresponding to round r is modeled by using binary variables x_b defined for each bid $b \in B^r$: variable x_b equals 1 if bid b wins; 0 otherwise. The constant parameter δ_{bk} is defined for each contract k and each bid b to indicate whether bid b covers contract k or not, i.e., $\delta_{bk} = 1$ if $k \in K_b$; 0 otherwise.

The WDP associated with round r can be formulated as follows:

$$(P1^r) : \min \sum_{b \in B^r} P_b x_b$$

$$\text{s.t.} \quad \sum_{b \in B^r} \delta_{bk} x_b = 1 \qquad \forall k \in K, \tag{1}$$

$$x_b \in \{0,1\} \qquad \forall b \in B^r. \tag{2}$$

$(P1^r)$ is a set partitioning formulation that minimizes the total cost paid by the buyers. Constraints (1) guarantee that each buyer contract is covered exactly once by one seller bid. Constraints (2) are integrality constraints. Notice that, given this formulation, a seller can win more than one bid assuming the bids submitted by a bidder at a given round are OR bids (Nisan, 2006).

Formulation $(P1^r)$, using set-partitioning constraints, has some drawbacks. Indeed, it is well known that set partitioning formulations are more difficult to solve than set covering and set packing ones. Moreover, in our particular case, bidders submit bids independently of one another in a way that maximizes their own utility functions. Hence, there is no guarantee that the set of bids considered at a given round will cover all buyers' contracts. To be feasible, a set partitioning formulation requires that there exists, at each round of the auction, a set of bids that not only cover all buyers' contracts but are also disjoint. Finally, a set partitioning formulation may yield negative dual values, that is, negative marginal costs to serve a contract.

In the following, we will rather consider a relaxed set covering formulation, $(P2^r)$, and assume that a contract can be "theoretically" covered more than once. Formulation $(P2^r)$ is identical to $(P1^r)$, except that equality constraints (1) are replaced by the following set of inequality constraints:

$$\sum_{b \in B^r} \delta_{bk} x_b \quad \geq \quad 1 \qquad \forall k \in K. \tag{3}$$

Obviously, by allowing two bids covering the same contract to win, the WDP is more likely to be feasible. It is noteworthy that, in practice, a contract covered twice will in fact be served only once by the "appropriate" bidder. Both winning bidders will however receive a payment as if they both were serving it. We will prove in Section 4 that if a contract is covered more than once (constraints 3 are not tight), the price of serving this contract is necessarily null.

Even though the risk of infeasibility is decreased for set covering formulations compared to set partitioning ones, there is no guarantee that the set of bids of a given round will cover all the contracts in K. To circumvent such problems, we assume that each buyer associates with each contract a *reserve price* representing the maximum price that it is ready to pay for the contract. These reserve prices are private information and are not permitted to change during the auction process. Buyers' reserve prices are assumed to be submitted as simple bids at each round of the auction. More precisely, the set of bids B^r considered in a given round will always include a subset of simple bids, $B^a = \{(k, M_k), k \in K\}$, where M_k is the reserve price associated with contract k. One should notice that these simple bids then can be used to circumvent infeasibilities in set partitioning formulations. However, we wanted them to be the last alternative for ensuring feasibility, once all sellers' bids are exploited.

2.2 Bidder Pricing Mechanism

Different bidder pricing schemes exist for combinatorial bidding (e.g., see Xia *et al.*, 2004, for more details). The well-known Vickrey-Clarke-Groves (VCG) mechanism, also known as second-price mechanism, was proposed to ensure the incentive-compatibility property implying that bidding truthfully is a dominant strategy for bidders (Vickrey, 1961; Clarke, 1971; Groves, 1973). VCG mechanisms present serious limitations, however, e.g., computational complexity, sensitivity to collusion and cheating, and the fact that they do not guarantee the budget-balance of the market and may give a seller a marginally small revenue (Rothkopf *et al.*, 1990; Sakurai *et al.*, 2000; Ausubel and Milgrom, 2002, 2006). The latter stands out since it can be shown that for the important case of exchanges (even non-combinatorial ones), budget-balance may not be achieved. Moreover, Ausubel and Milgrom (2006) showed that the VCG auction loses its dominant-strategy property when bidders face effective budget constraints.

Another, simpler pricing mechanism, which has been largely considered in the literature, is the so-called *first-price* mechanism in which each winning bidder "receives" (in the case of reverse auctions) exactly the amount requested in its bid. Day and Raghavan (2007) consider that first-price mechanisms for sealed-bid auctions may lead to inefficient outcomes since incomplete information about the preferences of other participants considerably complicates the task of determining the maximum price that should be submitted to secure a particular bundle of items. Indeed, the shortcomings pointed out by Day and Raghavan (2007) can be circumvented by considering price-directed iterative auctions. In such iterative auctions, pricing information is revealed to bidders at each round to help them construct promising bids for the next rounds. This price information may take several forms. A first and easiest alternative is to assume that the price of a package represents the sum of the prices of the items it contains. These are called *linear* or *additive* prices. A second alternative, that takes into account the possible synergy between items, assigns a price to the bundle as a whole. In this case, the bundle price may be *anonymous* (i.e., it does not depend on the bidder), or *discriminatory* implying a different bundle price for different participants.

In this paper, we consider iterative auctions with a first-price rule. Formally, a seller winning a bid b must receive an amount exactly equal to P_b, the price asked in its bid. The object of the paper being more about buyer-pricing problem once the auction process is finished, we will not go into more details regarding price feedback for iterative auctions. We refer the reader to the papers by Kwasnica *et al.* (2005) and Bichler *et al.* (2009) for an overview on this topic.

2.3 Buyer Cost-Sharing Problem

At the end of the auction process, once the bidder-pricing problem is solved, the "artificial" auctioneer knows exactly the total amount that must be paid by the buyers. Formally, this amount is given by $C(A) = \sum_{b \in B^R} P_b x_b^*$, where A denotes the set of buyers and x^* is an optimal solution of the WDP $(P2^R)$ in the final round R. The question is: what is the amount that must be paid by each buyer? In other words, how the total cost, $C(A)$, should be shared among buyers?

Cost-sharing problems are commonly encountered in cooperative game theory. In cooperative games, different agents interact and try to form coalitions. A coalition is defined as a subset of players (or agents) that cooperate in order to obtain gains (see, e.g., Young, 1994, for more details on cooperative game theory concepts). In our context, we assume that by participating in the same auction, buyers obtain gains when compared to the case where each buyer participates in the auction separately. In other words, if each buyer runs a one-sided auction with the same set of sellers, then the total amount paid by all buyers with their one-sided auctions is greater than the total amount they would pay if they participate all together in the same auction. Such situations are common in procurement markets. For transportation markets, for example, by putting together different lanes (i.e., origin-destination pairs), carriers can make interesting offers that minimize empty movements and other related costs (Caplice and Sheffi, 2006). Thus, shippers would generally pay a total transportation cost that is lower than the amount they would have paid had they submitted their lanes separately.

Cost-sharing methods proposed in cooperative game theory imply the determination of the cost associated with some or all possible coalitions of players. The cost associated with a coalition represents the cost the coalition would incur working on its own. In traditional cooperative games, such a cost is determined by solving an optimization problem restricted to the members of the coalition. In our case, the problem is more complex. In fact, the total cost paid by all buyers (the grand coalition) is determined through a multi-round auction process. Determining the cost incurred by a subset of buyers is not an easy task. Ideally, one would run multiple multi-round auctions, one for each sub-coalition of buyers. Clearly, such a process is impossible in practice because: (1) Sellers are not disposed to run such auctions: they see no interest in doing this (the interest is more for buyers) (2) Running a multitude of auctions is computationally intractable and would take a long time.

In this paper, we propose a new approach for determining cost allocations that combines some results obtained for linear prices in one-sided combinatorial auctions and some key-concepts of cooperative game theory. More precisely, we consider a contract-pricing approach in which prices are defined for single contracts rather than buyers. Recall that this is done to ensure the proposal-neutral property so as the cost allocated to a buyer does not depend on its identity. Afterward, if p_k denotes the price assigned to contract k, and $K(a)$ is the set of contracts requested by buyer a, then the total amount that must be paid by buyer a is $C(a) = \sum_{k \in K(a)} p_k$.

