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Stochastic Modelling of Free-Floating Car-Sharing Systems

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Abstract. Car-sharing systems (CSSs) have gained popularity during the last decade as a flexible, efficient and ecological alternative mode of transportation. But for the operator, managing such systems is far to be simple. Due to heterogeneity of demand and also randomness, the user may face a lack of resources: no car or no parking space available. And the operator has to design the system in order to improve it. The total number of cars impact the performance of the system. We address the dimensioning issue. For that, mathematical models are needed. In many cities, two systems coexist: station-based and free-floating. The latter gives more flexibility to the user both to take or return the car. But he can reserve only the car for a short period, and not the parking space, as the car is parked on public space with no specific parking spaces. The car reservation is here to help the user. The aim of the paper is to study its influence on the system behavior. This study focuses on Communauto's Montreal free-floating car-sharing system (FFCSS). Data analysis investigates the main features of the system based on user preferences. It allows proposing a mathematical modelling. Then we present two analytical approaches. First the mean-field method could be used for different variants, and w gives first insights on the optimal fleet size in a homogeneous framework. Second the general inhomogeneous model is described as a closed Jackson network with blocking-rerouting policy. We prove that its state at stationarity is given by a product-form distribution. It allows in future work to obtain an explicit large-scale representation of the system which can be used both theoretically or numerically for optimization purposes.

Keywords: Keywords: car-sharing, carsharing, free-floating, travel behavior, simulation.

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Contents

1 Introduction

The goal of this work is to study the free-floating car-sharing system of Montreal with the perspective to optimize such a system. Here, optimizing the system means minimizing the number of unhappy users. This will be detailed in the following.

As in some main cities, the car-sharing system of Montreal consists of two parts. The first one is a station-based system. In this system, stations are spread throughout the city, where only cars from the car-sharing system can park. There are only round trips thus each car in the system is assigned to a station. When a user takes a car, he has to park it back in the same station. This guarantees users an available parking space at the end of the trip. In such a system, the only possible cause of unhappiness for a user is if he does not find an available car at its departure station.

The station-based system may coexist with a free-floating system. The paper focuses on this type of system. In a free-floating system, there are no stations, and cars can be taken or left anywhere in a predetermined area, called the service area. As far as we know, this paper is the first stochastic analysis of such a system. The dynamics of this system are quite different from those of the station-based system, as a car does not need to be parked in the vicinity of station where it has been picked up, but it can be parked anywhere in the service area. Therefore, the user cannot know in advance whether there will be parking space available at its destination, so there are two sources of unhappiness for the user: if he cannot find either a car in the area where he wants to start his trip and or a free parking space close enough to his destination.

The first part of the paper consists in analyzing data related to the free-floating car-sharing system of Montreal, in order to determine its main characteristics. The data is provided by Communauto, the car-sharing operator in Montreal. This data concerns transactions, for example the time the user enters the system, the booking duration, the travelled distance or the trip duration.

The second part of the paper deals with modelling the free-floating car-sharing system of Montreal based on the characteristics previously observed. The analysis of the mathematical model aimed at determining the behavior of the system and deriving the optimal fleet size, which is the total number of cars to introduce in the system to minimize the number of unhappy users. The results are validated by simulations. For that a Python simulator is written and implemented with values of parameters from data analysis. Moreover it gives insight on some features which cannot be taken into account in the mathematical model. In particular simulations with a more realistic trip distribution than the exponential distribution are presented.

The outline of the paper is the following. After the introduction in Section 1, related works are presented in Section 2. Section 3 is devoted to data analysis which the stochastic models are introduced and studied in Section 4. Validations by simulation and a model discussion are presented in Section 5. Section 6 deals with the conclusion and perspectives for future work.

2 Methodology and related work

2.1 General context

Several models related to queuing theory have been proposed for car-sharing systems. The most often studies about car-sharing systems in the literature are devoted to station-based systems. However, as it is already present in the literature (see [8]), a free-floating system can be reduced to a station-based system by dividing the service area in smaller zones, which can be considered as stations. What we investigate is an optimal size of the system, which means determining the number of cars that needs to be made available per zone in order to minimize the number of unhappy users.

Here is a state of the art of the models which have already been studied in the field of vehicle-sharing. It is interesting to note the differences between the characteristics of our model, which describes the free-floating car-sharing in Montreal, and those of the models previously described in the literature. The relevant differences are those regarding the capacities (finite or infinite) of the stations or zones, and the possibilities of booking before a trip: booking the car before the trip starts, booking the parking space at the destination, no booking. Moreover in real-world systems as the free-floating car-sharing system of Montreal, a user can book a car before his trip but he does not necessarily need to book it, he can take any available car in the service area and start a trip with it. Moreover, if he books a car, he can cancel the booking before starting the trip. Therefore, in our model, we introduce a probability of booking a car, and a probability of cancelling the trip if the car is booked. Moreover, we consider models with zones having both infinite and finite capacity (the finite capacity being, of course, the realistic framework).

