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The Exact Solution of Several Classes of Inventory-Routing Problems

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Abstract. In order to be competitive companies need to take advantage of synergetic interactions between different decision areas. Two of these are related to the distribution and inventory management processes. Inventory-Routing Problems (IRPs) arise when inventory and routing decisions must be taken simultaneously, which yields a difficult combinatorial optimization problem. In this paper, we propose a branch-and-cut algorithm for the exact solution of several classes of IRPs. Specifically, we solve the multi-vehicle IRP with a homogeneous and a heterogeneous fleet, the IRP with transshipment options, and the IRP with added consistency features. We perform an extensive computational analysis on benchmark instances.

Keywords. Inventory-routing problem (IRP), transshipment, adaptive large neighborhood search (ALNS), heuristic.

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1 Introduction

Inventory-routing problems (IRPs) arise in distribution and inventory management contexts in which a supplier coordinates the replenishment process of a number of customers. This is the case of vendor-managed inventory systems. Under this strategy, the supplier decides when to visit its customers, how much to deliver to each of them and how to combine them into vehicle routes. Applications include the distribution of liquified natural gas and ship routing problems [3, 9], distribution of raw material to the paper industry [12], and food distribution to supermarket chains [17], among others. A recent literature survey was carried out by Andersson et al. [4].

The IRP has received considerable attention in the last decade. The single vehicle case was solved exactly by branch-and-cut by Archetti et al. [5] and Solyalı and Süral [21, 22]. A number of heuristics have also been proposed [6, 8, 10]. The multi-vehicle IRP (MIRP) was modeled and solved heuristically by Coelho et al. [11] by means of a matheuristic embedding a large neighborhood search mechanism. Coelho et al. [11] have also proposed different consistency features for the MIRP. The resulting problems were also solved heuristically. Adulyasak et al. [1] have recently proposed studied different MILP formulations for the multi-vehicle Production Routing Problem (PRP) of which the MIRP is a special case. They have developed and compared several versions of a branch-and-cut algorithm and they have devised an optimization-based adaptive large neighborhood search to compute an initial upper bound. Their algorithm can solve the MIRP exactly as a special case of the PRP.

1.1 The classical MIRP

The classical MIRP is defined on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V} = \{0, ..., n\}$ is the vertex set and $\mathcal{A} = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}$ is the arc set. Vertex 0 represents the supplier and the vertices of $\mathcal{V}' = \mathcal{V} \setminus \{0\}$ represent customers. Both the supplier and customers incur unit inventory holding costs h_i per period ($i \in \mathcal{V}$), and each customer has an inventory holding capacity C_i. The length of the planning horizon is p and, at each time period $t \in \mathcal{T} = \{1, ..., p\}$, the quantity of product made available at the supplier is r^t. We assume the supplier has sufficient inventory to meet all the demand during the planning horizon and that inventories are not allowed to be negative, i.e. all the demand must be met at the same period in which it occurs and there is no backlogging option. At the beginning of the planning horizon the decision maker knows the current inventory level of the supplier and of the customers (I_0^0 and I_i^0), and receives the information on the demand d_i^t of each customer i for each time period t. Throughout the paper, we assume that the quantity r^t becoming available at the supplier in period t can be used for deliveries to customers in the same period, and that the quantities q_i^t received by customer i in period t can be used to meet the demand in that period. A set $\mathcal{K} = \{1, ..., K\}$ of vehicles are available. We denote by Q_k the capacity of vehicle k. Each vehicle is able to perform one route per time period, from the supplier to a subset of customers.

A routing cost c_{ij} is associated with arc $(i, j) \in \mathcal{A}$. The objective of the problem is to minimize the total cost while meeting the demand for each customer in each period.

1.2 The MIRP with additional features

There exist several variants of this basic version of the MIRP. In order to increase the quality of service, additional features may be added to reflect a number of concerns, e.g. workforce management [14, 20, 23] and regularity of service [11]. To this end, Coelho et al. [11] have incorporated several consistency features in order to increase the quality of service while remaining cost effective. These are now summarized:

1. Quantity consistency: limiting the quantity of products delivered to a customer by customer-dependent intervals.
2. Vehicle filling rate: only allowing a vehicle to be used if its load exceeds a minimum utilization rate.
3. Order-up-to level (OU): this common IRP constraint states that a customer's inventory capacity if filled whenever the customer is visited by a vehicle.
4. Driver consistency: extending the work of Groër et al. [15] to the MIRP, this feature requires that each customer be assigned to only one driver.
5. Driver partial consistency: a relaxation of the previous consistency feature to allow some of the deliveries not to be subject to it.
6. Visit spacing: this feature imposes a temporal space between consecutive visits to the same customer.

Moreover, as a means of increasing the flexibility and reducing the total inventory-distribution cost, Coelho et al. [10] have proposed the use of planned transshipments which entail inventory relocations between customers in order to reduce the total inventory holding cost as well as sharing the risk of a stockout. Transshipment networks were further studied by Lien et al. [16] and their impact on inventory control was investigated by Nonås and Jörnsten [18].

1.3 Scientific contribution and organization of the paper

In this paper we provide a unified model and a branch-and-cut algorithm capable of solving all of the above mentioned classes of MIRPs. Specifically, we compute the first exact solutions for the heterogeneous MIRP, the first exact solutions for the six consistency features just described, as well as exact solutions for the special case of the MIRP with a homogeneous fleet.

Our purpose is to propose a branch-and-cut algorithm capable of solving several classes of MIRPs. One interesting feature of this algorithm is that it can be directly implemented within the CPLEX framework and does not require the development of sophisticated decomposition procedures. We view this feature as an advantage and perhaps as the start of a trend in optimization given the strength and sophistication of the CPLEX software. An important by-product of our work is to make the solutions of several benchmark IRP instances available to the scientific community.

The remainder of the paper is organized as follows. In Section 2 we propose an undirected model and in Section 3 we describe our branch-and-cut algorithm. Extensive computational results are presented in Section 4, followed by conclusions in Section 5.

2 Mathematical formulations

In this section we provide mathematical formulations for all the variants of the MIRP under consideration.

2.1 Undirected model for the heterogeneous MIRP

Assuming that the transportation cost matrix is symmetric, we work with an undirected formulation in order to reduce the number of variables. Thus, the model works with the undirected routing variables x_{ij}^{kt} equal to the number of times edge (i,j) with $i<j$ is used on the route of vehicle k in period t. We also introduce variables y_i^{kt} equal to one if and only if node i (the supplier or a customer) is visited by vehicle k in period t, and zero otherwise. Let I_i^t denote the inventory level at vertex $i \in \mathcal{V}$ at the end of period $t \in \mathcal{T}$. We denote by q_i^{kt} the quantity of product delivered from the supplier using vehicle k to customer i in time period t. Assuming a maximum level (ML) inventory policy, the problem can then be formulated as

$$\text{minimize} \sum_{i\in\mathcal{V}}\sum_{t\in\mathcal{T}} h_i I_i^t + \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{V},i<j}\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} c_{ij}x_{ij}^{kt}, \tag{1}$$

subject to

$$I_0^t = I_0^{t-1} + r^t - \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{V}'} q_i^{kt} \quad t\in\mathcal{T} \tag{2}$$

$$I_0^t \geq 0 \quad t\in\mathcal{T} \tag{3}$$

$$I_i^t = I_i^{t-1} + \sum_{k\in\mathcal{K}} q_i^{kt} - d_i^t \quad i\in\mathcal{V}' \quad t\in\mathcal{T} \tag{4}$$

$$I_i^t \geq 0 \quad i\in\mathcal{V} \quad t\in\mathcal{T} \tag{5}$$

$$I_i^t \leq C_i \quad i\in\mathcal{V} \quad t\in\mathcal{T} \tag{6}$$

$$\sum_{k\in\mathcal{K}} q_i^{kt} \leq C_i - I_i^{t-1} \quad i\in\mathcal{V}' \quad t\in\mathcal{T} \tag{7}$$

$$q_i^{kt} \leq C_i y_i^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{8}$$

$$\sum_{i \in \mathcal{V}'} q_i^{kt} \leq Q_k y_0^{kt} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{9}$$

$$\sum_{j \in \mathcal{V}, i<j} x_{ij}^{kt} + \sum_{j \in \mathcal{V}, j<i} x_{ji}^{kt} = 2y_i^{kt} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{10}$$

$$\sum_{i \in \mathscr{S}} \sum_{j \in \mathscr{S}, i<j} x_{ij}^{kt} \leq \sum_{i \in \mathscr{S}} y_i^{kt} - y_m^{kt} \quad \mathscr{S} \subseteq \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{11}$$

for some $m \in \mathscr{S}$

$$q_i^{kt} \geq 0 \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{12}$$

$$x_{i0}^{kt} \in \{0, 1, 2\} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{13}$$

$$x_{ij}^{kt} \in \{0, 1\} \quad i, j \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{14}$$

$$y_i^{kt} \in \{0, 1\} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{15}$$

Constraints (2) define the inventory at the supplier while constraints (3) prevent stockouts at the supplier; constraints (4) and (5) are similar and apply to the customers. Constraints (6) impose maximal inventory level at the customers. Constraints (7) and (8) link the quantities delivered to the routing variables. In particular, they only allow a vehicle to deliver products to a customer if the customer is visited by this vehicle. Constraints (9) ensure the vehicle capacities are respected while constraints (10) and (11) are degree constraints and subtour elimination constraints, respectively. Constraints (12)−(15) enforce integrality and non-negativity conditions on the variables.

2.2 Valid inequalities

Archetti et al. [5] have introduced several classes of inequalities for the IRP. Some of these are also valid for the MIRP, both for the OU and the ML cases, namely (17), (18), (20), (22)−(24). We list them here, adapted to account for multiple vehicles:

$$x_{i0}^{kt} \leq 2y_i^{kt} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{16}$$

$$x_{ij}^{kt} \leq y_i^{kt} \quad i, j \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{17}$$

$$y_i^{kt} \leq y_0^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{18}$$

$$\sum_{k \in \mathcal{K}} \sum_{l=1}^{t} y_i^{kt} \geq \left\lceil \left(\sum_{k \in \mathcal{K}} \sum_{l=1}^{t-1} d_i^{kt} - I_i^0 \right) / C_i \right\rceil \quad i \in \mathcal{V} \quad t \in \mathcal{T}. \tag{19}$$

Constraints (16) and (17) are referred to as logical inequalities. They enforce the condition that if the supplier is the successor of a customer in the route of vehicle k in period t, i.e. $x_{i0}^{kt} = 1$ or 2, then i must be visited by the same vehicle, i.e. $y_i^{kt} = 1$. A similar reasoning is applied to customer j in inequalities (17). Constraints (18) include the supplier in the route of vehicle k if any customer is visited by that vehicle in that period. Constraints (19) ensure that customer i is visited at least the number of times corresponding to the right-hand side of the inequality. This inequality is only valid if the fleet is homogeneous.

Finally, we also tighten this formulation by imposing the following symmetry breaking constraints valid for the case where the vehicle fleet is homogeneous:

$$y_0^{kt} \leq y_0^{k-1,t} \quad k \in \mathcal{K} \backslash \{1\} \quad t \in \mathcal{T} \tag{20}$$

$$y_i^{kt} \leq \sum_{j<i} y_j^{k-1,t} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \backslash \{1\} \quad t \in \mathcal{T}. \tag{21}$$

Constraints (20) ensure that vehicle k cannot leave the depot if vehicle $k-1$ is not used. This symmetry breaking rule is then extended to the customer vertices by constraints (21) which state that if customer i is assigned to vehicle k in period t, then vehicle $k-1$ must serve a customer with an index smaller than i in the same period. These constraints are inspired from those proposed by Fischetti et al. [13] for the capacitated vehicle routing problem and by Albareda-Sambola et al. [2] for a plant location problem.

2.3 The MIRP with transshipment

Following [10], the MIRP with transshipments is modeled with extra variables w_{ij}^t representing the quantity transshipped from $i \in \mathcal{V}$ to $j \in \mathcal{V}'$ during period $t \in \mathcal{T}$. Then the following term is added to the objective function (1):

$$\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}'} \sum_{t \in \mathcal{T}} \beta c_{ij} w_{ij}^t, \tag{22}$$

where β represents the fraction of the transportation cost that is due to every unit transshipped from i to j. Constraints (2) and (4) are then replaced by

$$I_0^t = I_0^{t-1} + r^t - \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{V}'} q_i^{kt} - \sum_{j \in \mathcal{V}'} w_{0j}^t \quad t \in \mathcal{T} \tag{23}$$

$$I_i^t = I_i^{t-1} + \sum_{k \in \mathcal{K}} q_i^{kt} - d_i^t - \sum_{j \in \mathcal{V}'} w_{ij}^t + \sum_{j \in \mathcal{V}} w_{ji}^t \quad i \in \mathcal{V}' \quad t \in \mathcal{T}. \tag{24}$$

2.4 The consistent MIRP

We now describe minor modifications to the model necessary to account for each of the consistency features already described.