Contract prices are determined at the end of the auction with respect to winning and losing bids by using exact and approximate procedures as will be described in Section 3. We prove in Section 4 that a simple extension of the existing linear-pricing procedures may yield, in some cases, an unbalanced budget and we propose a simple way to ensure the budget-balance property. Besides, contract prices obtained by the exact or the approximate procedures may be not unique. Considering that buyers will gain by participating in the same combinatorial auction, we propose in Section 5 to fix contract prices using unicity procedures that are inspired by the nucleolus and the Shapley value methods, two concepts that are traditionally used in cooperative games.

3 Contract Prices

Linear prices must be coherent with the set of losing and winning bids to explain why some bids win and others lose. In our context, we are only interested in contract prices at the end of the auction, i.e., at the final round R. At this round, solving model $(P2^R)$ yields two sets of bids: the set of winning bids W^R and the set of losing bids E^R. Clearly $W^R \cup E^R = B^R$ where B^R is the set of all bids considered at round R. Let p_k denote the price associated with contract k. Recall that the bidder-pricing scheme that we adopted follows a first-price or a "receive-as-bid" rule. Hence, contract prices must ideally satisfy the following constraints:

$$\sum_{k \in K} \delta_{bk} p_k = P_b \qquad \forall b \in W^R, \tag{4}$$

$$\sum_{k \in K} \delta_{bk} p_k \leq P_b \qquad \forall b \in E^R. \tag{5}$$

3.1 Exact Contract Prices

When the WDP is a pure linear program, we can easily prove that the system of equations (4)-(5) is always feasible. In such cases, contract prices are derived from the optimal values of dual variables. In fact, consider the following linear relaxation of model ($P2^R$):

$$
\begin{aligned}
(LP2^R) : \min \sum_{b \in B^R} P_b x_b & & & \\
\text{s.t.} \quad \sum_{b \in B^R} \delta_{bk} x_b & \geq 1 & & \forall k \in K, \\
x_b & \geq 0 & & \forall b \in B^R.
\end{aligned}
$$

Its dual, ($DP2^R$), is given by:

$$
\begin{aligned}
(DP2^R) : \max \sum_{k \in K} \Pi_k & & & \\
\text{s.t.} \quad \sum_{k \in K} \delta_{bk} \Pi_k & \leq P_b & & \forall b \in B^R, \\
\Pi_k & \geq 0 & & \forall k \in K.
\end{aligned}
$$

By duality theory, a feasible primal solution x and a feasible dual solution π are optimal if and only if they satisfy the following complementary slackness conditions:

$$(P_b - \sum_{l \in L} \delta_{bk} \Pi_k) x_b = 0 \qquad \forall b \in B^R, \tag{6}$$

$$(1 - \sum_{b \in B} \delta_{bk} x_b) \Pi_k = 0 \qquad \forall k \in K. \tag{7}$$

Primal conditions (6) ensure that when a bid b wins (i.e., when $x_b = 1$), then $\sum_{k \in K} \delta_{bk} \Pi_k = P_b$, which is equivalent to the "receive-as-bid" rule (4). Dual constraints in model ($DP2^R$) ensure inequalities (5) for losing bids. Thus, if the optimal solution of the linear relaxation of model ($P2^R$) is integral, the corresponding optimal dual solution, $(\Pi^*_k)_{k \in K}$, represents the prices of serving contracts $k \in K$. This remains true when the WDP is an integer problem for which the linear relaxation yields an integer solution. Many previous researches aimed to determine some of these particular WDP (e.g. Rothkopf *et al.*, 1998; de Vries and Vohra, 2003).

For general WDP, the correspondence between contract prices and optimal dual values is no longer possible. Furthermore, there is no guarantee that the linear system (4)-(5) is always feasible (e.g., Kwasnica *et al.*, 2005; Shabalin *et al.*, 2005; Bichler *et al.*, 2009). In such cases, one can approximate contract prices by adequate linear programs that take back the main constraints of the dual as will be explained in the next section.

3.2 Approximate Contract Prices

In this section, we propose to adapt some of the LP-based approximation procedures proposed in the literature to the context of procurement markets. In such approaches, equality constraints (4) modeling the "receive-as-bid" rule are maintained as hard constraints, while inequality constraints (5), representing also the dual constraints in model $(DP2^R)$, are permitted to be violated. The objective is thus to determine contract prices such that these violations (also called "distortions") are reduced as much as possible. This is done by solving a series of LP models. In these models, a variable p_k is defined for each contract k to represent the price allocated to it. In addition, a continuous variable Δ_b is defined for each losing bid $b \in E$ to model the possible deviation with regard to the corresponding constraint (5) (the index of the final round 'R' is omitted here to simplify the presentation). As in the RAD auction proposed by Kwasnica *et al.* (2005), we consider an objective function that minimizes the maximum of all these distortions. A continuous variable, Z, representing the upper bound over all distortions is thus introduced. The LP problem approximating contract prices is thus given by:

$$(AP^0) \min Z \tag{8}$$

$$\text{s.t.} \quad \sum_{k\in K} \delta_{bk} p_k = P_b \qquad \forall b \in W, \tag{9}$$

$$\sum_{K\in K} \delta_{bk} p_k - \Delta_b \leq P_b \qquad \forall b \in E, \tag{10}$$

$$\Delta_b - Z \leq 0 \qquad \forall b \in E, \tag{11}$$

$$p_k \geq 0 \qquad \forall k \in K, \tag{12}$$

$$\Delta_b \geq 0 \qquad \forall b \in E. \tag{13}$$

Objective function (8) minimizes the maximum distortion. Constraints (9) ensure the "receive-as-bid" rule. Constraints (10) aim at satisfying inequalities (5) with some deviation. Constraints (11) express the fact that Z represents an upper bound for distortions. Finally, constraints (12) and (13) are non-negativity constraint on price and distortion variables.

Kwasnica *et al.* (2005) proved that improvement may still be possible with respect to the optimal solution (Z^*, Δ^*, p^*) of model (AP^0) when $Z^* \neq 0$. In fact, there may exist constraints (10) corresponding to some losing bids $b \in E$ for which $\Delta_b^* < Z^*$ and can be further lowered. They propose to lower distortions iteratively by solving a series of LP problems. An adaptation of their approach to our context consists in solving sequentially a number of LP problems, the problem solved at iteration 0 being (AP^0). The LP problem considered at iteration t, denoted hereafter (AP^t), is similar to its predecessor (AP^{t-1}) except for some inequality constraints (10) of (AP^{t-1}) that are replaced by equality constraints with fixed distortions. This is done for the losing bids for which the distortions cannot be further improved, i.e., the set of bids b for which the optimal distortion is equal to the optimal objective function. Thus, distortions are fixed progressively, inequality constraints are replaced by equality constraints, and problems are solved until no distortion can be further improved.

Formally, let $\hat{E}^t$ denote the set of distortions fixed at iteration $(t-1)$ and $\hat{\Delta}_b^t$ their corresponding values. The problem to be solved at iteration t is given by:

$$(AP^t) : \min Z$$

$$\text{s.t.} \quad \sum_{k \in K} \delta_{bk} p_k = P_b \qquad \forall b \in W, \tag{14}$$

$$\sum_{k \in K} \delta_{bk} p_k = P_b + \hat{\Delta}_b^t \qquad \forall b \in \hat{E}^t, \tag{15}$$

$$\sum_{k \in K} \delta_{bk} p_k - \Delta_b \leq P_b \qquad \forall b \in E \setminus \hat{E}^t, \tag{16}$$

$$\Delta_b - Z \leq 0 \qquad \forall b \in E \setminus \hat{E}^t, \tag{17}$$

$$p_k \geq 0 \qquad \forall k \in K, \tag{18}$$

$$\Delta_b \geq 0 \qquad \forall b \in E \setminus \hat{E}^t. \tag{19}$$

Both exact and approximate contract-pricing phases may yield multiple feasible solutions. Kwasnica *et al.* (2005) propose a procedure that fixes prices in a way that helps bidders construct potentially winning bids for subsequent rounds by solving the *threshold* problem, i.e., situations where a set of bidders desiring to acquire small packages must coordinate their efforts to outbid another single bidder bidding on the larger combination of all these packages. They propose a series of LP models that sequentially maximize the minimum price of the contracts in each winning bundle. Shabalin *et al.* (2005) showed that the procedures proposed by Kwasnica *et al.* (2005) may still not guarantee unique prices. They propose a procedure that sequentially minimizes the maximum of all prices and then minimizes the sum of the prices of the items in the optimal set.

In our context, price unicity is a fundamental issue in determining the payment each buyer individually has to make when the auction ends. Fixing contract prices in case of multiple feasible solutions must thus take into account the interests of the buyers rather than of the bidders. The object of the next sections is to propose procedures for fixing contract prices in a way that allocates fair payments to buyers, and ensures that the total payment allocated to buyers is equal to the total price asked by winning sellers. We will prove in the next section that this budget-balance property is not always satisfied when using a contract-pricing approach.