2.2 Modelling with queuing networks

The few stochastic models in the literature for car-sharing systems are related to queuing networks. So are the models we propose in this paper for the free-floating car-sharing system in Montreal. Because the number of cars, which are the customers of the networks, is fixed, the networks are Gordon-Newell networks, also called (closed) Jackson networks in [6]. Such networks, in particular their dynamics and their invariant measure known from [1], are presented in [11].

In [5] (see [3] for details and proofs), models are proposed for studying the former car-sharing system in Paris, called Autolib' (20011-2018). The system is an electric station-based one. A first model does not take into account the booking of the cars prior to the trips, but all the users book a parking space at their destination station when they pick-up their car. The analysis proves that, when the number of stations and cars become large together, the system reaches an equilibrium where the state of a station is determined, more precisely the number of reserved parking slots and available cars. Indeed its distribution is that of a tandem of queues with finite total capacity which has a product-form invariant measure. In that case, the measure depends on two parameters given by a fixed point equation in dimension two. The derivation, even the fact that it exists and is unique, is not simple and implicit functions are involved. The product-form expression is then used to determine the optimal fleet size of the system, by minimizing the proportion of problematic stations. By definition, a problematic station is either an empty station, where no car is available, or a full station, where no parking spaces are available. s The optimal fleet size and the corresponding proportion of problematic stations are computed when the system is large in different scenarios (light traffic, which means low demand and heavy traffic, which means high demand). An extension of this model is proposed in [3] as a second model, with a double booking: booking of both the car and the parking space at the destination station before the trip. All the users book both of them. A main difficulty in the model analysis of this paper is exploiting the large-scale invariant measure for explicit further derivations, as its expression relies on parameters given by an implicit function in dimension two.

In [6], a closed queuing network is used to model a bike-sharing system. It is proved that, for a fixed number of vehicles, the invariant measure of the model interpreted as a closed queuing network with blocking-rerouting procedure, is product-form. The approach is quite different from the one in [3] as the system is modeled by a queuing network, whereas in [3], for large-scale behavior, each station is identified to a queuing system when using a mean-field approach. Besides, the product-form measure in [6] concerns the whole system for a fixed number of zones, whereas the product-form measure in [3] holds for one station when the number of stations tends to infinity.

In this paper we use both approaches to analyse two different models. With arguments similar to [3], we also obtain the large-scale behavior of one zone (i.e. station) when the numbers of zones and cars tend to infinity together, but our product-form invariant measure is more explicit, depending on a fixed point equation in dimension one and not in dimension two. Using the expression of the invariant measure, the derivation of the optimal fleet size is difficult as the simplification that the optimum corresponds to some parameter equal to 1 in [3] does not hold here. Thus the computation of the optimal fleet size is just provided for a small mean booking duration. An inhomogeneous model as the one in [6] gives a more accurate invariant measure, as this measure is obtained for any fixed number of stations and cars and a general inhomogeneous framework (arrival rates, mean booking and trip durations, routing matrix), but it cannot be used as it is and one also has to proceed to further asymptotics. However, even if the tools are similar to those in [3] and [6], the system described in this paper is quite different. In the free-floating car-sharing system of Montreal, the parking space in the destination zone cannot be booked, just the car before the trip. In [3] the authors consider either only booking the destination parking space, either booking both the car and the destination parking space, and never only booking the car before the trip. As to [6], no booking system is considered. This paper is the first to give a stochastic analysis of such a system. And [3] and [6] are used as a strong basis for our analysis even if arguments have to be adapted.

Note that the first analysis devoted to vehicle-sharing systems is in [7]. It is a pioneer paper, with no booking prior to the trip, stations with infinite capacity and asymptotics when the number of cars gets large and the number of stations is fixed, which is not the natural scaling for real-world systems. This model with infinite capacity is also analyzed in [6] with the scaling where both numbers of stations and cars tend to infinity. And both papers [7] and [6] aim at computing the minimum number of vehicles to achieve a certain level of service, which is slightly different from what is studied in our paper. Indeed, the level of service is the proportion of users who find an available vehicle in a given zone. So the system is only optimized according to the proportion of users who find an available vehicle, without considering that users have also to find an available parking space close to their destination. Therefore, this optimization may be useless for real-world systems, as an optimal system should not only have vehicles available but should also enable users to find a parking space close enough to their destination. Some studies use the same framework. See [2] for example.

Other studies relying on operation research or machine-learning to optimize the system based on location or redistribution can be cited in the large literature about car-sharing. See [12] and references therein.