2.4.1 Quantity consistency

In order to ensure that all quantities delivered to the customers lie within an interval $[g_l, g_u]$ around a target value, e.g. the average demand of the customer over the planning horizon, it suffices to add the following constraint to the model:

$$y_i^{kt} g_l \sum_{t \in \mathcal{T}} d_i^t / p \leq q_i^{kt} \leq y_i^{kt} g_u \sum_{t \in \mathcal{T}} d_i^t / p \qquad i \in \mathcal{V}', k \in \mathcal{K}, t \in \mathcal{T}. \tag{25}$$

2.4.2 Vehicle filling rate

To avoid solutions containing vehicles routes with very small loads, one can ensure that a vehicle is only dispatched if it is at least γ percent filled, which is achieved through the following constraint:

$$\sum_{i \in \mathcal{V}'} q_i^{kt} \geq \gamma y_0^{kt} Q_k \qquad k \in \mathcal{K}, t \in \mathcal{T}. \tag{26}$$

2.4.3 Order-up-to policy

This is a common inventory policy by which a delivery should fill the customer's inventory capacity. The OU policy is enforced through the constraints

$$q_i^{kt} \geq C_i y_i^{kt} - I_i^{t-1} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}. \tag{27}$$

2.4.4 Driver consistency

Another way to increase quality of the service is to ensure that the same driver always visits the same customer. This is modeled with an extra binary variable z_i^k equal to 1 if and only if vehicle k visits customer i, and the following three sets of constraints:

$$\sum_{k \in \mathcal{K}} z_i^k = 1 \qquad i \in \mathcal{V}' \tag{28}$$

$$y_i^{kt} \leq z_i^k \qquad i \in \mathcal{V}', k \in \mathcal{K}, t \in \mathcal{T} \tag{29}$$

$$z_i^k \in \{0, 1\} \qquad i \in \mathcal{V}', k \in \mathcal{K}. \tag{30}$$

Constraints (28) ensure that exactly one vehicle is assigned to each customer over the planning horizon. Constraints (29) allow deliveries only from the vehicle assigned to the customer.

2.4.5 Driver partial consistency

A relaxation of the previous feature is achieved by the driver partial consistency policy. This can be handled in a number of ways. As in Coelho et al. [11] we add to the objective function a penalty term proportional to the number of extra vehicles assigned to each customer and we introduce a binary variable s_i^k indicating whether an extra vehicle k is assigned to customer i. We then impose the following sets of constraints to the model:

$$\sum_{k \in \mathcal{K}} z_i^k = 1 \qquad i \in \mathcal{V}' \tag{31}$$

$$y_i^{kt} \leq z_i^k + s_i^k \qquad i \in \mathcal{V}', k \in \mathcal{K}, t \in \mathcal{T} \tag{32}$$

$$s_i^k, z_i^k \in \{0,1\} \qquad i \in \mathcal{V}', k \in \mathcal{K}. \tag{33}$$

Constraints (31) assign a first vehicle to each customer, while constraints (32) allow additional vehicles to be assigned to the same customer. We then add a penalty term

$$\alpha \sum_{i \in \mathcal{V}'} \sum_{k \in \mathcal{K}} s_i^k \tag{34}$$

to the objective function (1). By adjusting the parameter α, one can control how restrictive the driver partial consistency policy will be.

2.4.6 Visit spacing

Adding the following constraints to the basic model will ensure that at least one visit will take place every (M_i+1) periods, and no more than one visit will take place in any (m_i+1) successive periods:

$$\sum_{k \in \mathcal{K}} \sum_{l=t}^{t+m_i} y_i^{kl} \leq 1 \qquad i \in \mathcal{V}', t \in \{1, ..., p - m_i\} \tag{35}$$

$$\sum_{k \in \mathcal{K}} \sum_{l=t}^{t+M_i} y_i^{kl} \geq 1 \qquad i \in \mathcal{V}', t \in \{1, ..., p - M_i\}. \tag{36}$$

3 Branch-and-cut algorithm

The MIRP is $\mathcal{NP}$-hard since it contains the VRP as a special case. If the instance size is not excessive, the undirected formulation of Section 2 can be solved exactly by branch-and-cut as follows. At a generic node of the search tree, a linear program defined by (1)$-$(10) is solved, a search for violated subtour elimination constraints (11) is performed, and some of these constraints are added to the current program which is then reoptimized. This process is repeated until a feasible or dominated solution is reached, or until there are no more cuts to be added. At this point branching on a fractional variable occurs.

3.1 Implementation features

We make a few remarks and comments regarding the implementation of the algorithm. We have implemented both the directed and the undirected formulations, but none outperforms the other significantly. We have opted for the edge formulation because it requires considerably fewer variables and this becomes a relevant issue on large instances. In order to solve the LP relaxation at each node we use the dual simplex algorithm. In our tests it has been shown to outperform the primal simplex method.

Some important differences between our implementation and that of Archetti et al. [5] are now described. Archetti et al. [5] have used the heuristic of Bertazzi et al. [8] to compute an upper bound at the root of the search tree. We, in contrast, do not compute an initial upper bound but apply an algorithm to further improve integer solutions found during the search, thus helping the identification of better solutions faster. This algorithm, described in Section 3.2, yields an approximation of the true routing costs. In addition, we generate all violated subtour elimination constraints (11), as opposed to the implementation of [5] which generates only the one with $m = \arg\max_j \{y_j^{kt}\}$. Finally, we take advantage of multi-core processors, running the MIP solver with up to six parallel threads. Since Archetti et al. [5] do not mention whether their implementation uses one or several cores, we assume they have run their code on a single core.

3.2 Solution improvement algorithm

The purpose of the Solution Improvement algorithm (SI), is to approximate the cost of a new solution resulting from vertex removals and reinsertions. It is solved whenever the branch-and-cut search identifies a new best solution. Using an idea proposed by Archetti et al. [7], we simplify and approximate the routing costs resulting from vertex removals and reinsertions as follows. Let a_i^{kt} represent the routing cost reduction if customer i is removed from the route of vehicle k at period t, which obviously visits customer i; let b_i^{kt} represent the routing cost if customer i is inserted in the route of vehicle k at period t, which obviously does not already visit customer i; finally, let r_i^{kt} be a binary parameter equal to 1 if and only if customer i is visited in the current route of vehicle k at period t. Also define the following binary variables: let u_i^{kt} be equal to 1 if and only if customer i is removed from the existing route of vehicle k at period t, and let v_i^{kt} be equal to 1 if and only if customer i is inserted in the route of vehicle k at period t. This subproblem is then to

$$\text{(SI)} \quad \text{minimize} \sum_{i \in \mathcal{V}} \sum_{t \in \mathcal{T}} h_i I_i^t - \sum_{i \in \mathcal{V}'} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} a_i^k t u_i^{kt} + \sum_{i \in \mathcal{V}'} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} b_i^{kt} v_i^{kt} \tag{37}$$

subject to (2)–(6) and to

$$q_i^{kt} \leq C_i - I_i^{t-1} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{38}$$

$$q_i^{kt} \leq (r_i^{kt} - u_i^{kt} + v_i^{kt}) C_i \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{39}$$

$$v_i^{kt} \leq 1 - r_i^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{40}$$

$$u_i^{kt} \leq r_i^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{41}$$

$$\sum_{i \in \mathcal{V}'} u_i^{kt} + \sum_{i \in \mathcal{V}'} v_i^{kt} \leq \varepsilon \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{42}$$

$$\sum_{i \in \mathcal{V}'} q_i^{kt} \leq Q_k \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{43}$$

$$q_i^{kt} \geq 0 \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \tag{44}$$

$$u_i^{kt}, v_i^{kt} \in \{0,1\} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}. \tag{45}$$

The objective function (37) minimizes the total inventory, removal and insertion cost. Constraints (38)−(39) enforce the ML policy. Constraints (40) ensure that if a customer is already present in a route, it cannot be reinserted in the same route. Likewise, constraints (41) guarantee that only those customers belonging to a route can be removed from it. Constraints (43) ensure that the vehicle capacity is respected. If the incumbent solution is changed by more than one customer, then this model only provides an approximation of the actual routing costs. For this reason, we have decided to limit the number of insertions and removals that could take place in the solution of SI, and we have added constraints (42) to limit the number of insertions and removals per route to a small value ε. The addition of the variants described in Sections 2.3 and 2.4 to the SI model is straightforward and is not described here.

We provide a sketch of the branch-and-cut scheme in Algorithm 1.

Algorithm 1 Proposed branch-and-cut algorithm

```
At the root node of the search tree, generate and insert all valid inequalities
(16)−(21) into the program.
Subproblem solution. Solve the LP relaxation of the node.
Termination check:
if there are no more nodes to evaluate then
   Stop.
else
   if The current solution is a new best solution then
      Apply the SI algorithm to the incumbent solution.
      if the SI algorithm yields an improved solution then
         Update the solution vector at the branch-and-cut level
      end if
   end if
   Select one node from the branch-and-bound tree.
end if
while the solution of the current LP relaxation contains subtours do
   Identify the connected components using the separation procedure of Pad-
   berg and Rinaldi [19].
   Add all violated subtour elimination constraints (11).
   Subproblem solution. Solve the LP relaxation of the node.
end while
if the solution of the current LP relaxation is integer then
   Go to the termination check.
else
   Branching: branch on one of the fractional variables.
   Go to the termination check.
end if
```

4 Computational results

The algorithm just described was coded in C++ using IBM Concert Technology and CPLEX 12.3 with six threads. All computations were executed on a grid of Intel Xeon™ processors running at 2.66 GHz with up to 48 GB of RAM installed per node, with the Scientific Linux 6.1 operating system.

To evaluate the performance of the algorithm, we have used the benchmark instance set for the single vehicle case created by Archetti et al. [5]. It was used in [1, 7, 8, 10, 11] to evaluate single and multi-vehicle algorithms for the IRP and is made up of instances with up to three time periods and 50 customers, and six time periods and 30 customers. These are labeled small-n-p-low or small-n-p-high, where n is the number of customers, p is the number of periods and the suffix low/high represents two levels of inventory holding costs. Optimal solution values are known from [5, 21] for the single vehicle case and from [11] for the multi-vehicle case. In order to account for multiple vehicle we follow the same procedure of [1, 11] by not changing the overall capacity, but by dividing the original vehicle capacity by the number of vehicles considered. A time limit of 12 hours (wall clock) was imposed on the solution of each instance of this small set.

We also apply our algorithm to a newer and larger instance set, containing 60 instances was proposed in [7], with up to six time periods and 200 customers. There are 10 instances of each size and they are described here as large-n-p-low or large-n-p-high as before. There are still no reported optimal solutions for this larger set. It was used to evaluate the heuristic algorithms of Archetti et al. [7] for the single vehicle case and of Coelho et al. [11] for the multi-vehicle one. We are then the first to provide optimal solution values and lower bounds for this larger set. A time limit of 24 hours (wall clock) was imposed on the solution of each instance of this large set. Detailed results on all instances are reported in the appendix.

4.1 MIRP solutions under the ML replenishment policy

We start our analysis by computing exact solutions for the MIRP under the ML replenishment policy. We provide in Table 1 the number of instances solved to optimality, average gaps between the best lower and upper bound obtained and average running times for up to five vehicles, including the single-vehicle case. Our algorithm was able to quickly obtain optimal solutions for the single-vehicle case. It is difficult if not impossible to make a clear comparison with the solution times obtained exactly by Archetti et al. [5] and heuristically by Archetti et al. [7] due to differences in CPLEX versions and on the hardware used. However, on instances with 30 customers and six time periods, our algorithm took on average 70 seconds compared to 1570 seconds of Archetti et al. [5] and 1922 seconds of Archetti et al. [7]. Our algorithm was also able to solve most of the short period instances with several vehicles.

Comparing the solutions obtained by our algorithm and by the heuristic of Coelho et al. [11], we are able to improve the majority of the best known solutions and we obtain good lower bounds for the ones not yet solved to optimality. Moreover, the sizes of the instances that our algorithm is able to solve to optimality are comparable to those solved by Adulyasak et al. [1] as a special case of the PRP but the gaps and solution times obtained by these authors appear to be slightly better than ours. These authors used the same CPLEX version as us, the same computer, but eight parallel cores instead of six. We note that their algorithm starts from a good initial upper bound whereas we implement ours from scratch. Finally, we report solutions for a larger number of vehicles and we solve several variants of the problem.

We have also obtained several optimal and new best solution values for the MIRP on the larger instances set. Out of the 60 instances, our algorithm was able to prove optimality for 17 of them and find new best solutions for 30 instances. For those that our algorithm was not able to solve exactly, we provide very tight lower bounds with respect to the heuristic solutions obtained in [7]. The average gap between our best lower bound and the upper bound provided by [7] is 1.7%. We provide in Table 2 the number of instances solved to optimality, the number of instances whose new best solution values has been identified, average gaps between the best lower and upper bound obtained and average running times.

We have also evaluated the performance of the algorithm on the larger instance set with two and three vehicles. Our algorithm has been able to find several new best solutions on the instances with 50 customers when compared to the heuristic solutions values provided by Coelho et al. [11]. Table 3 summarizes these results providing the number of instances with new best solution values, average gaps between the best lower and upper bound obtained and average running times.