In the following, let $\mathcal{P} = \{(p_k)_{k\in K}\}$ denote the set of contract-price vectors deduced from either the exact or the approximate contract-pricing phase.

4 Recovering budget-balance

This section deals with the budget-balance property and thus assumes, throughout, that contract prices are unique (i.e., $|\mathcal{P}^*| = 1$).

The contract-pricing approach we consider in this paper attributes to each buyer a a payment $P(a)$ equal to $\sum_{k\in K(a)} p_k$. Thus, the total amount allocated to buyers with such an approach is:

$$P(A) = \sum_{a\in A} P(a) = \sum_{a\in A} \sum_{k\in K(a)} p_k = \sum_{k\in K} p_k.$$

Besides, the total amount that must be paid by all buyers at the end of the auction process is:

$$C(A) = \sum_{b\in B^R} P_b x_b^* = \sum_{b\in W} P_b,$$

where x^* denotes the optimal solution obtained by solving model $(P2^R)$. Thus, the budget is balanced if the two amounts $P(A)$ and $C(A)$ are equal. We prove next that this is not always true.

Proposition 1

A contract-pricing approach satisfies the budget-balance property if and only if at least one of the following conditions is met:

- The set of winning bids at the final round is such that each contract is covered exactly once.
- If a contract is covered more than once at the final round, then its price is null.

Proof: Recall that, in Section 2.1, we choose to formulate the winner determination problem as a set covering model thus allowing a contract to be covered more than once. We have also noticed that, even though a contract is covered twice by two seller bids, it will be served only once in practice and the winning sellers will receive payments as if they both served it. Let K^+ denote the set of contracts covered more than once in the final solution of the WDP $(P2^R)$, that is, $\forall k \in K^+, \sum_{b\in B^R} \delta_{bk} x_b^* > 1$. For each $k \in K$, define μ_k as the number of times the contract k is over covered. That is, $\mu_k = \sum_{b\in B^R} \delta_{bk} x_b^* - 1$, or written differently

$\sum_{b \in B^R} \delta_{bk} x_b^* - \mu_k = 1$. Clearly, $\forall k \in K^+, \mu_k > 0$ and $\forall k \in K \setminus K^+, \mu_k = 0$. The total payment allocated to buyers with the contract-pricing approach is thus given by:

$$\begin{aligned} P(A) &= \sum_{k \in K} p_k \\ &= \sum_{k \in K} p_k \left(\sum_{b \in B^R} \delta_{bk} x_b^* - \mu_k \right) \\ &= \sum_{k \in K} p_k \sum_{b \in B^R} \delta_{bk} x_b^* - \sum_{k \in K^+} p_k \mu_k \\ &= \sum_{b \in B^R} x_b^* \left(\sum_{k \in K} \delta_{bk} p_k \right) - \sum_{k \in K^+} p_k \mu_k. \end{aligned}$$

Recall that $x_b^* = 1, \forall b \in W$ and $x_b^* = 0, \forall b \notin W$. Moreover, $\sum_{k \in K} \delta_{bk} p_k = P_b, \forall b \in W$. Thus:

$$\sum_{b \in B^R} x_b^* \left(\sum_{k \in K} \delta_{bk} p_k \right) = \sum_{b \in W} \sum_{k \in K} \delta_{bk} p_k = \sum_{b \in W} P_b = C(A).$$

Thus, $P(A) = C(A) - \sum_{k \in K^+} p_k \mu_k$. Hence, one can assert that a contract-pricing approach satisfies the budget-balanced property if and only if at least one of the following conditions is met:

- $K^+ = \emptyset$ implying that each contract is covered only once.
- $K^+ \neq \emptyset$ and $\forall k \in K^+, p_k = 0$ (Since $\mu_k > 0, \forall k \in K^+$ and $\forall k \in K, p_k \geq 0$).□

Corollary 1 When the WDP has the integrality property, a contract-pricing approach will always satisfy the budget-balance property.

Proof: When the WDP has the integrality property, contract prices correspond to values of the optimal dual variables $(\Pi_k^*)_{k \in K}$ as shown in Section 3.1. In this case, if a contract k is served more than once (i.e., the corresponding covering constraint in $(LP2^R)$ is slack), then by dual conditions (7), $\Pi_k^* = 0$, which means that the price of serving this contract is zero. It is worth mentioning that budget balance in this case is a classical result in duality theory since, at optimum, primal and dual objective values must be identical. □.

Corollary 2 When the WDP is modeled as a set partitioning problem, a contract-pricing approach will always satisfy the budget-balance property.

Proof: In this case, each contract is covered exactly once □.

In the following, the budget balance property is ensured by simply adding to approximate models (AP^t), the budget balance constraint:

$$\sum_{k \in K} p_k = \sum_{b \in W} P_b. \tag{20}$$

5 Price Unicity

When participating together in the same procurement auction, buyers offer a large choice to sellers to combine contracts and to propose low ask prices in their bids. Consequently, buyers will realize gains when compared to the case where each buyer participates on its own in the same auction. This can be be considered as a form of collaboration between buyers, a concept addressed in cooperative game theory. In cooperative games, a set of players, also called agents, collaborate in order to obtain gains. A cooperative game is generally described by a rule that computes the gain realized by any coalition of players once it is formed. Cooperative game theory deals with sharing gains, or costs, among the players of the game.

In our context, the proposal-neutral property implies considering contracts rather than buyers when determining cost distribution. In other words, being present at the same time in the same market is a form of collaboration between contracts. This collaboration is of course primarily due to combinatorial bidding. Hence, we are in presence of a cooperative game in which the players are the contracts and the total cost to be shared is $C(A)$, the output of the auction process. The assignment of these individual costs must inevitably be consistent with the set of losing and winning bids arguing why some bids win and others not. In other words, the prices affected to contracts must lie within set $\mathcal{P}$. In the following, we propose four procedures to ensure the unicity of contract prices in situations where multiple solutions exist in $\mathcal{P}$. These procedures are inspired by two cooperative game theory concepts: the *nucleolus* and the *Shapley value* (see Boyer *et al.*, 2006, for example, for a survey on cost-sharing procedures).

5.1 Nucleolus-based procedures

The nucleolus is a classical cooperative game theory concept that aims at maximizing the social welfare of the least-satisfied coalition of players (e.g., Granot *et al.*, 1998). Let N be the set of all the players of the game, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be the cost incurred by each player $i, i = 1, \ldots, n$ output by some cost-sharing procedure. For each sub-coalition S of players, let $g(\alpha, S)$ represent its "satisfaction" with allocation α. In general, a coalition satisfaction is measured through the *marginal gain* obtained by sub-coalition S with allocation α. Formally, $g(\alpha, S) = C(S) - \sum_{i \in S} \alpha_i$, where $C(S)$ is the total cost incurred by sub-coalition S if working on its own. Define also $g(\alpha)$ as the vector of the $g(\alpha, S)$ values for the $2^n - 2$ possible sub-coalitions of players, placed in a nondecreasing order with respect to their values. The nucleolus is defined as the unique allocation α^* that lexicographically maximizes $g(\alpha)$. In other words, α^* is the allocation that maximizes the lowest coalition gain, the second lowest gain, etc. The unicity procedures we propose next are inspired by this concept of maximizing the satisfaction of the least satisfied coalitions without properly applying the nucleolus method in its classical definition.

In our context, we consider a game in which the players are the contracts and a cost allocation consists in choosing a feasible price vector p among the set of vectors $\mathcal{P}$. To circumvent the difficultly related to enumerating and determining all coalition costs, we consider only single-player coalitions, i.e., coalitions that are composed of a unique contract. Moreover, we prove next that $\mathcal{P}$ is a bounded convex set implying thus that a feasible price for a contract k necessarily lies within a finite interval $[\underline{p}_k, \overline{p}_k]$, where $\underline{p}_k$, respectively $\overline{p}_k$, denotes the minimum, respectively the maximum, price that could be assigned to contract k within set $\mathcal{P}$. When prices are chosen arbitrarily, a contract is likely to be attributed its maximum, minimum or any other price within $[\underline{p}_k, \overline{p}_k]$. Based on this, we propose two ways for defining the satisfaction associated with a contract k and an allocation $\tilde{p} \in \mathcal{P}$, denoted here after by $g(\tilde{p}, k)$. First, $g(\tilde{p}, k)$ can be computed as the marginal gain obtained by contract k with respect to its maximum price. Formally, $g(\tilde{p}, k) = \overline{p}_k - \tilde{p}_k$. In this case, we consider that when no appropriate unicity procedure is used, a contract is assigned its maximum price $\overline{p}_k$. Second, $g(\tilde{p}, k)$ can be represented by the distance between $\tilde{p}_k$ and the minimum cost that could be assigned to k within $\mathcal{P}$, $\underline{p}_k$. In this case, we assume that a contract receives its minimum price when arbitrary choices are made. It is more appropriate then to speak of *marginal loss* rather than marginal gain since the price assigned to the contract with allocation $\tilde{p}$ is necessarily greater than or equal to its minimum price. The marginal loss of a contract k with allocation $\tilde{p}$ is given by: $l(\tilde{p}, k) = \tilde{p}_k - \underline{p}_k$.