4.2 MIRP solutions for homogeneous and heterogeneous vehicle fleets

In a second set of experiments, we have studied the impact of symmetry breaking constraints on the solution of homogeneous instances. We have also introduced a new set of heterogeneous instances in order to make comparisons with the homogeneous case. To our knowledge we are the first to solve heterogeneous instances of the MIRP. To this end, we have run a small subset of instances for the all three cases with three and four vehicles. We have opted not to change the overall capacity when the vehicles are heterogeneous, but to split it differently among the vehicles. For $K = 3$ the first vehicle accounts for 50% of the original capacity, the second vehicle holds 30% of the original capacity and the third vehicle has 20% of the original capacity. For $K = 4$ the percentage of the original capacity of each vehicle is 40, 25, 20 and 15. Average results are shown in Table 4. They indicate that imposing symmetry breaking constraints in the homogeneous MIRP has a significant effect on the reduction of the optimality gap and on computing times. As a result, more instances can be solved optimally. Heterogeneous MIRPs are much easier to solve than homogeneous instances without symmetry breaking constraints. However, they are

Table 1: Computational results on the small instances set

Instance	K = 1						K = 2						K = 3						K = 4						K = 5		
	Our algorithm			Adulyasak et al. [1]			Our algorithm			Adulyasak et al. [1]			Our algorithm			Adulyasak et al. [1]			Our algorithm			Adulyasak et al. [1]			Our algorithm		
	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)
small-5-3-low	5	0.00	0.0	5	0.00	0.0	5	0.00	3.8	5	0.00	0.1	5	0.00	4.6	5	0.00	0.3	5	0.00	4.0	–	–	–	5	0.00	0.0
small-10-3-low	5	0.00	0.6	5	0.00	0.1	5	0.00	7.6	5	0.00	1.2	5	0.00	17.4	5	0.00	6.7	5	0.00	40.8	–	–	–	5	0.00	47.8
small-15-3-low	5	0.00	1.2	5	0.00	0.2	5	0.00	11.8	5	0.00	3.3	5	0.00	31.4	5	0.00	29.2	5	0.00	119.0	–	–	–	5	0.00	1195.6
small-20-3-low	5	0.00	1.4	5	0.00	1.5	5	0.00	24.4	5	0.00	21.1	5	0.00	220.8	5	0.00	237.7	5	0.00	5544.4	–	–	–	3	2.66	14619.2
small-25-3-low	5	0.00	4.6	5	0.00	2.8	5	0.00	31.6	5	0.00	55.7	5	0.00	574.2	5	0.00	639.9	5	0.00	4665.8	–	–	–	2	4.22	26720.2
small-30-3-low	5	0.00	4.8	5	0.00	9.6	5	0.00	61.8	–	–	–	5	0.00	1285.8	5	0.00	1746.9	2	2.90	29714.8	5	0.00	22427.6	1	9.78	39793.6
small-35-3-low	5	0.00	4.8	5	0.00	4.1	5	0.00	56.0	–	–	–	5	0.00	1935.8	5	0.00	2224.3	2	5.49	31756.2	3	2.40	31040.2	0	12.68	43060.4
small-40-3-low	5	0.00	7.6	5	0.00	24.2	5	0.00	525.0	–	–	–	5	0.00	9092.0	5	0.00	5569.1	0	8.05	43010.4	1	5.80	41757.3	0	19.90	34767.0
small-45-3-low	5	0.00	10.2	5	0.00	27.5	5	0.00	3867.8	–	–	–	2	0.86	31805.2	5	0.00	18549.5	1	11.90	34721.6	1	8.70	36126.0	0	22.71	43046.2
small-50-3-low	5	0.00	62.6	5	0.00	138.7	4	0.15	10796.6	–	–	–	0	12.40	42930.4	0	4.90	43200.0	0	26.98	42998.6	0	20.7	43200.0	0	35.19	43043.4
small-5-3-high	5	0.00	0.2	5	0.00	0.0	5	0.00	2.8	5	0.00	0.2	5	0.00	2.6	5	0.00	0.3	5	0.00	4.8	–	–	–	5	0.00	4.6
small-10-3-high	5	0.00	0.2	5	0.00	0.1	5	0.00	5.8	5	0.00	1.2	5	0.00	12.6	5	0.00	7.6	5	0.00	54.0	–	–	–	5	0.00	75.2
small-15-3-high	5	0.00	1.0	5	0.00	0.3	5	0.00	11.6	5	0.00	3.3	5	0.00	26.0	5	0.00	24.7	5	0.00	89.4	–	–	–	5	0.00	723.2
small-20-3-high	5	0.00	3.6	5	0.00	2.1	5	0.00	23.8	5	0.00	17.3	5	0.00	217.4	5	0.00	227.6	5	0.00	7779.4	–	–	–	5	0.00	723.2
small-25-3-high	5	0.00	3.8	5	0.00	2.1	5	0.00	30.8	5	0.00	31.1	5	0.00	1013.6	5	0.00	646.1	5	0.00	9781.2	–	–	–	5	0.00	723.2
small-30-3-high	5	0.00	9.0	5	0.00	10.1	5	0.00	70.4	–	–	–	5	0.00	1623.4	5	0.00	966.9	2	1.11	27419.4	5	0.00	13813.8	1	4.41	38874.0
small-35-3-high	5	0.00	6.6	5	0.00	4.6	5	0.00	65.6	–	–	–	5	0.00	2696.0	5	0.00	1624.8	2	1.73	27955.2	3	0.80	27507.0	0	6.44	43012.6
small-40-3-high	5	0.00	13.6	5	0.00	32.3	5	0.00	478.5	–	–	–	5	0.00	6312.4	5	0.00	5638.0	2	2.15	39507.8	1	1.9	38848.5	0	5.70	37707.0
small-45-3-high	5	0.00	16.6	5	0.00	40.1	5	0.00	1595.0	–	–	–	2	0.40	32820.6	5	0.00	13913.3	1	4.24	34789.4	1	3.00	35684.3	0	8.62	43102.8
small-50-3-high	5	0.00	48.8	5	0.00	156.9	5	0.00	4431.6	–	–	–	0	4.39	42990.8	0	1.50	43200.0	0	9.73	43031.6	0	6.9	43200.0	0	15.38	43057.8
small-5-6-low	5	0.00	3.2	5	0.00	0.2	5	0.00	9.0	5	0.00	2.5	5	0.00	56.6	5	0.00	94.7	5	0.00	77.6	–	–	–	4	3.84	8804.6
small-10-6-low	5	0.00	7.8	5	0.00	0.8	5	0.00	607.4	5	0.00	86.2	4	0.98	14578.6	4	0.50	15658.4	2	4.38	30232.4	–	–	–	3	7.22	14876.4
small-15-6-low	5	0.00	22.4	5	0.00	5.4	5	0.00	555.2	5	0.00	451.0	4	0.36	18761.4	4	0.80	20951.5	0	5.81	42292.8	–	–	–	3	6.29	14126.6
small-20-6-low	5	0.00	40.6	5	0.00	29.3	5	0.00	8642.4	5	0.00	2542.6	1	7.11	42751.8	0	8.20	43200.0	0	14.71	43098.2	–	–	–	0	22.22	42445.8
small-25-6-low	5	0.00	54.0	5	0.00	49.3	4	0.24	19002.0	5	0.00	12045.5	0	8.76	43047.4	0	10.00	43200.0	0	17.79	42537.2	–	–	–	0	33.19	42281.8
small-30-6-low	5	0.00	96.8	5	0.00	177.6	1	1.74	36841.8	–	–	–	0	12.52	43079.6	–	–	–	0	37.87	42684.6	–	–	–	0	62.73	30445.4
small-5-6-high	5	0.00	1.6	5	0.00	0.1	5	0.00	9.0	5	0.00	2.3	5	0.00	38.0	5	0.00	60.0	5	0.00	51.2	–	–	–	4	16.84	1777.2
small-10-6-high	5	0.00	6.4	5	0.00	0.8	5	0.00	57.6	5	0.00	70.2	4	0.71	14611.2	4	0.60	16801.0	2	3.22	25466.6	–	–	–	0	5.52	37589.2
small-15-6-high	5	0.00	22.4	5	0.00	4.6	5	0.00	351.0	5	0.00	408.0	4	0.48	12470.4	4	0.20	16821.1	0	3.26	39944.6	–	–	–	0	7.11	42801.2
small-20-6-high	5	0.00	28.6	5	0.00	17.8	5	0.00	4035.8	5	0.00	1932.5	0	3.82	42985.6	1	3.80	40411.7	0	9.29	37441.0	–	–	–	0	13.81	42965.0
small-25-6-high	5	0.00	43.2	5	0.00	24.4	5	0.00	10160.2	5	0.00	8555.0	1	4.58	39241.4	0	4.80	43200.0	0	29.51	34401.6	–	–	–	0	24.32	41063.8
small-30-6-high	5	0.00	70.0	5	0.00	98.8	3	0.67	28788.8	–	–	–	0	6.20	42963.8	–	–	–	0	32.53	34447.4	–	–	–	0	18.11	43042.8
Average	5	0.00	18.7	5	0.00	27.1	4.75	0.08	4098.8	–	–	–	3.50	1.98	15318.7	–	–	–	2.37	7.27	23614.5	–	–	–	1.90	11.52	28594.8

Table 2: Computational results on the larger instance set, $K = 1$

Instance	# solved	# new best	gap (%)	time (s)
large-50-6-high	10	10	0.00	10272.4
large-100-6-high	0	7	0.88	86400.0
large-200-6-high	0	0	18.30	86077.8
large-50-6-low	7	9	0.15	36103.5
large-100-6-low	0	4	3.75	86400.0
large-200-6-low	0	0	32.76	86400.0
Average	2.83	5.00	9.30	65275.6

Table 3: Computational results on the larger instance set, $K = 2$ and 3

Instance	$K = 2$			$K = 3$		
	# best	gap (%)	time (s)	# best	gap (%)	time (s)
large-50-6-high	10	4.00	86400.0	0	15.72	86400.0
large-100-6-high	1	32.57	86400.0	0	56.09	86400.0
large-50-6-low	10	10.94	86400.0	0	36.22	86400.0
large-100-6-low	0	66.49	86400.0	0	77.56	86400.0
Average	5.25	28.50	86400.0	0.00	46.39	86400.0

more difficult than homogeneous instances with these constraints.

4.3 The IRP with transshipment

We also provide exact solutions for the IRP with transshipment. Heuristic solutions are available for its single vehicle case from Coelho et al. [10] and in Table 5 we show the number of instances solved to optimality, the number of new best known solutions, the average gap of our algorithm and the average running time. Out of the 160 instances tested, our algorithm was able to match the solution values on 61 and improved the solution values on 99. It was able to provide optimal solution for most of the instances and outperforms the only heuristic available for this problem.

4.4 The MIRP with consistency features

Finally, in order to evaluate the true cost of imposing consistency features, we have solved to optimality MIRP instances with quantity consistency, the driver consistency and the OU features. The first two were shown to be some of the least costly forms of consistency studied by Coelho et al. [11], while the OU inventory policy has been widely studied in the IRP and similar problems [5, 7, 6, 8, 10, 11, 1]. One can now assess for the first time their true impact in terms of cost in a multi-vehicle environment. Results are summarized in Table 6, in which we show the average gap between the best lower and upper bounds and the average running time in seconds. For the sake of comparison we also present

Table 4: Computational results on the MIRP with an homogeneous fleet (with and without symmetry breaking constraints) and with an heterogeneous fleet

K	Instance	Homogeneous, with symmetry breaking			Homogeneous, without symmetry breaking			Heterogeneous		
		# solved	gap (%)	time (s)	# solved	gap (%)	time (s)	# solved	gap (%)	time (s)
3	small-5-3-low	5	0.00	4.6	5	0.00	3.0	5	0.00	2.6
	small-10-3-low	5	0.00	17.4	5	0.00	47.6	5	0.00	14.0
	small-15-3-low	5	0.00	31.4	5	0.00	2218.8	5	0.00	121.6
	small-20-3-low	5	0.00	220.8	2	5.03	13698.4	5	0.00	161.6
	small-25-3-low	5	0.00	574.2	3	1.30	21467.6	4	1.49	15681.2
	small-30-3-low	5	0.00	1285.8	3	4.71	18211.6	2	3.36	19044.2
	small-35-3-low	5	0.00	1935.8	1	5.67	30285.0	4	2.90	18449.8
	Average	5	0.00	581.4	3.42	2.38	12276.0	4.28	1.10	7639.28
4	small-5-3-low	5	0.00	4.0	5	0.00	31.0	5	0.00	4.6
	small-10-3-low	5	0.00	40.8	3	1.82	9346.8	5	0.00	25.0
	small-15-3-low	5	0.00	119.0	1	5.48	18586.6	5	0.00	1082.0
	small-20-3-low	5	0.00	5544.4	0	14.12	27856.2	4	2.26	9040.8
	small-25-3-low	5	0.00	4665.8	0	16.32	28606.2	3	4.68	24302.0
	small-30-3-low	2	2.90	29714.8	0	12.61	43200.0	1	8.60	38771.0
	small-35-3-low	2	5.49	31756.2	0	12.11	43200.0	1	5.46	43200.0
	Average	4.14	1.19	10263.5	1.28	8.92	24403.8	3.42	3.00	16632.2

the average percentage increase in cost when the upper bounds are compared to those of the basic case without the consistency feature, as presented in Table 1.

5 Conclusions

We have developed a unified branch-and-cut algorithm for the exact solution of several classes of IRPs. Our algorithm is able to solve the classical IRP with one or several vehicles, under different inventory policies and with a homogeneous or a heterogeneous fleet of vehicles. We have also solved instances with flexibility and consistency features. Flexibility is achieved through the use of transshipments; the corresponding instances were also solved exactly for the first time. Consistency features were added by imposing some regularity in the inventory management and distribution processes. Extensive computational results on benchmark instances confirm the success of the proposed algorithm.

References

[1] Y. Adulyasak, J.-F. Cordeau, and R. Jans. Formulations and branch-and-cut algorithms for multi-vehicle production and inventory routing problems. Technical Report G-2012-14, GERAD, Montréal, Canada, 2012.

[2] M. Albareda-Sambola, E. Fernández, and G. Laporte. A computational comparison of several models for the exact solution of the capacity and distance constrained plant location problem. *Computers & Operations Research*, 38(8):1109–1116, 2011.