Proposition 2 The set $\mathcal{P}$ of feasible contract prices output either by the exact or the approximate contract-pricing phase is a bounded convex set.

Proof: When the WDP has the integrality property, we established in Section 3.1 that $\mathcal{P}$ is defined by the constraints defining the dual of the LP relaxation of the WDP which can be written as:

$$\begin{aligned} \sum_{k \in K} \delta_{bk} p_k &= P_b \qquad \forall b \in W, \\ \sum_{k \in K} \delta_{bk} p_k &\leq P_b \qquad \forall b \in E, \\ p_k &\geq 0 \qquad \forall k \in K. \end{aligned}$$

When the approximate phase is necessary, $\mathcal{P}$ is defined by the set of constraints:

$$\sum_{k \in K} \delta_{bk} p_k = P_b \qquad \forall b \in W, \tag{21}$$

$$\sum_{k \in K} \delta_{bk} p_k = P_b + \Delta_b^* \qquad \forall b \in \hat{E}, \tag{22}$$

$$\sum_{k \in K} \delta_{bk} p_k \leq P_b \qquad \forall b \in E \setminus \hat{E}, \tag{23}$$

$$p_k \geq 0 \qquad \forall k \in K, \tag{24}$$

where Δ^* is the vector of optimal distortions output by the approximate contract-pricing phase (refer to Section 3.2). Thus, in both definitions, $\mathcal{P}$ consists of a set of linear inequalities implying that it is a polyhedron and thus a convex set (Nemhauser and Wolsey, 1988).

Moreover, recall that a reserve price M_k is associated with each contract k and that simple bids $B^a = \{(k, M_k), k \in K\}$ are considered at each round of the auction to ensure the feasibility of the WDP. Hence, $\forall p \in \mathcal{P}$ and $\forall k \in K$:

- If (k, M_k) is a winning bid at the final round R, then $p_k = M_k$.
- If (k, M_k) is a losing bid at the final round R, then:
  - if the WDP has the integrality property, $\mathcal{P}$ is defined by dual constraints and we have $0 \leq p_k \leq M_k$.
  - Otherwise, $\mathcal{P}$ is defined by the set of constraints (21)-(24) and we have either : $0 \leq p_k \leq M_k$ or $p_k = \Delta^*_{(k,M_k)} + M_k$

In all cases, we prove that $\forall p \in \mathcal{P}$ and $\forall k \in K$, either $0 \leq p_k \leq M_k$ or p_k equals a finite value. Thus $\mathcal{P}$ is bounded. $\square$

Based on the above definitions of marginal gain and marginal loss, two cost allocation procedures are derived, each determining the "nucleolus" with respect to its respective definition of "player satisfaction".

5.1.1 Procedure 1: maximizing the minimal marginal gain

The satisfaction associated with a contract k with an allocation $\tilde{p}$ is given by its marginal gain $g(\tilde{p}, k) = \overline{p}_k - \tilde{p}_k$, where $\overline{p}_k$ is the maximum price that can be attributed to contract k given the set of vector prices $\mathcal{P}$. These maximum prices are obtained by solving a series of LP problems, one for each contract k in K. The LP problem corresponding to contract k is given by: $\{\overline{p}_k = \max p_k$ s.t. $p \in \mathcal{P}$ and $\sum_{k \in K} p_k = \sum_{b \in W} P_b\}$.

The object of Procedure 1 is to assign prices to contracts within set $\mathcal{P}$ in a way that maximizes the minimum distance with respect to contract maximum prices. This is done iteratively until all contract prices are fixed. One avoids thus situations where some contracts are assigned their maximum prices where as other contracts receive their minimum prices. To use a cooperative game theory jargon, let $g(\tilde{p})$ be the vector of the $g(\tilde{p}, k) = \overline{p}_k - \tilde{p}_k$ values obtained with allocation $\tilde{p}$ for the $|K|$ single-contract sub-coalitions, these values being placed in non-decreasing order. The procedure we propose consists in finding the allocation, or equivalently the vector $\tilde{p}$ in $\mathcal{P}$, that lexicographically maximizes $g(\tilde{p})$. This, in fact, reduces to iteratively solving a series of LP problems. The LP problem solved at iteration 0 is given by:

$$(PRO1^0) : \max w \tag{25}$$

$$\text{s.t.} \quad p \in \mathcal{P}, \tag{26}$$

$$\sum_{k \in K} p_k = \sum_{b \in W} P_b, \tag{27}$$

$$p_k + w \leq \overline{p}_k, \qquad \forall k \in K \tag{28}$$

$$w \geq 0. \tag{29}$$

In this model, continuous variable w represents the minimal contract marginal gain given the set of multiple price vectors $\mathcal{P}$ and the budget-balance constraint (27). The objective function (25) maximizes this minimal gain. At each iteration t, a contract k for which constraint (28) was tight at the previous iteration $(t-1)$ is allocated a fixed price $\tilde{p}_k$. More precisely, $\tilde{p}_k = \overline{p}_k - w^{*,t-1}$, where $w^{*,t-1}$ is the optimal objective function resulting from solving the LP model at iteration $(t-1)$. The set of contracts for which prices are fixed is denoted by $\tilde{K}$. The process continues until all contract prices are fixed, i.e., $\tilde{K} = K$.

Formally, the LP model considered at iteration t is formulated as follows:

$$
\begin{aligned}
(PRO1^t) : \max w & & & \\
\text{s.t.} \quad p & \in & \mathcal{P}, & \\
\sum_{k \in K} p_k & = & \sum_{b \in W} P_b, & \\
p_k & = & \overline{p}_k - w^{*,t-1} & \quad \forall k \in \tilde{K}, \\
p_k + w & \leq & \overline{p}_k & \quad \forall k \in K \setminus \tilde{K}, \\
w & \geq & 0. &
\end{aligned}
$$

5.1.2 Procedure 2: minimizing the maximal marginal loss

This procedure is based on the same idea as Procedure 1 except that the coalition satisfaction is defined in a different way. For this approach, a contract k is fully satisfied with a given allocation $\tilde{p} \in \mathcal{P}$ if it is assigned its minimum cost $\underline{p}_k$. We thus defined the concept of marginal loss of a single-contract coalition k, with an allocation $\tilde{p}$, as the difference between its price with allocation $\tilde{p}$, $\tilde{p}_k$, and its minimal price $\underline{p}_k$. As in Procedure 1, the minimum prices $\underline{p}_k, k \in K$ are obtained by solving a series of LP problems one for each contract $k \in K$. The LP problem associated with a contract $k \in K$ is given by: $\{\underline{p}_k = \min p_k \text{ s.t. } p \in \mathcal{P}$ and $\sum_{k \in K} p_k = \sum_{b \in W} P_b\}$. Marginal losses of all contracts are then lexicographically minimized by iteratively solving a series of LP problems. The problem solved at iteration 0 is given by:

$$
\begin{aligned}
(PRO2^0) : \min \beta & & & & (30)\\
\text{s.t.} \quad p & \in & \mathcal{P}, & & (31)\\
\sum_{k \in K} p_k & = & \sum_{b \in W} P_b, & & (32)\\
p_k - \beta & \leq & \underline{p}_k & \quad \forall k \in K, & (33)\\
\beta & \geq & 0, & & (34)
\end{aligned}
$$

where β represents the maximal deviation with respect to the minimum price for all contracts. At each iteration t, the contracts k for which constraints (33) are tight are allocated a fixed price $\tilde{p}_k = \underline{p}_k + \beta^{*,t}$, where $\beta^{*,t}$ is the optimal objective function obtained after solving the model of iteration t. The process is iterated until all prices are fixed.