Table 5: Computational results on the IRP with transshipments

Instance	# solved	# best	gap (%)	time (s)
small-5-3-low	5	0	0.00	0.2
small-10-3-low	5	0	0.00	0.8
small-15-3-low	5	0	0.00	1.2
small-20-3-low	5	0	0.00	4.0
small-25-3-low	5	2	0.00	8.4
small-30-3-low	5	3	0.00	12.6
small-35-3-low	5	5	0.00	29.0
small-40-3-low	5	5	0.00	58.6
small-45-3-low	5	5	0.00	107.0
small-50-3-low	5	5	0.00	520.0
small-5-3-high	5	0	0.00	0.2
small-10-3-high	5	0	0.00	0.8
small-15-3-high	5	1	0.00	1.2
small-20-3-high	5	1	0.00	3.4
small-25-3-high	5	3	0.00	6.4
small-30-3-high	5	3	0.00	11.4
small-35-3-high	5	5	0.00	23.8
small-40-3-high	5	5	0.00	57.4
small-45-3-high	5	5	0.00	121.6
small-50-3-high	5	5	0.00	579.2
small-5-6-low	5	0	0.00	0.6
small-10-6-low	5	3	0.00	3.8
small-15-6-low	5	5	0.00	14.0
small-20-6-low	5	5	0.00	348.0
small-25-6-low	5	5	0.00	6183.4
small-30-6-low	4	5	0.57	31961.0
small-5-6-high	5	0	0.00	0.6
small-10-6-high	5	3	0.00	2.8
small-15-6-high	5	5	0.00	9.8
small-20-6-high	5	5	0.00	279.8
small-25-6-high	5	5	0.00	11677.6
small-30-6-high	4	5	0.07	34255.6
Average	4.93	3.09	0.02	2696.3

Table 6: Average computational results on the small instances set under consistency features

Instance	Quantity consistency			Driver consistency			OU		
	gap (%)	time (s)	% increase	gap (%)	time (s)	% increase	gap (%)	time (s)	% increase
small-5-3-low	0.00	2.8	8.58	0.00	4.0	3.18	0.00	0.6	10.77
small-10-3-low	0.00	12.6	0.35	0.00	26.2	0.48	0.00	14.2	5.33
small-15-3-low	0.00	29.0	0.00	0.00	596.6	0.00	0.00	36.4	6.64
small-20-3-low	0.00	112.6	0.00	3.32	13839.8	0.00	0.00	431.0	9.73
small-25-3-low	0.00	387.4	0.00	3.61	13592.4	0.11	0.00	2752.4	9.59
small-30-3-low	0.00	696.2	0.00	3.12	24191.4	0.09	0.77	10886.2	8.61
small-35-3-low	0.00	1139.0	0.00	0.51	25433.6	0.00	1.76	14799.0	9.53
Average	0.00	339.9	1.27	1.50	11097.7	0.55	0.36	4131.4	8.60

[3] H. Andersson, M. Christiansen, and K. Fagerholt. Transportation planning and inventory management in the LNG supply chain. In E. Bjørndal, M. Bjørndal, P. M. Pardalos, and M. Rönnqvist, editors, *Energy, Natural Resources and Environmental Economics*, pages 223–248. Springer, New York, 2010.

[4] H. Andersson, A. Hoff, M. Christiansen, G. Hasle, and A. Løkketangen. Industrial aspects and literature survey: Combined inventory management and routing. *Computers & Operations Research*, 37(9):1515–1536, 2010.

[5] C. Archetti, L. Bertazzi, G. Laporte, and M. G. Speranza. A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Science*, 41(3):382–391, 2007.

[6] C. Archetti, L. Bertazzi, G. Paletta, and M. G. Speranza. Analysis of the maximum level policy in a production-distribution system. *Computers & Operations Research*, 12(38):1731–1746, 2011.

[7] C. Archetti, L. Bertazzi, A. Hertz, and M. G. Speranza. A hybrid heuristic for an inventory routing problem. *INFORMS Journal on Computing*, 24 (1):101–116, 2012.

[8] L. Bertazzi, G. Paletta, and M. G. Speranza. Deterministic order-up-to level policies in an inventory routing problem. *Transportation Science*, 36 (1):119–132, 2002.

[9] M. Christiansen. Decomposition of a combined inventory and time constrained ship routing problem. *Transportation Science*, 33(1):3–16, 1999.

[10] L. C. Coelho, J.-F. Cordeau, and G. Laporte. The inventory-routing problem with transshipment. *Computers & Operations Research*, 39(11):2537–2548, 2012.

[11] L. C. Coelho, J.-F. Cordeau, and G. Laporte. Consistency in multi-vehicle inventory-routing. *Transportation Research Part C*, 24(1):270–287, 2012.

[12] S. Dauzère-Pérès, A. Nordli, A. Olstad, K. Haugen, U. Koester, M. P. Olav, G. Teistklub, and A. Reistad. Omya Hustadmarmor optimizes its supply chain for delivering calcium carbonate slurry to European paper manufacturers. *Interfaces*, 37(1):39–51, 2007.

[13] M. Fischetti, J. J. Salazar-González, and P. Toth. Experiments with a multi-commodity formulation for the symmetric capacitated vehicle routing problem. In *Proceedings of the 3rd Meeting of the EURO Working Group on Transportation*, pages 169–173, Barcelona, Spain, 1995.

[14] P. Francis, K. Smilowitz, and M. Tzur. Flexibility and complexity in periodic distribution problems. *Naval Research Logistics*, 54(2):136–150, 2007.

[15] C. Groër, B. L. Golden, and E. A. Wasil. The consistent vehicle routing problem. *Manufacturing & Service Operations Management*, 11(4):630–643, 2009.

[16] R. W. Lien, S. M. R. Iravani, K. Smilowitz, and M. Tzur. An efficient and robust design for transshipment networks. *Production and Operations Management*, 20(5):699–713, 2011.

[17] A. Mercer and X. Tao. Alternative inventory and distribution policies of a food manufacturer. *Journal of the Operational Research Society*, 47(6): 755–765, 1996.

[18] L. M. Nonås and K. Jörnsten. Optimal solution in the multi-location inventory system with transshipments. *Journal of Mathematical Modelling and Algorithms*, 6(1):47–75, 2007.

[19] M. W. Padberg and G. Rinaldi. A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. *SIAM Review*, 33(1):60–100, 1991.

[20] K. Smilowitz, M. Nowak, and T. Jiang. Workforce management in periodic delivery operations. *Transportation Science*, 2012. doi: 10.1287/trsc.1120.0407.

[21] O. Solyalı and H. Süral. A branch-and-cut algorithm using a strong formulation and an a priori tour based heuristic for an inventory-routing problem. *Transportation Science*, 45(3):335–345, 2011.

[22] O. Solyalı and H. Süral. The one-warehouse multi-retailer problem: reformulation, classification and computational results. *Annals of Operations Research*, pages 1–25, 2012. doi: http://dx.doi.org/10.1007/s10479-011-1022-0.

[23] H. Zhong, R. W. Hall, and M. M. Dessouky. Territory planning and vehicle dispatching with driver learning. *Transportation Science*, 41(1):74–89, 2007.

A Computational results on the small instances set, ML inventory policy

Table 7: Computational results on the small instances set, $p = 3$, low inventory cost, $K = 1$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3low-1n5	1235.92	1235.92	0.00	0.0
H3low-2n5	988.66	988.66	0.00	0.0
H3low-3n5	1758.02	1758.02	0.00	0.0
H3low-4n5	1397.29	1397.29	0.00	0.0
H3low-5n5	999.42	999.42	0.00	0.0
H3low-1n10	1743.07	1743.07	0.00	0.0
H3low-2n10	2229.25	2229.25	0.00	1.0
H3low-3n10	1871.14	1871.14	0.00	0.0
H3low-4n10	1773.00	1773.00	0.00	0.0
H3low-5n10	1938.18	1938.18	0.00	2.0
H3low-1n15	2131.04	2131.04	0.00	0.0
H3low-2n15	2131.58	2131.58	0.00	0.0
H3low-3n15	2463.68	2463.68	0.00	0.0
H3low-4n15	2151.94	2151.94	0.00	1.0
H3low-5n15	2160.59	2160.59	0.00	5.0
H3low-1n20	2267.32	2267.32	0.00	0.0
H3low-2n20	2497.90	2497.90	0.00	0.0
H3low-3n20	2590.48	2590.48	0.00	0.0
H3low-4n20	3122.31	3122.31	0.00	6.0
H3low-5n20	2849.90	2849.90	0.00	1.0
H3low-1n25	2840.92	2840.92	0.00	0.0
H3low-2n25	3014.56	3014.56	0.00	11.0
H3low-3n25	3050.40	3050.40	0.00	1.0
H3low-4n25	3078.67	3078.67	0.00	11.0
H3low-5n25	2954.96	2954.96	0.00	0.0
H3low-1n30	3427.78	3427.78	0.00	2.0
H3low-2n30	3328.94	3328.94	0.00	9.0
H3low-3n30	3471.86	3471.86	0.00	0.0
H3low-4n30	3321.48	3321.48	0.00	13.0
H3low-5n30	2914.60	2914.60	0.00	0.0
H3low-1n35	3346.12	3346.12	0.00	1.0
H3low-2n35	3541.71	3541.71	0.00	19.0
H3low-3n35	3811.78	3811.78	0.00	2.0
H3low-4n35	3229.34	3229.34	0.00	1.0
H3low-5n35	3315.26	3315.26	0.00	1.0
H3low-1n40	3702.14	3702.14	0.00	3.0
H3low-2n40	3832.09	3832.09	0.00	21.0
H3low-3n40	3874.62	3874.62	0.00	1.0
H3low-4n40	3534.80	3534.80	0.00	2.0
H3low-5n40	3575.46	3575.46	0.00	11.0
H3low-1n45	3950.86	3950.86	0.00	2.0
H3low-2n45	3702.72	3702.72	0.00	1.0
H3low-3n45	3968.04	3968.04	0.00	2.0
H3low-4n45	3998.26	3998.26	0.00	9.0
H3low-5n45	3717.54	3717.54	0.00	37.0
H3low-1n50	4047.18	4047.18	0.00	42.0
H3low-2n50	4512.96	4512.96	0.00	126.0
H3low-3n50	4451.44	4451.44	0.00	61.0
H3low-4n50	4405.84	4405.84	0.00	6.0
H3low-5n50	4218.37	4218.37	0.00	78.0

Table 8: Computational results on the small instances set, $p = 3$, high inventory cost, $K = 1$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3high-1n5	2108.34	2108.34	0.00	0.0
H3high-2n5	1767.06	1767.06	0.00	0.0
H3high-3n5	2973.00	2973.00	0.00	0.0
H3high-4n5	1981.04	1981.04	0.00	1.0
H3high-5n5	2170.04	2170.04	0.00	0.0
H3high-1n10	4510.61	4510.61	0.00	0.0
H3high-2n10	4504.61	4504.61	0.00	0.0
H3high-3n10	4031.40	4031.40	0.00	0.0
H3high-4n10	3933.46	3933.46	0.00	0.0
H3high-5n10	4709.79	4709.79	0.00	1.0
H3high-1n15	5589.70	5589.70	0.00	0.0
H3high-2n15	5443.34	5443.34	0.00	0.0
H3high-3n15	6300.86	6300.86	0.00	0.0
H3high-4n15	4977.58	4977.58	0.00	1.0
H3high-5n15	4867.53	4867.53	0.00	4.0
H3high-1n20	6859.02	6859.02	0.00	0.0
H3high-2n20	7087.74	7087.74	0.00	2.0
H3high-3n20	7354.68	7354.68	0.00	0.0
H3high-4n20	6952.79	6952.79	0.00	14.0
H3high-5n20	7874.26	7874.26	0.00	2.0
H3high-1n25	8227.86	8227.86	0.00	0.0
H3high-2n25	8765.72	8765.72	0.00	8.0
H3high-3n25	9382.42	9382.42	0.00	1.0
H3high-4n25	8452.93	8452.93	0.00	10.0
H3high-5n25	10081.40	10081.40	0.00	0.0
H3high-1n30	12066.90	12066.90	0.00	2.0
H3high-2n30	10941.30	10941.30	0.00	24.0
H3high-3n30	12122.40	12122.40	0.00	1.0
H3high-4n30	9687.10	9687.10	0.00	16.0
H3high-5n30	9773.90	9773.90	0.00	2.0
H3high-1n35	11659.90	11659.90	0.00	2.0
H3high-2n35	10466.80	10466.80	0.00	20.0
H3high-3n35	13776.50	13776.50	0.00	2.0
H3high-4n35	10307.40	10307.40	0.00	6.0
H3high-5n35	10847.80	10847.80	0.00	3.0
H3high-1n40	13364.90	13364.90	0.00	8.0
H3high-2n40	11317.80	11317.80	0.00	32.0
H3high-3n40	13598.90	13598.90	0.00	3.0
H3high-4n40	11353.40	11353.40	0.00	8.0
H3high-5n40	13070.20	13070.20	0.00	17.0
H3high-1n45	14179.10	14179.10	0.00	24.0
H3high-2n45	13142.20	13142.20	0.00	3.0
H3high-3n45	14843.60	14843.60	0.00	8.0
H3high-4n45	13574.50	13574.50	0.00	17.0
H3high-5n45	13587.30	13587.30	0.00	31.0
H3high-1n50	14577.30	14577.30	0.00	53.0
H3high-2n50	15001.60	15001.60	0.00	40.0
H3high-3n50	15279.50	15279.50	0.00	43.0
H3high-4n50	16517.00	16517.00	0.00	42.0
H3high-5n50	15678.70	15678.70	0.00	66.0

Table 9: Computational results on the small instances set, $p = 6$, low inventory cost, $K = 1$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6low-1n5	3187.30	3187.30	0.00	3.0
H6low-2n5	2565.92	2565.92	0.00	3.0
H6low-3n5	4489.83	4489.83	0.00	2.0
H6low-4n5	3174.35	3174.35	0.00	5.0
H6low-5n5	2267.10	2267.10	0.00	3.0
H6low-1n10	4141.53	4141.53	0.00	15.0
H6low-2n10	5044.63	5044.63	0.00	7.0
H6low-3n10	4506.83	4506.83	0.00	4.0
H6low-4n10	4823.53	4823.53	0.00	2.0
H6low-5n10	4545.98	4545.98	0.00	11.0
H6low-1n15	5389.08	5389.08	0.00	20.0
H6low-2n15	5418.47	5418.47	0.00	28.0
H6low-3n15	5897.68	5897.68	0.00	23.0
H6low-4n15	5335.01	5335.01	0.00	18.0
H6low-5n15	5052.51	5052.51	0.00	23.0
H6low-1n20	6114.04	6114.04	0.00	18.0
H6low-2n20	5957.31	5957.31	0.00	42.0
H6low-3n20	6784.06	6784.06	0.00	27.0
H6low-4n20	7309.54	7309.54	0.00	76.0
H6low-5n20	6961.82	6961.82	0.00	40.0
H6low-1n25	7052.06	7052.06	0.00	53.0
H6low-2n25	7231.75	7231.75	0.00	59.0
H6low-3n25	7514.57	7514.57	0.00	51.0
H6low-4n25	7462.08	7462.08	0.00	45.0
H6low-5n25	7048.40	7048.40	0.00	62.0
H6low-1n30	8052.73	8052.73	0.00	80.0
H6low-2n30	7629.99	7629.99	0.00	106.0
H6low-3n30	8136.21	8136.21	0.00	98.0
H6low-4n30	7501.80	7501.80	0.00	72.0
H6low-5n30	7228.63	7228.63	0.00	128.0