5.2 Shapley value-based procedures

The Shapley value is a very common cost-sharing procedure in cooperative game theory essentially based on the so-called *incremental costs* (Shapley, 1953; Shapley and Shubik, 1969). The incremental cost is defined for a coalition of players S and a player $i \notin S$ as the additional cost incurred by the new coalition $S \cup i$ when player i joins S. Formally, if $C(S)$ denotes the cost incurred by a coalition S in the given game, the incremental cost associated with coalition S and player i is given by: $\overline{C}(S, i) = C(S \cup i) - C(S)$. Assume that we consider a cost-sharing procedure in which each player pays its incremental cost for joining the players already in the game. For example, consider a game with three players, i_1, i_2 and i_3. Assume that player i_1 is the first player of the game, i_2 is the second player to join the game and player i_3 is the last one. Then, player i_1 is allocated a cost $C(\{i_1\})$, player i_2 is allocated a cost $C(\{i_1, i_2\}) - C(\{i_1\})$, and player i_3 a cost $C(\{i_1, i_2, i_3\}) - C(\{i_1, i_2\})$. Obviously, one should consider all the alternatives for the order of arrival of the players in the game. The Shapley value indeed assumes that this order of arrival is random and the probability that a player joins first, second, third, etc. a coalition is the same for all players. The cost allocated to a player i is computed as the expected mean value of its incremental costs for the different orders of arrival. More generally, the cost allocated to a player i in a game including a set N of players is given by:

$$c_i = \sum_{S \subseteq N : i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} [C(S) - C(S \setminus \{i\})],$$

or equivalently,

$$c_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [C(S \cup \{i\}) - C(S)].$$

Recall that in our context, we are given a set of multiple contract prices $\mathcal{P}$, and we need to fix prices in a unique way so as to derive fair payments for buyers. We propose hereafter an approach that adapts the Shapley rule concept to our problem.

Unlike traditional games, computing a cost associated with each coalition of contracts is not straightforward in our case. Moreover, when the number of contracts is too large, enumerating all possible sub-coalitions becomes time consuming. To circumvent such a difficulty, we propose to adapt the Shapley value concept to a series of restricted games. A restricted game is defined for each subset of contracts K_b corresponding to a winning bid b in W^R and the corresponding total cost to be shared is P_b, the price asked in bid b. We refer to such a game $G(K_b)$.

This idea of separating the original large game $G(K)$ into small sub-games is due to the "receive-as-bid" pricing scheme adopted for the auction. In fact, the total cost $C(A)$ to be shared among the contracts corresponds to the sum of the asked prices of the winning bids. Thus, assigning costs to contracts may be reduced to sharing each asked price P_b of a winning bid b in W^R among the contracts covered by this bid. Applying the Shapley value to these separate sub-games $G(K_b), b \in W$, obviously reduces the number of sub-coalitions to be considered and thus the combinatorial dimension of the problem when compared to the case where the whole game $G(K)$ is considered.

Formally, consider a sub-game $G(K_b)$ associated with a given winning bid $b \in W^R$. The Shapley rule assumes that one can determine the cost incurred by each sub-coalition of players in K_b. Let S be a sub-coalition of K_b composed of l contracts $k_1, k_2, ..., k_l$. One way of determining the cost incurred by S when working on its own is to determine the maximum total price that could be attributed to $\{k_1, k_2, ..., k_r\}$ given that individual prices are in $\mathcal{P}$.

Formally, the cost associated with S with this first method, denoted by $C^1(S)$, is given by: $C^1(S) = Max\{\sum_{i=1}^{l} p_{k_i} : p \in \mathcal{P} \text{ and } \sum_{k\in K} p_k = \sum_{b\in W} P_b\}$.

Thus, the price allocated to a contract $k \in K_b$ when considering cost function C^1 is given by:

$$p_k^1 = \sum_{S\subset K_b : k\in S} \frac{(|S|-1)!(|K_b|-|S|)!}{|K_b|!}[C^1(S) - C^1(S\setminus\{k\})].$$

A second way of determining the cost incurred by a sub-coalition $S = \{k_1, k_2, ..., k_l\}$ of K_b is to compute the total minimum price that could allocated to $k_1, k_2, ..., k_l$ within $\mathcal{P}$. That is, $C^2(S) = Min\{\sum_{i=1}^{l} p_{k_i} : p \in \mathcal{P} \text{ and } \sum_{k\in K} p_k = \sum_{b\in W} P_b\}$. The price allocated to a contract $k \in K_b$, in this case, is given by:

$$p_k^2 = \sum_{S\subset K_b : k\in S} \frac{(|S|-1)!(|K_b|-|S|)!}{|K_b|!}[C^2(S) - C^2(S\setminus\{k\})].$$

It is well known that, when applied to a given game, the Shapley value results in a unique cost distribution (Shapley, 1953; Shapley and Shubik, 1969). In our case, we apply the Shapley value concept to a series of sub-games $G(K_b)$ for each winning bid b. These winning bids are output by a WDP that we formulated as a set covering problem enabling thus a contract to be covered more than once. In other words, it may happen that a contract k is covered by two winning bids b^1 and b^2 in the final round. This would imply that contract k is allocated two different prices, a first price deduced from applying the Shapley rule to game $G(K_{b^1})$ and a second price derived from game $G(K_{b^2})$. We prove next, that the Shapley-based procedures we propose yield in fact unique contract prices.

Proposition 3 The Shapley-based procedure using either cost functions C^1 or C^2 determines contract prices in a unique way.

Proof: The proof is based on the observation that when a contract k is covered more than once, its price is necessarily equal to 0.

In fact, when the WDP has the integrality property, by duality theory, it is well known that the dual variable associated with a slack constraint is necessary null (refer to the proof of Proposition 1).

For general WDPs, we established in the proof of Proposition 1 (Section 4) that $P(A) = C(A) - \sum_{k \in K^+} p_k \mu_k$, where $P(A)$ is the total price to be paid by buyers with a contract-pricing approach, $C(A)$ is the total payment that must be done to winning bidders and μ_k is the number of times a contract k is over covered. Moreover, in order to satisfy the budget balance property, we incorporated the equality constraint $C(A) = P(A)$ in all unicity models. Thus, $\sum_{k \in K^+} p_k \mu_k = 0$.

Recall that $\forall k \in K, p_k \geq 0$ and in case where $K^+ \neq \emptyset$, $\mu_k > 0, \forall k \in K^+$. Thus $\forall k \in K^+, p_k = 0$. □

Proposition 4 When considering the Shapley-based procedure, cost functions C^1 and C^2 yield the same prices allocation. That is, $\forall k \in K$, $p_k^1 = p_k^2$.

Proof: Consider a contract $k \in K$ and let W_k be the set of winning bids covering it.

- If $|W_k| > 1$, the proof of Proposition 3 shows that $p_k = 0$ independently of the contract-pricing approach used. Thus, $p_k^1 = p_k^2 = 0$.
- If $|W_k| = 1$, let $b = (K_b, P_b)$ denote the winning bid covering contract k, i.e., $k \in K_b$.
  - If K_b includes only one contract (i.e. b is a simple bid), the receive-as-bid rule implies that $p_k = P_b$ and there is no need to apply the unicity procedures.
  - Consider thus the non-trivial case where $|K_b| > 1$. Cost function C^1 yields a price p_k^1 for k equal to :

$$p_k^1 = \sum_{S \subset K_b : k \in S} \frac{(|S|-1)!(|K_b|-|S|)!}{|K_b|!} [C^1(S) - C^1(S \setminus \{k\})] \tag{35}$$

Let S, be a subset of K_b including k.
We have, $C^1(S) = Max\{\sum_{k \in S} p_k : p \in \mathcal{P}$ and $\sum_{k \in K} p_k = \sum_{b \in W} P_b\}$.
Recall that, given the receive-as-bid rule, $\sum_{k \in K_b} = P_b$.