Table 10: Computational results on the small instances set, $p = 6$, high inventory cost, $K = 1$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high-1n5	5789.35	5789.35	0.00	1.0
H6high-2n5	4883.77	4883.77	0.00	2.0
H6high-3n5	6643.29	6643.29	0.00	3.0
H6high-4n5	5076.88	5076.88	0.00	2.0
H6high-5n5	4377.71	4377.71	0.00	0.0
H6high-1n10	8480.17	8480.17	0.00	8.0
H6high-2n10	8347.44	8347.44	0.00	5.0
H6high-3n10	8321.68	8321.68	0.00	3.0
H6high-4n10	8474.26	8474.26	0.00	2.0
H6high-5n10	9386.03	9386.03	0.00	14.0
H6high-1n15	12052.60	12052.60	0.00	33.0
H6high-2n15	11823.50	11823.50	0.00	18.0
H6high-3n15	13305.70	13305.70	0.00	16.0
H6high-4n15	10479.30	10479.30	0.00	31.0
H6high-5n15	10054.10	10054.10	0.00	14.0
H6high-1n20	14266.50	14266.50	0.00	30.0
H6high-2n20	14477.80	14477.80	0.00	24.0
H6high-3n20	14319.30	14319.30	0.00	25.0
H6high-4n20	14390.30	14390.30	0.00	36.0
H6high-5n20	15556.70	15556.70	0.00	28.0
H6high-1n25	15487.70	15487.70	0.00	42.0
H6high-2n25	16502.50	16502.50	0.00	58.0
H6high-3n25	17833.30	17833.30	0.00	57.0
H6high-4n25	16194.00	16194.00	0.00	25.0
H6high-5n25	18552.40	18552.40	0.00	34.0
H6high-1n30	22837.90	22837.90	0.00	73.0
H6high-2n30	19876.80	19876.80	0.00	54.0
H6high-3n30	23096.90	23096.90	0.00	57.0
H6high-4n30	17509.80	17509.80	0.00	62.0
H6high-5n30	18731.80	18731.80	0.00	104.0

Table 11: Computational results on the small instances set, $p = 3$, low inventory cost, $K = 2$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3low-1n5	1396.33	1396.33	0.00	1.0
H3low-2n5	1177.49	1177.49	0.00	4.0
H3low-3n5	2438.02	2438.02	0.00	11.0
H3low-4n5	1717.29	1717.29	0.00	1.0
H3low-5n5	1220.21	1220.21	0.00	2.0
H3low-1n10	2263.19	2263.19	0.00	5.0
H3low-2n10	2809.86	2809.86	0.00	10.0
H3low-3n10	2220.46	2220.46	0.00	11.0
H3low-4n10	2482.06	2482.06	0.00	5.0
H3low-5n10	2159.18	2159.18	0.00	7.0
H3low-1n15	2297.02	2297.02	0.00	13.0
H3low-2n15	2554.10	2554.10	0.00	14.0
H3low-3n15	2800.00	2800.00	0.00	9.0
H3low-4n15	2513.69	2513.69	0.00	11.0
H3low-5n15	2610.59	2610.59	0.00	12.0
H3low-1n20	2917.30	2917.30	0.00	19.0
H3low-2n20	2664.98	2664.98	0.00	14.0
H3low-3n20	2818.62	2818.62	0.00	9.0
H3low-4n20	3560.62	3560.62	0.00	64.0
H3low-5n20	3417.18	3417.18	0.00	16.0
H3low-1n25	3133.28	3133.28	0.00	11.0
H3low-2n25	3501.14	3501.14	0.00	77.0
H3low-3n25	3471.82	3471.82	0.00	28.0
H3low-4n25	3247.61	3247.61	0.00	28.0
H3low-5n25	3506.70	3506.70	0.00	14.0
H3low-1n30	3803.78	3803.78	0.00	61.0
H3low-2n30	3645.20	3645.20	0.00	38.0
H3low-3n30	3616.18	3616.18	0.00	17.0
H3low-4n30	3598.08	3598.08	0.00	147.0
H3low-5n30	3215.86	3215.86	0.00	46.0
H3low-1n35	3597.90	3597.90	0.00	47.0
H3low-2n35	3854.60	3854.60	0.00	82.0
H3low-3n35	4184.06	4184.06	0.00	83.0
H3low-4n35	3441.32	3441.32	0.00	37.0
H3low-5n35	3593.64	3593.64	0.00	31.0
H3low-1n40	3992.46	3992.46	0.00	68.0
H3low-2n40	4180.07	4180.07	0.00	2109.0
H3low-3n40	4105.58	4105.58	0.00	112.0
H3low-4n40	3736.78	3736.78	0.00	46.0
H3low-5n40	3895.46	3895.46	0.00	290.0
H3low-1n45	4077.86	4077.86	0.00	34.0
H3low-2n45	4183.68	4183.68	0.00	106.0
H3low-3n45	4121.72	4121.72	0.00	39.0
H3low-4n45	4457.26	4457.26	0.00	2677.0
H3low-5n45	3911.98	3911.98	0.00	16483.0
H3low-1n50	4567.16	4567.16	0.00	3544.0
H3low-2n50	4842.92	4842.92	0.00	4755.0
H3low-3n50	4705.74	4705.74	0.00	2584.0
H3low-4n50	4662.84	4662.84	0.00	147.0
H3low-5n50	4567.43	4534.27	0.73	42953.0

Table 12: Computational results on the small instances set, $p = 3$, high inventory cost, $K = 2$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3high-1n5	2265.21	2265.21	0.00	0.0
H3high-2n5	1969.31	1969.31	0.00	3.0
H3high-3n5	3653.00	3653.00	0.00	6.0
H3high-4n5	2301.04	2301.04	0.00	3.0
H3high-5n5	2372.36	2372.36	0.00	2.0
H3high-1n10	5032.05	5032.05	0.00	8.0
H3high-2n10	5080.54	5080.54	0.00	6.0
H3high-3n10	4371.94	4371.94	0.00	6.0
H3high-4n10	4643.24	4643.24	0.00	5.0
H3high-5n10	4930.79	4930.79	0.00	4.0
H3high-1n15	5755.54	5755.54	0.00	11.0
H3high-2n15	5853.37	5853.37	0.00	9.0
H3high-3n15	6646.62	6646.62	0.00	12.0
H3high-4n15	5338.36	5338.36	0.00	10.0
H3high-5n15	5317.53	5317.53	0.00	16.0
H3high-1n20	7508.76	7508.76	0.00	23.0
H3high-2n20	7254.98	7254.98	0.00	16.0
H3high-3n20	7592.14	7592.14	0.00	8.0
H3high-4n20	7380.59	7380.59	0.00	52.0
H3high-5n20	8441.54	8441.54	0.00	20.0
H3high-1n25	8521.52	8521.52	0.00	10.0
H3high-2n25	9258.10	9258.10	0.00	73.0
H3high-3n25	9804.92	9804.92	0.00	26.0
H3high-4n25	8631.33	8631.33	0.00	28.0
H3high-5n25	10633.10	10633.10	0.00	17.0
H3high-1n30	12449.90	12449.90	0.00	53.0
H3high-2n30	11258.10	11258.10	0.00	55.0
H3high-3n30	12277.10	12277.10	0.00	27.0
H3high-4n30	9963.98	9963.98	0.00	165.0
H3high-5n30	10063.50	10063.50	0.00	52.0
H3high-1n35	11922.30	11922.30	0.00	53.0
H3high-2n35	10765.60	10765.60	0.00	79.0
H3high-3n35	14147.50	14147.50	0.00	65.0
H3high-4n35	10522.10	10522.10	0.00	80.0
H3high-5n35	11127.00	11127.00	0.00	51.0
H3high-1n40	13660.40	13660.40	0.00	68.0
H3high-2n40	11665.70	11665.70	0.00	1809.0
H3high-3n40	13830.10	13830.10	0.00	338.0
H3high-4n40	11590.40	11590.40	0.00	67.0
H3high-5n40	13390.40	13390.40	0.00	111.0
H3high-1n45	14307.30	14307.30	0.00	105.0
H3high-2n45	13640.90	13640.90	0.00	308.0
H3high-3n45	15008.20	15008.20	0.00	68.0
H3high-4n45	14033.50	14033.50	0.00	4939.0
H3high-5n45	13792.40	13792.40	0.00	2555.0
H3high-1n50	15097.10	15097.10	0.00	1037.0
H3high-2n50	15342.90	15342.90	0.00	1853.0
H3high-3n50	15520.90	15520.90	0.00	1534.0
H3high-4n50	16775.10	16775.10	0.00	532.0
H3high-5n50	16027.70	16027.70	0.00	17202.0

Table 13: Computational results on the small instances set, $p = 6$, low inventory cost, $K = 2$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6low-1n5	3775.68	3775.68	0.00	8.0
H6low-2n5	3185.19	3185.19	0.00	9.0
H6low-3n5	5962.76	5962.76	0.00	11.0
H6low-4n5	3805.98	3805.98	0.00	12.0
H6low-5n5	2891.85	2891.85	0.00	5.0
H6low-1n10	5667.41	5667.41	0.00	93.0
H6low-2n10	6512.43	6512.43	0.00	784.0
H6low-3n10	5260.44	5260.44	0.00	2088.0
H6low-4n10	6198.49	6198.49	0.00	46.0
H6low-5n10	5137.52	5137.52	0.00	26.0
H6low-1n15	5987.40	5987.40	0.00	208.0
H6low-2n15	6152.89	6152.89	0.00	134.0
H6low-3n15	7011.20	7011.20	0.00	1677.0
H6low-4n15	6126.71	6126.71	0.00	132.0
H6low-5n15	6364.99	6364.99	0.00	625.0
H6low-1n20	7388.80	7388.80	0.00	2276.0
H6low-2n20	6358.66	6358.66	0.00	547.0
H6low-3n20	7473.61	7473.61	0.00	4130.0
H6low-4n20	8310.80	8310.80	0.00	2738.0
H6low-5n20	8509.86	8509.86	0.00	33521.0
H6low-1n25	7461.55	7461.55	0.00	735.0
H6low-2n25	8309.70	8309.70	0.00	12284.0
H6low-3n25	8690.21	8690.21	0.00	38196.0
H6low-4n25	7884.33	7884.33	0.00	796.0
H6low-5n25	8532.58	8430.69	1.19	42999.0
H6low-1n30	9056.82	8972.12	0.94	42906.0
H6low-2n30	8449.94	8214.35	2.79	43045.0
H6low-3n30	8442.43	8442.43	0.00	12000.0
H6low-4n30	8225.35	8220.13	0.06	43200.0
H6low-5n30	8045.41	7648.23	4.94	43026.0

Table 14: Computational results on the small instances set, $p = 6$, high inventory cost, $K = 2$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high-1n5	6379.56	6379.56	0.00	8.0
H6high-2n5	5498.64	5498.64	0.00	8.0
H6high-3n5	8108.03	8108.03	0.00	12.0
H6high-4n5	5718.02	5718.02	0.00	9.0
H6high-5n5	4999.24	4999.24	0.00	8.0
H6high-1n10	9996.96	9996.96	0.00	78.0
H6high-2n10	9814.04	9814.04	0.00	109.0
H6high-3n10	9071.05	9071.05	0.00	53.0
H6high-4n10	9864.35	9864.35	0.00	32.0
H6high-5n10	10018.40	10018.40	0.00	16.0
H6high-1n15	12624.70	12624.70	0.00	148.0
H6high-2n15	12568.40	12568.40	0.00	94.0
H6high-3n15	14450.10	14450.10	0.00	1265.0
H6high-4n15	11267.70	11267.70	0.00	68.0
H6high-5n15	11374.40	11374.40	0.00	180.0
H6high-1n20	15540.40	15540.40	0.00	3232.0
H6high-2n20	14940.30	14940.30	0.00	580.0
H6high-3n20	14986.80	14986.80	0.00	767.0
H6high-4n20	15423.00	15423.00	0.00	2807.0
H6high-5n20	17096.50	17096.50	0.00	12793.0
H6high-1n25	15954.80	15954.80	0.00	197.0
H6high-2n25	17577.40	17577.40	0.00	20332.0
H6high-3n25	19024.30	19024.30	0.00	11288.0
H6high-4n25	16663.10	16663.10	0.00	3253.0
H6high-5n25	20003.30	20003.30	0.00	15731.0
H6high-1n30	23859.40	23620.20	1.00	43185.0
H6high-2n30	20675.90	20675.90	0.00	28999.0
H6high-3n30	23416.80	23416.80	0.00	1547.0
H6high-4n30	18275.20	18275.20	0.00	27297.0
H6high-5n30	19576.90	19120.50	2.33	42916.0