Thus, $\sum_{k\in S} p_k = P_b - \sum_{k\in K_b\setminus S} p_k$ and

$$\begin{aligned}
C^1(S) &= Max\{\sum_{k\in S} p_k : p\in\mathcal{P} \text{ and } \sum_{k\in K} p_k = \sum_{b\in W} P_b\} \\
&= Max\{P_b - \sum_{k\in(K_b\setminus S)} p_k : p\in\mathcal{P} \text{ and } \sum_{k\in K} p_k = \sum_{b\in W} P_b\} \\
&= P_b - Min\{\sum_{k\in(K_b\setminus S)} p_k : p\in\mathcal{P} \text{ and } \sum_{k\in K} p_k = \sum_{b\in W} P_b\} \\
&= P_b - C^2(K_b\setminus S)
\end{aligned}$$

Replacing $C^1(S)$ by $(P_b - C^2(K_b\setminus S))$ in equation (35), we obtain:

$$\begin{aligned}
p_k^1 &= \sum_{S\subset K_b:k\in S} \frac{(|S|-1)!(|K_b|-|S|)!}{|K_b|!}[(P_b - C^2(K_b\setminus S)) - (P_b - C^2((K_b\setminus S)\cup\{k\}))] \\
&= \sum_{S\subset K_b:k\in S} \frac{(|S|-1)!(|K_b|-|S|)!}{|K_b|!}[C^2((K_b\setminus S)\cup\{k\}) - C^2(K_b\setminus S)] \\
&= \sum_{S'\subset K_b:k\notin S'} \frac{(|K_b\setminus S'|-1)!|S'|!}{|K_b|!}[C^2(S'\cup\{k\}) - C^2(S')] \\
&= \sum_{S'\subset K_b\setminus\{k\}} \frac{(|K_b|-|S'|-1)!|S'|!}{|K_b|!}[C^2(S'\cup\{k\}) - C^2(S')] \qquad (36)
\end{aligned}$$

Equality (36) corresponds in fact to the equivalent formulation of the Shapley value rewritten with cost function C^2 for contract k. Thus, $p_k^1 = p_k^2$. □

In the following, the Shapley-based procedure described above using cost function C^1 (or equivalently C^2) is referred to as Procedure 3.

Relying on the observations made in this section, both nucleolus-based and Shapley-based procedures can be accelerated by fixing some contracts prices before running them. First, the set of contracts K^+ can be easily deduced from solving the WDP at the final round R (determining slack constraints (3)). The prices of all contracts in K^+ are then fixed to 0 independently of the procedure used. This would especially be beneficial for the Shapley-based procedures for which the size of the grand coalition has a major impact on computing performances. Second, one can fix the price of each contract k for which $\underline{p}_k = \overline{p}_k$: for these contracts, prices are already unique.

6 Computational Experiments

In previous sections, we proposed three cost-allocation procedures, Procedures 1, 2 and 3, for bilateral markets where one-sided combinatorial auctions are used as trading mechanisms. These procedures are conceived for satisfying three properties: proposal neutral, budget balance and fairness. In this section, we define criteria for evaluating the quality of the allocation yielded by the proposed procedures. These performance measures are then used to compare the results obtained with the three procedures on a large set of instances.

6.1 Criteria for Evaluating Cost-Allocation Procedures

In order to satisfy the proposal-neutral property, the proposed procedures are based on a contract-pricing approach in which prices are assigned to contracts in a unique way and the payment to be done by a buyer corresponds to the sum of the prices of the contracts it requested. The procedures differ from one another in the way contract prices are fixed when multiple contract-price vectors are yielded by either the exact or the approximate contract-pricing phase (i.e., set $\mathcal{P}$ includes more than one element). However, one should keep in mind that these procedures will finally be used to determine the price to be paid by each buyer individually given the set of contracts it submitted. Relying on this, we propose to compare the proposed procedures on a buyer rather than a contract level.

One can determine for each buyer $a \in A$, the minimum and maximum prices it could pay given the set of multiple contract price vectors $\mathcal{P}$. The minimum price is given by:

$$\underline{C}(a) = \min\{\sum_{k \in K(a)} p_k \text{ s.t. } p \in \mathcal{P} \text{ and } \sum_{k \in K} p_k = \sum_{b \in W} P_b\} \qquad \forall a \in A,$$

and the maximum price is given by:

$$\overline{C}(a) = \max\{\sum_{k \in K(a)} p_k \text{ s.t. } p \in \mathcal{P} \text{ and } \sum_{k \in K} p_k = \sum_{b \in W} P_b\} \qquad \forall a \in A.$$

Observe that set $\mathcal{P}$ is bounded. Consequently, this minimum and maximum values are necessarily finite.

Hence, within $\mathcal{P}$, a buyer a would pay on average

$$\hat{C}(a) = \frac{\underline{C}(a) + \overline{C}(a)}{2},$$

which, represents $\frac{\hat{C}(a)}{C(A)}\%$ of the total ask price. In the following, this percentage is referred to as the *buyer central payment*.

To measure the fairness of a given unicity procedure, we propose to compute, for each buyer, the *relative gap* between the cost allocated to it with this procedure and its central payment. For a procedure χ and a buyer $a \in A$, this relative gap is given by:

$$G(\chi, a) = \frac{\frac{\hat{C}(a)}{C(A)} - \frac{\chi(a)}{C(A)}}{\frac{\hat{C}(a)}{C(A)}} = \frac{\hat{C}(a) - \chi(a)}{\hat{C}(a)}.$$

To compare Procedures 1, 2 and 3, we compute for each the average relative gap it yields with respect to buyers central payments as well as the corresponding standard deviation. In other words, we consider the following measures:

$$\hat{E}(\chi) = \frac{1}{|A|} \sum_{a \in A} G(\chi, a).$$

$$\hat{\sigma}(\chi) = \sqrt{\frac{\sum_{a \in A} (G(\chi, a) - \hat{E}(\chi))^2}{|A|}}.$$

Given these two measures, one can have a good idea on how the payments allocated to buyers are spread around their central payments. More precisely, a procedure for which both the average relative gap and the standard deviation take small values assigns costs to buyers that are close to their central payments without favoring one buyer over another. On the opposite, a procedure for which the standard deviation is large, even though it results in a relatively small average relative gap, suggests that this procedure assigns low costs to some buyers relatively to their central payments and very large costs to other ones.

6.2 Problem Tests

One-sided one-shot combinatorial auction instances can be generated easily using the CATS generator developed by Leyton-Brown *et al.* (2000). CATS consists in a suite of "distribution families for generating realistic, economically motivated combinatorial bids". Combinatorial bids are constructed with respect to some adjacency relationships between the auctioned items arising in real-life situations.

In our context, we opted for the *PATHS* distribution since it better fits procurement markets. For this distribution, complementarities are based on adjacency in space. Buyers can thus be viewed as shippers who need to move some commodities from some origins to some destinations, bidders are the carriers who make offers in form of bids to win shipping contracts, and the auctioned items are the origin-destination pairs.

We used the CATS suite as a black box to which we passed as input the following parameters:

- the number of auctioned items (i.e., the number of contracts, $|K|$),
- the number of bids to generate ($|B|$).

Since the generator yields one-sided combinatorial auctions for unilateral markets, to handle the bilateral aspect, we fixed the number of participating buyers, $|A|$, and randomly assigned contracts (the ones already defined in CATS) to them.

We tested Procedures 1, 2 and 3 on different problem settings (21 in total) obtained with CATS through varying the number of auctioned items, $|K|$, the number of bids, $|B|$, and the number of buyers $|A|$. A total of 2100 single-round combinatorial auctions were generated, 100 instances for each of the 21 problem settings. These instances were such that winning bids include no more than seven contracts. In fact, the purpose of this experimental study was to compare the quality of the solutions yielded by the proposed unicity procedures. Obviously, when the number of contracts in a winning bid is relatively large, the Shapley value-based procedures would require large computing times since the number of sub-coalitions grows exponentially with the number of contracts in winning bids. Such an assumption remains realistic in some trading contexts where the number of contracts won by a given seller may be restricted, thus avoiding one seller winning "the lion's share" (Caplice and Sheffi, 2006).

Table 1 hereafter describes the instances considered. As mentioned in Section 6.1, one can determine, for each buyer $a \in A$ and each instance i, the minimum and maximum prices, $[\underline{C^i}(a), \overline{C^i}(a))]$, that it could pay given the set of multiple contract prices. Hence, to measure the variation in the payments that could be allocated to a buyer a, one can simply compute the ratio $V(a,i) = \frac{(\overline{C^i}(a) - \underline{C^i}(a))}{C(A)}$ which gives this variation relatively to the total cost. Obviously, a large variation in a buyer payment makes the problem of choosing good cost allocation procedures more relevant.

Thus, in order to describe the instances considered, we determine for each problem setting $(|K|, |B|, |A|)$, the variation in the payments that could be attributed to a buyer on average for an instance. These amounts are reported in the column denoted by "A.V.". Formally, the average variation associated with a problem setting $(|K|, |B|, |A|)$ is given by:

$$A.V.(|K|, |B|, |A|) = \frac{1}{100} \sum_{i=1}^{100} \sum_{a \in A} \frac{(\overline{C^i}(a) - \underline{C^i}(a))}{|A| \times C(A)}.$$

In the second column denoted by "PIP", we report the percentage of instances for which the winner determination problem has the integrality property. For these instances, contract prices correspond to the optimal dual values and there is no need to use the approximate contract-pricing phase.