Table 15: Computational results on the small instances set, $p = 3$, low inventory cost, $K = 3$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3low-1n5	1430.51	1430.51	0.00	1.0
H3low-2n5	1582.69	1582.69	0.00	12.0
H3low-3n5	2997.44	2997.44	0.00	1.0
H3low-4n5	2291.17	2291.17	0.00	6.0
H3low-5n5	1513.76	1513.76	0.00	3.0
H3low-1n10	2732.61	2732.61	0.00	21.0
H3low-2n10	3470.15	3470.15	0.00	18.0
H3low-3n10	2649.26	2649.26	0.00	16.0
H3low-4n10	3183.36	3183.36	0.00	17.0
H3low-5n10	2461.11	2461.11	0.00	15.0
H3low-1n15	2783.77	2783.77	0.00	23.0
H3low-2n15	2757.82	2757.82	0.00	21.0
H3low-3n15	3072.84	3072.84	0.00	24.0
H3low-4n15	2886.32	2886.32	0.00	34.0
H3low-5n15	3260.59	3260.59	0.00	55.0
H3low-1n20	3605.72	3605.72	0.00	196.0
H3low-2n20	2908.48	2908.48	0.00	35.0
H3low-3n20	3064.78	3064.78	0.00	20.0
H3low-4n20	4088.88	4088.88	0.00	667.0
H3low-5n20	4124.24	4124.24	0.00	186.0
H3low-1n25	3503.38	3503.38	0.00	83.0
H3low-2n25	3952.08	3952.08	0.00	2214.0
H3low-3n25	4068.66	4068.66	0.00	222.0
H3low-4n25	3659.14	3659.14	0.00	113.0
H3low-5n25	4120.26	4120.26	0.00	239.0
H3low-1n30	4251.64	4251.64	0.00	1069.0
H3low-2n30	4102.37	4102.37	0.00	1475.0
H3low-3n30	3820.66	3820.66	0.00	115.0
H3low-4n30	3958.58	3958.58	0.00	3297.0
H3low-5n30	3588.64	3588.64	0.00	473.0
H3low-1n35	4080.60	4080.60	0.00	2463.0
H3low-2n35	4221.73	4221.73	0.00	1102.0
H3low-3n35	4725.86	4725.86	0.00	1417.0
H3low-4n35	4011.04	4011.04	0.00	2438.0
H3low-5n35	4034.70	4034.70	0.00	2259.0
H3low-1n40	4532.84	4532.84	0.00	13369.0
H3low-2n40	4544.37	4544.37	0.00	19595.0
H3low-3n40	4310.24	4310.24	0.00	749.0
H3low-4n40	3994.38	3994.38	0.00	911.0
H3low-5n40	4339.54	4339.54	0.00	10836.0
H3low-1n45	4537.30	4537.30	0.00	29437.0
H3low-2n45	4744.96	4684.66	1.27	43052.0
H3low-3n45	4267.42	4267.42	0.00	203.0
H3low-4n45	4921.16	4860.16	1.24	43131.0
H3low-5n45	4140.74	4067.02	1.78	43203.0
H3low-1n50	6017.66	4801.53	20.21	42832.0
H3low-2n50	5913.05	5158.88	12.75	43014.0
H3low-3n50	5380.35	4906.64	8.80	42842.0
H3low-4n50	5377.14	4984.34	7.31	42936.0
H3low-5n50	5390.00	4694.25	12.91	43028.0

Table 16: Computational results on the small instances set, $p = 3$, high inventory cost, $K = 3$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3high-1n5	2298.73	2298.73	0.00	0.0
H3high-2n5	2369.93	2369.93	0.00	4.0
H3high-3n5	4191.26	4191.26	0.00	3.0
H3high-4n5	2874.10	2874.10	0.00	3.0
H3high-5n5	2663.66	2663.66	0.00	3.0
H3high-1n10	5506.09	5506.09	0.00	13.0
H3high-2n10	5743.25	5743.25	0.00	13.0
H3high-3n10	4808.06	4808.06	0.00	17.0
H3high-4n10	5335.29	5335.29	0.00	12.0
H3high-5n10	5224.45	5224.45	0.00	8.0
H3high-1n15	6242.90	6242.90	0.00	16.0
H3high-2n15	6071.26	6071.26	0.00	24.0
H3high-3n15	6926.18	6926.18	0.00	14.0
H3high-4n15	5705.16	5705.16	0.00	16.0
H3high-5n15	5967.25	5967.25	0.00	60.0
H3high-1n20	8165.42	8165.42	0.00	229.0
H3high-2n20	7499.50	7499.50	0.00	48.0
H3high-3n20	7840.54	7840.54	0.00	22.0
H3high-4n20	7919.09	7919.09	0.00	313.0
H3high-5n20	9149.16	9149.16	0.00	475.0
H3high-1n25	8893.82	8893.82	0.00	53.0
H3high-2n25	9708.34	9708.34	0.00	4250.0
H3high-3n25	10404.90	10404.90	0.00	379.0
H3high-4n25	9049.07	9049.07	0.00	167.0
H3high-5n25	11250.90	11250.90	0.00	219.0
H3high-1n30	12908.90	12908.90	0.00	2409.0
H3high-2n30	11686.50	11686.50	0.00	1965.0
H3high-3n30	12488.00	12488.00	0.00	102.0
H3high-4n30	10287.10	10287.10	0.00	844.0
H3high-5n30	10450.40	10450.40	0.00	2797.0
H3high-1n35	12396.00	12396.00	0.00	3380.0
H3high-2n35	11125.30	11125.30	0.00	513.0
H3high-3n35	14694.50	14694.50	0.00	1157.0
H3high-4n35	11092.90	11092.90	0.00	6014.0
H3high-5n35	11568.30	11568.30	0.00	2416.0
H3high-1n40	14224.10	14224.10	0.00	9173.0
H3high-2n40	12015.60	12015.60	0.00	6950.0
H3high-3n40	14040.40	14040.40	0.00	1973.0
H3high-4n40	11849.00	11849.00	0.00	1540.0
H3high-5n40	13835.10	13835.10	0.00	11926.0
H3high-1n45	14771.00	14771.00	0.00	34500.0
H3high-2n45	14257.20	14116.30	0.99	43015.0
H3high-3n45	15170.00	15170.00	0.00	387.0
H3high-4n45	14503.70	14389.30	0.79	43055.0
H3high-5n45	14012.20	13978.90	0.24	43146.0
H3high-1n50	16115.80	15169.10	5.87	43101.0
H3high-2n50	16304.20	15745.50	3.43	43046.0
H3high-3n50	16132.10	15602.10	3.29	42862.0
H3high-4n50	17616.40	17084.20	3.02	43064.0
H3high-5n50	17182.70	16091.40	6.35	42881.0

Table 17: Computational results on the small instances set, $p = 6$, low inventory cost, $K = 3$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6low-1n5	4657.15	4657.15	0.00	30.0
H6low-2n5	4221.00	4221.00	0.00	80.0
H6low-3n5	7703.53	7703.53	0.00	37.0
H6low-4n5	4495.31	4495.31	0.00	21.0
H6low-5n5	3879.09	3879.09	0.00	115.0
H6low-1n10	7119.24	7119.24	0.00	22691.0
H6low-2n10	8276.23	7868.99	4.92	37276.0
H6low-3n10	6185.29	6185.29	0.00	10369.0
H6low-4n10	7483.28	7483.28	0.00	1672.0
H6low-5n10	5824.09	5824.09	0.00	885.0
H6low-1n15	6861.07	6861.07	0.00	15096.0
H6low-2n15	7107.77	7107.77	0.00	28077.0
H6low-3n15	8158.42	8012.93	1.78	42951.0
H6low-4n15	7157.60	7157.60	0.00	2216.0
H6low-5n15	7605.73	7605.73	0.00	5467.0
H6low-1n20	8804.32	8354.27	5.11	42907.0
H6low-2n20	6928.07	6928.07	0.00	41904.0
H6low-3n20	8461.61	7933.11	6.25	43019.0
H6low-4n20	10177.10	8999.35	11.57	42853.0
H6low-5n20	10598.10	9261.09	12.62	43076.0
H6low-1n25	8469.87	8302.80	1.97	43102.0
H6low-2n25	9779.99	8413.32	13.97	43002.0
H6low-3n25	10659.10	9043.48	15.16	43096.0
H6low-4n25	8810.72	8736.15	0.85	43076.0
H6low-5n25	10408.10	9174.62	11.85	42961.0
H6low-1n30	10883.80	9077.31	16.60	43207.0
H6low-2n30	9445.46	8857.73	6.22	42982.0
H6low-3n30	9492.10	8652.74	8.84	43219.0
H6low-4n30	9444.94	8169.34	13.51	43142.0
H6low-5n30	9348.98	7719.84	17.43	42848.0

Table 18: Computational results on the small instances set, $p = 6$, high inventory cost, $K = 3$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high-1n5	7259.05	7259.05	0.00	34.0
H6high-2n5	6530.35	6530.35	0.00	72.0
H6high-3n5	9862.89	9862.89	0.00	20.0
H6high-4n5	6405.74	6405.74	0.00	26.0
H6high-5n5	5972.78	5972.78	0.00	38.0
H6high-1n10	11440.90	11440.90	0.00	27693.0
H6high-2n10	11584.80	11171.10	3.57	42840.0
H6high-3n10	9982.68	9982.68	0.00	963.0
H6high-4n10	11168.50	11168.50	0.00	1244.0
H6high-5n10	10702.10	10702.10	0.00	316.0
H6high-1n15	13517.60	13517.60	0.00	6559.0
H6high-2n15	13512.70	13512.70	0.00	2127.0
H6high-3n15	15622.90	15251.30	2.38	43080.0
H6high-4n15	12296.00	12296.00	0.00	1313.0
H6high-5n15	12610.70	12610.70	0.00	9273.0
H6high-1n20	16977.30	16536.20	2.60	42896.0
H6high-2n20	15505.00	15386.70	0.76	43022.0
H6high-3n20	15940.40	15545.90	2.48	42961.0
H6high-4n20	17243.50	16027.70	7.05	43020.0
H6high-5n20	19047.50	17861.90	6.22	43029.0
H6high-1n25	16911.30	16828.50	0.49	42843.0
H6high-2n25	19270.40	17648.50	8.42	43227.0
H6high-3n25	20786.60	19086.90	8.18	43188.0
H6high-4n25	17593.30	17593.30	0.00	23895.0
H6high-5n25	21936.90	20665.30	5.80	43054.0
H6high-1n30	25657.30	23629.20	7.90	43128.0
H6high-2n30	22061.00	20984.40	4.88	42890.0
H6high-3n30	24390.20	23618.40	3.16	42917.0
H6high-4n30	19460.40	18092.30	7.03	42964.0
H6high-5n30	21036.60	19350.40	8.02	42920.0

Table 19: Computational results on the small instances set, $p = 3$, low inventory cost, $K = 4$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3low-1n5	1601.57	1601.57	0.00	7.0
H3low-2n5	1812.65	1812.65	0.00	2.0
H3low-3n5	3603.74	3603.74	0.00	5.0
H3low-4n5	2631.58	2631.58	0.00	3.0
H3low-5n5	1676.40	1676.40	0.00	3.0
H3low-1n10	3261.94	3261.94	0.00	31.0
H3low-2n10	4271.02	4271.02	0.00	106.0
H3low-3n10	2968.74	2968.74	0.00	19.0
H3low-4n10	3683.17	3683.17	0.00	14.0
H3low-5n10	2872.09	2872.09	0.00	34.0
H3low-1n15	3166.44	3166.44	0.00	98.0
H3low-2n15	3396.66	3396.66	0.00	66.0
H3low-3n15	3757.36	3757.36	0.00	138.0
H3low-4n15	3200.18	3200.18	0.00	52.0
H3low-5n15	3671.07	3671.07	0.00	241.0
H3low-1n20	4148.00	4148.00	0.00	8665.0
H3low-2n20	3128.05	3128.05	0.00	203.0
H3low-3n20	3645.48	3645.48	0.00	862.0
H3low-4n20	4773.73	4773.73	0.00	15735.0
H3low-5n20	4764.78	4764.78	0.00	2257.0
H3low-1n25	3949.73	3949.73	0.00	1046.0
H3low-2n25	4501.34	4501.34	0.00	13308.0
H3low-3n25	4687.62	4687.62	0.00	6081.0
H3low-4n25	4044.84	4044.84	0.00	906.0
H3low-5n25	4672.82	4672.82	0.00	1988.0
H3low-1n30	4720.58	4643.47	1.63	42990.0
H3low-2n30	4429.49	4429.49	0.00	18467.0
H3low-3n30	4142.12	4142.12	0.00	1240.0
H3low-4n30	4618.97	4087.33	11.51	42927.0
H3low-5n30	4028.06	3974.00	1.34	42950.0
H3low-1n35	4368.96	4368.96	0.00	15974.0
H3low-2n35	4624.20	4624.20	0.00	13713.0
H3low-3n35	5390.98	5020.31	6.88	43113.0
H3low-4n35	4675.38	4071.09	12.92	43132.0
H3low-5n35	4539.64	4193.28	7.63	42849.0
H3low-1n40	5142.30	4398.24	14.47	42929.0
H3low-2n40	5136.10	4540.65	11.59	42903.0
H3low-3n40	4665.60	4571.66	2.01	43139.0
H3low-4n40	4543.98	4401.36	3.14	43138.0
H3low-5n40	4778.69	4345.82	9.06	42943.0
H3low-1n45	5215.32	4598.71	11.82	40299.0
H3low-2n45	5878.08	4573.29	22.20	43098.0
H3low-3n45	4514.50	4514.50	0.00	4172.0
H3low-4n45	5804.92	4814.47	17.06	42956.0
H3low-5n45	4448.83	4074.05	8.42	43083.0
H3low-1n50	6918.43	4956.97	28.35	43024.0
H3low-2n50	7169.05	5379.86	24.96	43043.0
H3low-3n50	5783.26	5008.61	13.39	42870.0
H3low-4n50	6848.93	4984.11	27.23	43154.0
H3low-5n50	8032.42	4739.66	40.99	42902.0

Table 20: Computational results on the small instances set, $p = 3$, high inventory cost, $K = 4$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3high-1n5	2472.11	2472.11	0.00	4.0
H3high-2n5	2599.64	2599.64	0.00	3.0
H3high-3n5	4807.52	4807.52	0.00	9.0
H3high-4n5	3210.24	3210.24	0.00	5.0
H3high-5n5	2824.50	2824.50	0.00	3.0
H3high-1n10	6021.09	6021.09	0.00	32.0
H3high-2n10	6539.31	6539.31	0.00	172.0
H3high-3n10	5126.96	5126.96	0.00	12.0
H3high-4n10	5834.68	5834.68	0.00	30.0
H3high-5n10	5644.39	5644.39	0.00	24.0
H3high-1n15	6611.31	6611.31	0.00	70.0
H3high-2n15	6705.87	6705.87	0.00	86.0
H3high-3n15	7607.68	7607.68	0.00	122.0
H3high-4n15	6017.46	6017.46	0.00	37.0
H3high-5n15	6375.39	6375.39	0.00	132.0
H3high-1n20	8717.83	8717.83	0.00	18778.0
H3high-2n20	7710.91	7710.91	0.00	137.0
H3high-3n20	8414.08	8414.08	0.00	1081.0
H3high-4n20	8589.28	8589.28	0.00	17237.0
H3high-5n20	9782.55	9782.55	0.00	1664.0
H3high-1n25	9287.90	9287.90	0.00	611.0
H3high-2n25	10259.10	10259.10	0.00	33516.0
H3high-3n25	11026.80	11026.80	0.00	8247.0
H3high-4n25	9436.93	9436.93	0.00	1461.0
H3high-5n25	11806.00	11806.00	0.00	5071.0
H3high-1n30	13410.20	13277.70	0.99	42970.0
H3high-2n30	12027.90	12027.90	0.00	7374.0
H3high-3n30	12823.60	12823.60	0.00	716.0
H3high-4n30	10875.50	10611.20	2.43	43028.0
H3high-5n30	10903.20	10671.60	2.12	43009.0
H3high-1n35	12698.30	12698.30	0.00	7699.0
H3high-2n35	11530.40	11530.40	0.00	3136.0
H3high-3n35	15394.90	15004.80	2.53	42900.0
H3high-4n35	11719.50	11244.40	4.05	42953.0
H3high-5n35	12033.90	11783.30	2.08	43088.0
H3high-1n40	14819.50	14157.70	4.47	42826.0
H3high-2n40	12580.80	12076.60	4.01	43181.0
H3high-3n40	14372.90	14372.90	0.00	31408.0
H3high-4n40	12329.30	12329.30	0.00	36907.0
H3high-5n40	14275.70	13947.60	2.30	43217.0
H3high-1n45	15639.20	14724.90	5.85	43214.0
H3high-2n45	15180.70	14079.30	7.26	42899.0
H3high-3n45	15451.70	15451.70	0.00	1894.0
H3high-4n45	15504.50	14478.30	6.62	43097.0
H3high-5n45	14305.00	14091.20	1.49	42843.0
H3high-1n50	17243.40	15516.50	10.01	43121.0
H3high-2n50	16917.20	15910.60	5.95	43026.0
H3high-3n50	17280.20	15597.20	9.74	43017.0
H3high-4n50	19489.30	17209.10	11.70	42925.0
H3high-5n50	18293.10	16232.50	11.26	43069.0

Table 21: Computational results on the small instances set, $p = 6$, low inventory cost, $K = 4$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6low-1n5	5506.19	5506.19	0.00	38.0
H6low-2n5	5125.83	5125.83	0.00	184.0
H6low-3n5	9320.73	9320.73	0.00	40.0
H6low-4n5	5085.81	5085.81	0.00	14.0
H6low-5n5	4913.41	4913.41	0.00	112.0
H6low-1n10	8422.59	7639.49	9.30	39553.0
H6low-2n10	9875.13	8954.74	9.32	34170.0
H6low-3n10	7255.57	7255.57	0.00	20118.0
H6low-4n10	8645.11	8645.11	0.00	14402.0
H6low-5n10	6639.54	6423.25	3.26	42919.0
H6low-1n15	7784.01	7269.93	6.60	42997.0
H6low-2n15	8073.80	7879.71	2.40	42889.0
H6low-3n15	9299.33	8334.00	10.38	39483.0
H6low-4n15	8422.43	8044.19	4.49	43059.0
H6low-5n15	8982.66	8519.93	5.15	43036.0
H6low-1n20	10466.50	8930.06	14.68	43047.0
H6low-2n20	7656.16	7146.28	6.66	42888.0
H6low-3n20	9483.95	8208.28	13.45	43175.0
H6low-4n20	11716.70	9324.25	20.42	43195.0
H6low-5n20	12590.50	10283.40	18.32	43186.0
H6low-1n25	9428.53	9064.78	3.86	43138.0
H6low-2n25	11914.70	8342.46	29.98	42908.0
H6low-3n25	12494.10	9089.88	27.25	40847.0
H6low-4n25	9919.66	9178.56	7.47	42892.0
H6low-5n25	12536.30	9978.67	20.40	42901.0
H6low-1n30	13363.20	9051.40	32.27	42843.0
H6low-2n30	10863.80	9117.65	16.07	43023.0
H6low-3n30	52341.40	8417.00	83.92	41539.0
H6low-4n30	11309.10	8288.79	26.71	43178.0
H6low-5n30	11326.60	7887.49	30.36	42840.0

Table 22: Computational results on the small instances set, $p = 6$, high inventory cost, $K = 4$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high-1n5	8111.31	8111.31	0.00	70.0
H6high-2n5	7411.33	7411.33	0.00	82.0
H6high-3n5	11475.30	11475.30	0.00	33.0
H6high-4n5	6985.40	6985.40	0.00	19.0
H6high-5n5	6996.36	6996.36	0.00	52.0
H6high-1n10	12752.40	11833.50	7.21	29909.0
H6high-2n10	13253.20	12188.00	8.04	29270.0
H6high-3n10	11067.90	11067.90	0.00	14067.0
H6high-4n10	12323.90	12323.90	0.00	11245.0
H6high-5n10	11477.50	11378.90	0.86	42842.0
H6high-1n15	14489.00	13732.70	5.22	36506.0
H6high-2n15	14514.70	14293.30	1.52	42948.0
H6high-3n15	16747.90	15921.70	4.93	34115.0
H6high-4n15	13464.90	13311.40	1.14	42933.0
H6high-5n15	13996.10	13508.70	3.48	43221.0
H6high-1n20	18559.60	17105.30	7.84	42860.0
H6high-2n20	16256.30	15523.20	4.51	28166.0
H6high-3n20	17089.40	15826.80	7.39	43161.0
H6high-4n20	19196.00	16076.50	16.25	30128.0
H6high-5n20	21180.50	18961.50	10.48	42890.0
H6high-1n25	17858.00	17450.60	2.28	42980.0
H6high-2n25	50986.40	16948.10	66.76	17087.0
H6high-3n25	52704.60	19027.00	63.90	26197.0
H6high-4n25	18649.40	18082.60	3.04	42828.0
H6high-5n25	24248.60	21447.50	11.55	42916.0
H6high-1n30	28755.20	23749.60	17.41	43209.0
H6high-2n30	23175.20	21576.10	6.89	42995.0
H6high-3n30	25690.10	23791.30	7.39	43181.0
H6high-4n30	21487.60	18408.90	14.32	43125.0
H6high-5n30	23170.70	19344.10	16.51	42890.0

Table 23: Computational results on the small instances set, $p = 3$, low inventory cost, $K = 5$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3low-1n5	1710.34	1710.34	0.00	3.0
H3low-2n5	2019.58	2019.58	0.00	6.0
H3low-3n5	3965.84	3965.84	0.00	5.0
H3low-4n5	3318.97	3318.97	0.00	6.0
H3low-5n5	2008.54	2008.54	0.00	3.0
H3low-1n10	3728.78	3728.78	0.00	230.0
H3low-2n10	4681.34	4681.34	0.00	110.0
H3low-3n10	3401.18	3401.18	0.00	68.0
H3low-4n10	4156.96	4156.96	0.00	33.0
H3low-5n10	2993.89	2993.89	0.00	28.0
H3low-1n15	3580.81	3580.81	0.00	1177.0
H3low-2n15	3889.43	3889.43	0.00	975.0
H3low-3n15	4133.60	4133.60	0.00	368.0
H3low-4n15	3572.53	3572.53	0.00	132.0
H3low-5n15	4256.15	4256.15	0.00	3326.0
H3low-1n20	4404.77	4404.77	0.00	1709.0
H3low-2n20	3344.11	3344.11	0.00	1050.0
H3low-3n20	4016.48	4016.48	0.00	2785.0
H3low-4n20	5248.12	4641.62	11.56	24596.0
H3low-5n20	5506.02	5410.33	1.74	42956.0
H3low-1n25	4095.20	4095.20	0.00	1241.0
H3low-2n25	5040.04	4369.83	13.30	43041.0
H3low-3n25	5280.32	5065.34	4.07	43054.0
H3low-4n25	4378.75	4378.75	0.00	3264.0
H3low-5n25	5372.70	5172.71	3.72	43001.0
H3low-1n30	5571.84	4529.26	18.71	42929.0
H3low-2n30	4879.36	4813.40	1.35	42953.0
H3low-3n30	4458.94	4458.94	0.00	26888.0
H3low-4n30	5044.12	4230.86	16.12	43105.0
H3low-5n30	4463.06	3896.28	12.70	43093.0
H3low-1n35	4823.24	4484.06	7.03	43108.0
H3low-2n35	5180.05	4892.34	5.55	43189.0
H3low-3n35	6252.49	5284.21	15.49	42853.0
H3low-4n35	6252.49	5284.21	15.49	42853.0
H3low-5n35	5330.98	4307.25	19.20	43047.0
H3low-1n40	6206.74	4474.75	27.91	43190.0
H3low-2n40	5864.94	4720.55	19.51	42839.0
H3low-3n40	5960.13	4448.56	25.36	1390.0
H3low-4n40	4895.50	4385.20	10.42	43219.0
H3low-5n40	5462.27	4572.73	16.29	43197.0
H3low-1n45	6113.30	4653.98	23.87	43196.0
H3low-2n45	7449.50	4687.68	37.07	43159.0
H3low-3n45	4816.82	4756.58	1.25	43073.0
H3low-4n45	7453.31	5012.47	32.75	42939.0
H3low-5n45	5089.89	4142.31	18.62	42864.0
H3low-1n50	8845.57	5181.86	41.42	43168.0
H3low-2n50	8284.56	5477.81	33.88	43179.0
H3low-3n50	6561.62	5123.89	21.91	42854.0
H3low-4n50	7833.30	5091.06	35.01	43141.0
H3low-5n50	8755.37	4928.37	43.71	42875.0

Table 24: Computational results on the small instances set, $p = 3$, high inventory cost, $K = 5$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H3high-1n5	2577.54	2577.54	0.00	6.0
H3high-2n5	2807.26	2807.26	0.00	4.0
H3high-3n5	5160.89	5160.89	0.00	6.0
H3high-4n5	3899.72	3899.72	0.00	4.0
H3high-5n5	3166.83	3166.83	0.00	3.0
H3high-1n10	6496.66	6496.66	0.00	138.0
H3high-2n10	6960.89	6960.89	0.00	100.0
H3high-3n10	5557.18	5557.18	0.00	83.0
H3high-4n10	6308.77	6308.77	0.00	29.0
H3high-5n10	5771.47	5771.47	0.00	26.0
H3high-1n15	7023.67	7023.67	0.00	823.0
H3high-2n15	7194.30	7194.30	0.00	1983.0
H3high-3n15	7981.65	7981.65	0.00	755.0
H3high-4n15	6388.14	6388.14	0.00	114.0
H3high-5n15	6963.56	6963.56	0.00	1366.0
H3high-1n20	8965.21	8965.21	0.00	1020.0
H3high-2n20	7910.50	7910.50	0.00	618.0
H3high-3n20	8787.48	8787.48	0.00	4279.0
H3high-4n20	9047.34	9018.30	0.32	43061.0
H3high-5n20	10523.80	10523.80	0.00	32363.0
H3high-1n25	9473.21	9473.21	0.00	695.0
H3high-2n25	10824.20	10399.50	3.92	42959.0
H3high-3n25	11625.70	11436.10	1.63	43159.0
H3high-4n25	9719.33	9719.33	0.00	1267.0
H3high-5n25	12541.60	12153.60	3.09	43194.0
H3high-1n30	14185.00	13351.20	5.88	43201.0
H3high-2n30	12475.60	12421.40	0.43	42930.0
H3high-3n30	13129.10	13129.10	0.00	22272.0
H3high-4n30	11650.60	10494.30	9.92	42889.0
H3high-5n30	11413.20	10747.80	5.83	43078.0
H3high-1n35	13202.20	12760.00	3.35	43194.0
H3high-2n35	12396.50	11588.90	6.51	42973.0
H3high-3n35	16195.10	15173.10	6.31	43048.0
H3high-4n35	12425.50	11295.20	9.10	42962.0
H3high-5n35	12647.90	11769.50	6.95	42886.0
H3high-1n40	15600.60	14283.80	8.44	43120.0
H3high-2n40	13068.00	12304.20	5.84	42965.0
H3high-3n40	14788.70	14351.80	2.95	43135.0
H3high-4n40	13006.10	12155.70	6.54	16312.0
H3high-5n40	14833.90	14134.60	4.71	43003.0
H3high-1n45	16255.20	14906.60	8.30	43200.0
H3high-2n45	16791.10	14237.00	15.21	43199.0
H3high-3n45	15783.60	15697.30	0.55	43129.0
H3high-4n45	16656.70	14635.10	12.14	43020.0
H3high-5n45	14941.20	13908.10	6.91	42966.0
H3high-1n50	18612.70	15703.00	15.63	43204.0
H3high-2n50	18278.90	15936.40	12.82	42940.0
H3high-3n50	18772.40	15884.90	15.38	42951.0
H3high-4n50	20105.50	17314.50	13.88	43002.0
H3high-5n50	20186.80	16311.20	19.20	43192.0