$(\|K\|,\|B\|,\|A\|)$	$A.V.$	PIP
(20,60,3)	3.61	19
(20,60,4)	2.98	11
(20,60,5)	2.91	15
(40,120,3)	2.34	8
(40,120,4)	3.23	2
(40,120,5)	3.05	1
(60,180,4)	3.39	1
(60,180,5)	2.62	2
(60,180,6)	2.48	0
(80,240,4)	3.15	0
(80,240,5)	2.80	1
(80,240,6)	2.40	2
(100,300,5)	2.30	0
(100,300,6)	2.34	0
(100,300,7)	1.98	1
(150,450,5)	2.31	0
(150,450,6)	2.09	0
(150,450,7)	1.77	0
(200,600,6)	1.90	0
(200,600,7)	1.48	0
(200,600,8)	1.56	0

Table 1: Description of problem tests

6.3 Results

The branch-and-bound algorithm of CPLEX 10.1 was applied to all formulations on a 1.8 GHz Pentium III PC. The results obtained with the three procedures are summarized in Tables 2 and 3. Table 2 displays for each procedure and each problem setting, the average values (on the 100 instances) of the performance measures described in Section 6.1. Thus, column $\hat{E}(.)$ associated with a problem setting $(|K|,|B|,|A|)$ and a procedure χ, reports the average relative gap with regard to buyers central payments over the 100 instances of $(|K|,|B|,|A|)$. That is,

$$\hat{E}((|K|,|B|,|A|),\chi) = \frac{1}{100}\sum_{i\in(|K|,|B|,|A|)} \hat{E}(\chi,i).$$

Similarly,

$$\hat{\sigma}((|K|,|B|,|A|),\chi) = \frac{1}{100}\sum_{i\in(|K|,|B|,|A|)} \hat{\sigma}(\chi,i).$$

One can see from Table 2 that Procedure 1 outperforms the other procedures in terms

Problem setting	Procedure 1		Procedure 2		Procedure 3	
	$\hat{E}(.)$	$\hat{\sigma}(.)$	$\hat{E}(.)$	$\hat{\sigma}(.)$	$\hat{E}(.)$	$\hat{\sigma}(.)$
(20,60,3)	- 0.06	1.12	- 0.06	4.29	- 0.03	0.48
(20,60,4)	0.04	1.92	0.32	7.02	- 0.10	0.59
(20,60,5)	- 0.05	1.94	0.57	8.00	- 0.13	0.66
(40,120,3)	0.32	3.39	0.81	6.81	- 0.08	0.59
(40,120,4)	0.09	1.78	0.26	5.27	- 0.05	0.83
(40,120,5)	0.02	2.28	0.31	6.50	- 0.11	1.04
(60,180,4)	0.00	1.44	0.20	4.48	- 0.05	0.64
(60,180,5)	0.02	2.71	0.50	7.60	- 0.13	0.83
(60,180,6)	0.10	2.42	0.43	7.17	- 0.04	1.04
(80,240,4)	0.06	1.43	0.29	3.80	- 0.02	0.67
(80,240,5)	0.14	1.78	0.29	4.56	- 0.04	0.67
(80,240,6)	0.19	3.28	0.86	8.03	- 0.03	1.04
(100,300,5)	0.19	4.42	1.26	10.39	- 0.20	1.10
(100,300,6)	0.04	0.11	0.16	0.09	- 0.04	11.95
(100,300,7)	0.29	3.41	0.28	7.44	- 0.02	0.85
(150,450,5)	0.11	1.61	- 0.08	4.55	0.09	0.62
(150,450,6)	0.03	1.64	0.01	4.09	0.05	0.58
(150,450,7)	0.29	0.20	0.66	0.15	0.06	35.13
(200,600,6)	0.03	1.54	- 0.13	3.71	0.01	0.43
(200,600,7)	0.29	2.60	- 0.04	5.67	0.04	0.41
(200,600,8)	0.03	1.83	- 0.05	5.19	0.05	0.51
Average	0.10	2.04	0.33	5.47	-0.04	2.89

Table 2: Average relative gap and average standard deviations for procedures 1, 2 and 3

of the quality of the allocations yielded. In fact, Procedure 1 yields an average relative gap $\hat{E}$ that varies between -0.06% (for problem setting $(20, 60, 3)$) and 0.29% (for problem settings $(100, 300, 7)$, $(150, 450, 7)$ and $(200, 600, 8)$) resulting in a total average of 0.10% for all problem settings. Moreover, the corresponding average standard deviation $\hat{\sigma}$ does not exceed 3.41% (problem setting $(100, 300, 7)$) with a minimum value of 0.11% (obtained for problem setting $(100, 300, 6)$). These results show that, for the instances considered, Procedure 1 sets contract prices in a way that ensures all buyers to pay an amount relatively close to their central payments. Although Procedures 2 and 3 yield an average relative gap that is also relatively small; 0.33% on average for Procedure 2 and -0.04% for Procedure 3, there is an important variation of the average standard deviations throughout the problem settings. Observe, for instance, the results obtained for Procedure 2 in problem setting $(100, 300, 5)$: the average relative gap equals 1.26% while the average standard deviation reaches 10.39%. This suggests that even though, with Procedure 2, a buyer, on average, pays an amount that is 1.26% lower than its corresponding central payment, the payments allocated vary considerably from one buyer to another. The same behavior is observed for

Procedure 3 in problem setting $(150, 450, 7)$ where the average standard deviation exceeds 35.13%.

Table 3 reports the average computing time, in seconds, required by each procedure for each problem setting. Procedures 1 and 2 need no more than 0.13 seconds on average. For large instances in which 600 contracts are considered, these computing times remain relatively small for both procedures: 0.46 sec. on average for Procedure 1 and 0.37 sec. on average for Procedure 2. When compared to Procedures 1 and 2, Procedure 3 shows worse computing times especially for the large problems where the average computing time exceeds 102.41 sec. However, on the $2,100$ instances considered, Procedure 3 requires no more than 19 sec. on average, which represents very small computing times. As a conclusion, all three procedures need relatively short computing times enabling one to use them in practice for real-life applications.

Problem setting	Procedure 1	Procedure 2	Procedure 3
(20,60,3)	0.01	0.01	0.01
(20,60,4)	0.01	0.01	0.01
(20,60,5)	0.01	0.01	0.01
(40,120,3)	0.03	0.02	0.08
(40,120,4)	0.02	0.02	0.09
(40,120,5)	0.02	0.02	0.09
(60,180,4)	0.04	0.03	0.73
(60,180,5)	0.04	0.03	1.03
(60,180,6)	0.04	0.04	0.77
(80,240,4)	0.09	0.07	3.89
(80,240,5)	0.09	0.07	4.82
(80,240,6)	0.09	0.07	4.67
(100,300,5)	0.14	0.11	12.13
(100,300,6)	0.11	0.10	11.95
(100,300,7)	0.13	0.10	11.69
(150,450,5)	0.30	0.23	45.10
(150,450,6)	0.30	0.23	44.65
(150,450,7)	0.20	0.15	35.13
(200,600,6)	0.28	0.21	50.02
(200,600,7)	0.27	0.22	49.91
(200,600,8)	0.46	0.37	102.41
Average	0.13	0.10	18.06

Table 3: Average solution times (in seconds) for Procedures 1 ,2 and 3

7 Conclusion

This paper considers a bilateral procurement market in which multiple buyers and sellers trade heterogeneous indivisible contracts. The market bilateral aspect is initially ignored and the trading process is modeled as a classical one-sided combinatorial reverse auction where sellers are the only bidders. At the end of the auction process, given the winning bids and the "receive-as-bid" pricing rule, one can determine the total price to be paid by all buyers. At this stage, the bilateral aspect is reconsidered to allocate costs to buyers individually.

Three cost allocation procedures were proposed. All these procedures are proposal-neutral in the sense that the cost allocated to a buyer is independent of its identity. To handle this, a contract-pricing approach was considered implying that prices are allocated to single contracts rather than buyers. The price to be paid by a buyer is then computed as the sum of the prices of the contracts it requested. Even though such an approach is closely related to the problem of determining linear prices in combinatorial auctions, this paper focuses on the unicity of contract prices. It presents a new approach for fixing prices when multiple feasible linear prices exist. Unlike previous unicity methods, we propose new procedures defined with respect to the interests of buyers rather than bidders. Moreover, these procedures are defined so as to ensure a balanced budget, that is, a total payment for buyers that is equal to the total ask price output by the auction process. Relying on the observation that simultaneous participation of the buyers is a form of collaboration, the proposed unicity procedures are inspired by two cooperative game theory concepts: the nucleolus and the Shapley value.

The proposed procedures were compared through a set of instances generated with varying the numbers of buyers, contracts, and submitted bids. The procedures were compared with respect to solution time and quality. The results obtained prove that the three procedures yield cost allocations in relatively short computing times. Regarding solution quality, we define two performance measures with respect to buyer central payments. A buyer central payment corresponds to the central value of its min-max payment interval defined over the set of multiple contract price vectors. For the instances considered, Procedure 1 obtains the best results.

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References

ABRACHE, J., BOURBEAU, B., CRAINIC, T.G. AND GENDREAU, M., 2004. A New Bidding Framework for Combinatorial E-Auctions. *Computers & Operations Research*, 31:1177–1203.

ABRACHE, J., CRAINIC, T.G., GENDREAU, M. AND REKIK, M., 2007. Combinatorial Auctions. *Annals of Operations Research*, 153(1):131–164.

ANANDALINGAN, G., DAY, R.W. AND RAGHAVAN, S., 2005. The Landscape of Electronic Market Design. *Management Science*, 51(3):316–327.

AUSUBEL, L.M. AND MILGROM, P., 2002. Ascending Auctions with Package Bidding. *Frontiers of Theoretical Economics*, 1:1–45.

AUSUBEL, L.M. AND MILGROM, P. The lovely but Lonely Vickrey Auction. In *Combinatorial Auctions*, chapter 1, pages 17–40. MIT Press, 2006.

BICHLER, M., SHABALIN, P. AND PIKOVSKY, A., 2009. A Computational Analysis of Linear Price Iterative Combinatorial Auction Formats. *Information Systems Research*, 20(1):33–59.

BOURBEAU, B., CRAINIC, T.G., GENDREAU, M. AND ROBERT, J., 2005. Design for Optimized Multi-lateral Multi-commodity Markets. *European Journal of Operational Research*, 163:503–529.

BOYER, M., MOREAUX, M. AND TRUCHON, M. Partage des coûts et tarification des infrastructures. Publication 2006MO-01, CIRANO, Canada, 2006.

CAPLICE, C. AND SHEFFI, Y. Combinatorial Auctions for Truckload Transportation. In *Combinatorial Auctions*, pages 539–571. MIT Press, 2006.

CAVALLO, R., PARKES, D.C., JUDA, A.I., KIRSH, A., KULESZA, A., LAHAIE, S., LUBIN, B., MICHAEL, L. AND SHNEIDMAN, J. TBBL: A Tree-Based Bidding Language for Iterative Combinatorial Exchanges. In *The Multidisciplinary Workshop on Advance in Preference Handling (IJCAI)*. 2005.

CHU, L. YONG, 2009. Truthful Bundle/Multiunit Double Auctions. *Management Science*, 55(7):1184–1198.

CLARKE, E.H., 1971. Multipart Pricing of Public Goods. *Public Choice*, 11:17–33.

Day, R.W. and Raghavan, S., 2007. Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions. *Management Science*, 53(9):1389–1406.

de Vries, S. and Vohra, S., 2003. Combinatorial Auctions: A Survey. *INFORMS Journal on Computing*, 15(3):284–309.

Fan, M., Srinivasan, S., Stallaert, J. and Whinston, A.B. *Electronic Commerce and the Revolution in Financial Markets*. South-western Thomson Learning, 2002.

Fan, M., Stallaert, J. and Whinston, A.B., 2000. The Internet and the Future of Financial Markets. *Communications of the ACM*, 43(11):83–88.

Granot, D., Granot, F. and Zhu, W.R., 1998. Characterization Sets for the Nucleolus. *Internationa Journal of Game Theory*, 27:359–374.

Groves, T., 1973. Incentives in Teams. *Econometrica*, 41:617–631.

Kalagnanam, J.R., Davenport, A.J. and Lee, H.S., 2001. Computational Aspects of Clearing Continuous Call Double Auctions with Assignment Constraints and Indivisible Demand. *Electronic Commerce Research Journal*, 1(3):221–237.

Kothari, A., Sandholm, S. and Suri, S. Solving Combinatorial Exchanges: Optimality via a Few Partial Bids. In *AAAI-02 workshop on AI for Intelligent Business*. Edmonton, Canada, 2002.

Kwasnica, A.M., Ledyard, J.O., Porter, D. and DeMartini, C., 2005. A New and Improved Design for Multiobject Iterative Auctions. *Management Science*, 51(3):419–434.

Ledyard, J.O., Olson, M., Porter, D., Swanson, J.A. and Torma, D.P., 2002. The First Use of a Combined Value Auction for Transportation Services. *Interfaces*, 32(5):2–14.

Leyton-Brown, K., Pearson, M. and Shoham, Y. Towards a Universal Test Suite for Combinatorial Auction Algorithms. In *Proceedings of the ACM Conference on Electronic Commerce (ACM-EC)*, pages 66–76. 2000.

Nemhauser, G. and Wolsey, L. *Integer and combinatorial optimization*. Wiley, New York, 1988.

Nisan, N. Bidding Languages for Combinatorial Auctions. In *Combinatorial Auctions*, chapter 9. MIT Press, 2006.

O'Neill, R.P., Stokiewicz, P., Hobbs, B.F., Rothkopf, M.H. and Stewart, W.R. Jr, 2005. Efficient Market-Clearing Prices in Markets with Nonconvexities. *European Journal of Operational Research*, 164:269–285.

Parkes, D.C., Kalagnanam, J. and Eso, M. Achieving Budget-Balance with Vickrey-Based Payment Schemes in Exchanges. In *Proc. 17th International Joint Conference on Artificial Intelligence*, pages 1161–1168. 2001.

Pekeč, A. and Rothkopf, M.H., 2003. Combinatorial Auction Design. *Management Science*, 49(11):1485–1503.

Rassenti, S.J., Smith, V.L. and Bulfin, R.L., 1982. A Combinatorial Auction Mechanism for Airport Time Slot Allocation. *Bell Journal Of Economics*, 13(2):402–417.

Rothkopf, M.H., Pekeč, A. and Harstad, R.M., 1998. Computationally manageable combinatorial auctions. *Management Science*, 44:1131–1147.

Rothkopf, M.H., Teisberg, T.J. and Kahn, E.P., 1990. Why are Vickrey Auctions Rare? *Journal of Political Economy*, 98:94–109.

Sakurai, Y, Yokoo, M and Kamei, K. An Efficient Approximate Algorithm for Winner Determination in Combinatorial Auctions. In ACM, editor, *EC'00*, pages 30–37. 2000.

Shabalin, P., Pikovsky, A. and Bichler, M. *An Analysis of Linear Prices in Iterative Combinatorial Auctions*. IBIS, Available at http://ibis.in.tum.de/staff/pikovsky/index.htm., 2005.

Shapley, L.S. A Value for n-persons Games. In *Contributions to the Theory of Games II*, pages 305–317. Princeton University Press, 1953.

Shapley, L.S. and Shubik, M., 1969. On Market Games. *Journal of Economic Theory*, 1:9–25.

Smith, T., Sandholm, T. and Simmons, R. Constructing and Clearing Combinatorial Exchanges Using Preference Elicitation. In *Proceedings of AAAI-02 workshop on Preferences in AI and CP: Symbolic Approaches*, pages 87–93. 2002.

Vickrey, W., 1961. Conterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance*, 16:8–37.

Walsh, W.E, Wellman, M.P. and Ygge, F. Combinatorial Auctions for Supply Chain Formation. In *Proc. ACM Conference on Electronic Commerce*, pages 260–269. 2000.

Xia, M., Koehler, G.J. and Whinston, A.B., 2004. Pricing Combinatorial Auctions. *European Journal of Operational Research*, 154:251–270.

Xia, M., Stallaert, J. and Whinston, A.B., 2005. Solving the Combinatorial Double Auction Problem. *European Journal of Operational Research*, 164(1):239–251.

Young, H.P. Cost Allocation. In R.J. Aumann and S. Hart, editors, *Handbook of Game Theory, Vol.II*, chapter 34, pages 1191–1235. Amsterdam, North Holland, 1994.