Table 25: Computational results on the small instances set, $p = 6$, low inventory cost, $K = 5$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6low-1n5	6445.68	6445.68	0.00	191.0
H6low-2n5	6009.40	6009.40	0.00	328.0
H6low-3n5	11282.70	11282.70	0.00	203.0
H6low-4n5	6285.00	6285.00	0.00	109.0
H6low-5n5	infeasible			
H6low-1n10	9993.84	8304.16	16.91	30550.0
H6low-2n10	11692.50	10042.20	14.11	22904.0
H6low-3n10	8240.86	8118.06	1.49	42911.0
H6low-4n10	10092.50	9568.94	5.18	42925.0
H6low-5n10	7230.24	6964.14	3.68	43202.0
H6low-1n15	8966.03	7497.79	16.37	42663.0
H6low-2n15	9073.30	8499.30	6.32	43070.0
H6low-3n15	10584.20	8810.77	16.75	19067.0
H6low-4n15	9553.43	8734.27	8.57	42933.0
H6low-5n15	10731.90	9171.31	14.54	39443.0
H6low-1n20	12030.00	9087.64	24.46	43060.0
H6low-2n20	8386.88	7352.43	12.33	43187.0
H6low-3n20	10963.20	8601.59	21.54	42901.0
H6low-4n20	13762.50	10002.60	27.32	43005.0
H6low-5n20	15268.00	11379.90	25.47	40076.0
H6low-1n25	10637.10	9573.27	10.00	43188.0
H6low-2n25	45300.10	8021.73	82.29	39063.0
H6low-3n25	15054.70	9966.92	33.80	43103.0
H6low-4n25	11129.10	9773.34	12.18	43152.0
H6low-5n25	14987.80	10839.60	27.68	42903.0
H6low-1n30	51574.10	8953.17	82.64	40403.0
H6low-2n30	50928.20	9345.86	81.65	17627.0
H6low-3n30	12583.70	9038.79	28.17	42912.0
H6low-4n30	13713.70	8639.45	37.00	42944.0
H6low-5n30	48267.70	7634.23	84.18	8341.0

Table 26: Computational results on the small instances set, $p = 6$, high inventory cost, $K = 5$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high-1n5	9042.98	9042.98	0.00	152.0
H6high-2n5	8298.86	8298.86	0.00	103.0
H6high-3n5	13399.10	13399.10	0.00	180.0
H6high-4n5	8179.58	8179.58	0.00	110.0
H6high-5n5	0.00	8179.58	0.00	110.0
H6high-1n10	14238.80	12625.00	11.33	27262.0
H6high-2n10	14949.60	13565.60	9.26	31894.0
H6high-3n10	12054.30	11927.90	1.05	42970.0
H6high-4n10	13754.90	13288.00	3.39	42993.0
H6high-5n10	12092.20	11783.30	2.56	42827.0
H6high-1n15	15610.30	13937.30	10.72	43130.0
H6high-2n15	15555.90	14977.00	3.72	43114.0
H6high-3n15	18008.90	16588.20	7.89	41795.0
H6high-4n15	14655.90	13991.80	4.53	42979.0
H6high-5n15	15657.20	14298.60	8.68	42988.0
H6high-1n20	20309.00	17497.90	13.84	42838.0
H6high-2n20	17053.70	15833.10	7.16	43091.0
H6high-3n20	18383.20	16183.60	11.97	43033.0
H6high-4n20	21217.90	16654.30	21.51	42955.0
H6high-5n20	23387.30	19983.30	14.55	42908.0
H6high-1n25	18928.70	18005.10	4.88	43172.0
H6high-2n25	54484.40	17299.60	68.25	33279.0
H6high-3n25	25317.60	20178.60	20.30	42869.0
H6high-4n25	20100.30	17913.30	10.88	42938.0
H6high-5n25	27041.50	22366.60	17.29	43061.0
H6high-1n30	30430.80	24185.40	20.52	43062.0
H6high-2n30	25457.60	22187.60	12.84	42943.0
H6high-3n30	27942.20	24141.70	13.60	43112.0
H6high-4n30	23677.10	18642.20	21.26	42908.0
H6high-5n30	25521.90	19821.10	22.34	43189.0

B Computational results on the large instances set, ML inventory policy

Table 27: Computational results on the large instances set, $K = 1$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high1n50	30189.40	30189.40	0.00	3056.0
H6high2n50	29790.00	29790.00	0.00	3334.0
H6high3n50	29790.90	29790.90	0.00	4020.0
H6high4n50	31518.30	31518.30	0.00	5737.0
H6high5n50	29240.40	29240.40	0.00	684.0
H6high6n50	31903.10	31903.10	0.00	28320.0
H6high7n50	29734.50	29734.50	0.00	13561.0
H6high8n50	25954.20	25954.20	0.00	21552.0
H6high9n50	30192.90	30192.90	0.00	20581.0
H6high10n50	31338.20	31338.20	0.00	1879.0
H6high1n100	57459.20	57212.70	0.43	86400.0
H6high2n100	53510.10	53076.40	0.81	86400.0
H6high3n100	58505.10	58183.40	0.55	86400.0
H6high4n100	51554.20	51511.00	0.08	86400.0
H6high5n100	57976.50	57867.70	0.19	86400.0
H6high6n100	55087.80	54843.00	0.44	86400.0
H6high7n100	56076.90	55712.90	0.65	86400.0
H6high8n100	56057.10	54729.00	2.37	86400.0
H6high9n100	59425.90	58086.80	2.25	86400.0
H6high10n100	56588.30	56034.30	0.98	86400.0
H6high1n200	136337.00	109774.00	19.48	86400.0
H6high2n200	141543.00	111501.00	21.22	86400.0
H6high3n200	123147.00	106760.00	13.31	86400.0
H6high4n200	129615.00	107705.00	16.90	86400.0
H6high5n200	126552.00	107869.00	14.76	86400.0
H6high6n200	136513.00	107862.00	20.99	86400.0
H6high7n200	111186.00	94778.20	14.76	86400.0
H6high8n200	115946.00	101109.00	12.80	86400.0
H6high9n200	136819.00	103598.00	24.28	54474.0
H6high10n200	142796.00	107888.00	24.45	86400.0
H6low1n50	9975.82	9901.42	0.75	86400.0
H6low2n50	10632.00	10632.00	0.00	2536.0
H6low3n50	10510.70	10510.70	0.00	1355.0
H6low4n50	10513.40	10513.40	0.00	60289.0
H6low5n50	10113.00	10113.00	0.00	2416.0
H6low6n50	10148.00	10113.60	0.34	86400.0
H6low7n50	9982.20	9982.20	0.00	14698.0
H6low8n50	10299.10	10252.80	0.45	86400.0
H6low9n50	10009.90	10009.90	0.00	6326.0
H6low10n50	9659.20	9659.20	0.00	3523.0
H6low1n100	15649.30	15500.80	0.95	86400.0
H6low2n100	14697.30	14376.50	2.18	86400.0
H6low3n100	16154.60	15341.30	5.03	86400.0
H6low4n100	14644.30	14565.30	0.54	86400.0
H6low5n100	15234.80	15096.60	0.91	86400.0
H6low6n100	15769.40	14995.70	4.91	86400.0
H6low7n100	15537.70	15097.20	2.83	86400.0
H6low8n100	15279.20	14794.80	3.17	86400.0
H6low9n100	17189.60	15298.50	11.00	86400.0
H6low10n100	16145.00	15176.00	6.00	86400.0
H6low1n200	32683.30	23652.00	27.63	86400.0
H6low2n200	34033.40	23827.70	29.99	86400.0
H6low3n200	33317.20	22921.80	31.20	86400.0
H6low4n200	34004.10	22087.10	35.05	86400.0
H6low5n200	35486.90	23412.70	34.02	86400.0
H6low6n200	33359.50	22902.70	31.35	86400.0
H6low7n200	32773.90	21087.30	35.66	86400.0
H6low8n200	33488.70	22493.00	32.83	79789.0
H6low9n200	35172.60	22807.40	35.16	86400.0
H6low10n200	34871.50	22774.60	34.69	86400.0

Table 28: Computational results on the large instances set, $K = 2$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high1n50	31782.60	30457.40	4.17	86400.0
H6high2n50	31120.90	30070.10	3.38	86400.0
H6high3n50	30439.00	29956.60	1.58	86400.0
H6high4n50	32721.80	31702.90	3.11	86400.0
H6high5n50	30114.60	29529.50	1.94	86400.0
H6high6n50	33495.10	32063.90	4.27	86400.0
H6high7n50	31839.50	29783.00	6.46	86400.0
H6high8n50	28551.60	26295.70	7.90	86400.0
H6high9n50	31415.00	30473.20	3.00	86400.0
H6high10n50	32787.80	31418.10	4.18	86400.0
H6high1n100	91823.10	57289.20	37.61	86400.0
H6high2n100	59860.60	53115.60	11.27	86400.0
H6high3n100	114040.00	58326.10	48.85	86400.0
H6high4n100	55343.70	51514.00	6.92	86400.0
H6high5n100	67259.20	57956.00	13.83	86400.0
H6high6n100	93171.20	54810.90	41.17	86400.0
H6high7n100	111776.00	55778.50	50.10	86400.0
H6high8n100	142585.00	54945.30	61.46	86400.0
H6high9n100	70292.50	58109.00	17.33	86400.0
H6high10n100	89314.90	56116.60	37.17	86400.0
H6low1n50	12209.00	10292.40	15.70	86400.0
H6low2n50	11819.60	11024.70	6.73	86400.0
H6low3n50	11237.70	10700.00	4.78	86400.0
H6low4n50	11740.10	10653.40	9.26	86400.0
H6low5n50	10855.50	10271.00	5.38	86400.0
H6low6n50	11662.00	10425.20	10.61	86400.0
H6low7n50	12012.00	10270.90	14.49	86400.0
H6low8n50	13379.10	10694.10	20.07	86400.0
H6low9n50	11301.50	10362.00	8.31	86400.0
H6low10n50	11475.40	9860.49	14.07	86400.0
H6low1n100	50283.70	15578.90	69.02	86400.0
H6low2n100	72766.60	14468.40	80.12	86400.0
H6low3n100	20836.90	15579.20	25.23	86400.0
H6low4n100	42634.20	14617.80	65.71	86400.0
H6low5n100	72016.20	15230.10	78.85	86400.0
H6low6n100	52944.90	14903.50	71.85	86400.0
H6low7n100	47628.30	15133.90	68.22	86400.0
H6low8n100	47359.60	15030.30	68.26	86400.0
H6low9n100	48666.60	15078.10	69.02	86400.0
H6low10n100	48398.00	15210.60	68.57	86400.0

Table 29: Computational results on the large instances set, $K = 3$

Instance	Upper bound	Lower bound	gap (%)	time (s)
H6high1n50	36497.10	30987.90	15.09	86400.0
H6high2n50	37327.90	30621.40	17.97	86400.0
H6high3n50	33123.90	30370.50	8.31	86400.0
H6high4n50	41324.70	31904.90	22.79	86400.0
H6high5n50	34069.50	29672.80	12.91	86400.0
H6high6n50	38639.90	32446.90	16.03	86400.0
H6high7n50	35794.90	30223.10	15.57	86400.0
H6high8n50	33535.30	26666.30	20.48	86400.0
H6high9n50	36437.20	30797.80	15.48	86400.0
H6high10n50	36330.20	31759.50	12.58	86400.0
H6high1n100	110146.00	57348.40	47.93	86400.0
H6high2n100	107683.00	53063.00	50.72	86400.0
H6high3n100	165084.00	58520.90	64.55	86400.0
H6high4n100	168527.00	51537.80	69.42	86400.0
H6high5n100	167986.00	58216.10	65.34	86400.0
H6high6n100	109981.00	54834.60	50.14	86400.0
H6high7n100	107437.00	55890.30	47.98	86400.0
H6high8n100	106218.00	55276.00	47.96	86400.0
H6high9n100	184721.00	58078.00	68.56	86400.0
H6high10n100	108231.00	55953.20	48.30	86400.0
H6low1n50	15532.90	10847.10	30.17	86400.0
H6low2n50	17992.20	11499.50	36.09	86400.0
H6low3n50	15425.30	11059.70	28.30	86400.0
H6low4n50	17314.70	10984.80	36.56	86400.0
H6low5n50	15650.60	10302.10	34.17	86400.0
H6low6n50	16468.40	10726.60	34.87	86400.0
H6low7n50	17899.40	10543.70	41.09	86400.0
H6low8n50	19411.30	10917.80	43.76	86400.0
H6low9n50	17758.70	10662.80	39.96	86400.0
H6low10n50	16003.00	10050.50	37.20	86400.0
H6low1n100	70570.90	15700.70	77.75	86400.0
H6low2n100	68421.00	14488.60	78.82	86400.0
H6low3n100	64859.60	15692.40	75.81	86400.0
H6low4n100	66067.20	14610.80	77.89	86400.0
H6low5n100	66507.60	15395.30	76.85	86400.0
H6low6n100	69776.70	15053.80	78.43	86400.0
H6low7n100	66053.50	15154.90	77.06	86400.0
H6low8n100	66338.10	15332.30	76.89	86400.0
H6low9n100	71529.50	15341.70	78.55	86400.0
H6low10n100	67376.90	15129.60	77.54	86400.0