2.3 Focus on the capacity

One of the difficulties brought by free-floating systems is that the maximum number of parking spaces in a zone, hereafter the capacity of the zone, can be variable. Indeed, the cars of the free-floating system are parked in the public space, sharing it with private cars. On the contrary, in a station-based system, the maximum number of parking spaces at a station, hereafter the capacity of the station, is a fixed number. As some papers in the literature consider a model with stations having finite capacity, as far as we know, no paper investigates a model with stations of a variable capacity. In the whole paper we consider that the zones have a fixed capacity, as a first attempt to investigate the system.

2.4 Analyzing data

From data analysis, most papers study the user behavior, also called *user preferences* in the literature. For more details about the data from Communauto, see [13], and also [9]. Note that no stochastic analysis has been done yet for a model defined from this data analysis. However a multi-agent model has been proposed and investigated by simulations in [10]. Some approximations have been made, as the fact that the trip length is proportional to the origin-destination distance. In fact, our data analysis shows a significant proportion of loop-trips. And even worse, the probability to return the car close to the departure point, which corresponds to a small origin-destination distance increases with the trip length (see Figure 5). It seems to prove that this is not quite accurate. We seek here to propose a model which takes into account the presence of loop-trips.

3 Data analysis

3.1 Data structure and data cleaning

3.1.1 Parameters

The data from Communauto is related to transactions over four years, from 2014 to 2018. A **transaction** is any operation related to the rental of a car in the car-sharing system. It can be for instance a booking, if the car is booked then the user decides to cancel the trip, a booking followed by a trip, or only a trip, if the user does not book his car before starting the trip. An important distinction needs to be made between bookings and trips. The definitions of the terms used throughout the report are the following.

- **A booking** corresponds to the period during which the car is booked and parked at a station, waiting for the user to come and start the trip, or cancel the trip.
- **A trip** corresponds to the period during which a car is driven. During a trip, the car is not parked in a zone.

In the data table, a line represents a transaction and every column a parameter related to the transaction. Some particularly interesting parameters are:

- *CarID:* a number which identifies the car involved in the transaction.
- *UserID:* a number which identifies the user involved in the transaction.
- *TripDate:* the time when the transaction begins. The transaction starts when a user books a car, or when the user takes a car without previously booking it.
- *BookDuration:* the booking duration.
- *Duration:* the whole transaction duration, which consists in the booking duration and the trip duration itself if the trip has not been cancelled.
- *FirstDrive:* the time between the moment the transaction starts and the moment the car leaves its parking space. If the car is taken directly without booking, *FirstDrive* should be equal to zero.
- *Distance:* the distance driven during the trip. A distance equal to zero means that the trip is cancelled, so the transaction is reduced to a booking.
- *StartOdo* and *EndOdo*: the odometer of the car respectively at the beginning and at the end of the trip.
- *StartLongitude*, *StartLatitude* and *EndLongitude*, *EndLongitude*: the GPS position of the car at the beginning and at the end of the transaction.

As the dataset is very large, the software R is used to analyze it.

The goal of the data analysis is to establish how the Montreal car-sharing system is used. In particular, as it works with a possibility of booking the car before using it, it is interesting to study how this possibility of booking is used in practice. In particular if the booking duration is significant compared to the trip duration, it means that it is interesting to propose a mathematical model with a probability to book a car. Likewise, if it is common that users cancel their trip, it is interesting to consider a probability to cancel the trip in the model.

3.1.2 The data cleaning process

As it is often the case in the field of data analysis, the first step consists in data cleaning.

It first takes the form of standard data manipulation, like removing duplicates, incoherent values (negative durations or distances) or handling missing values. For instance, the fields *EndLongitude* and *EndLatitude* are often missing, but they can be deduced from the starting point of the same car for the next transaction of that car. The incoherent values removed corresponds to less than 1% of the data.

Moreover, in order to compare the booking durations and the trip durations, another data manipulation performed is the conversion of the dates of the beginning of the trip in times and dates using the *chron* package of R. The *chron* package is very convenient as it allows mathematical operations with data having a date/time format ("$mm/dd/yyyy \quad h:m:s$"). By converting every duration in fractions of days, trip completion times are available in a date-time format.

The more specific data-cleaning processes are detailed hereafter.

Handling overlaps

The first difficulty is to handle the duration fields *BookDuration*, *Duration* and *FirstDrive* in order to obtain a coherent dataset. For instance, the lines for which *BookDuration*, *FirstDrive* or *Duration* for a given car (identified by its ID) are removed if they overlap with the next transaction of the same car. Such overlaps may not occur in reality but could appear in the data because of a default of synchronisation of the chronometers in the cars. This requires a function to remove transactions which overlapped with other transactions.

It leads to remove 0.04% of the data.

Correcting zero distances

Another incoherence in the dataset is the presence of some distances which are equal to zero while they correspond to quite long trips (in duration). Looking closer, we realize that those distances correspond to cases in which the odometer of the car is blocked, for some reason. Indeed, when taking the transactions for a given car chronologically, we can see that, in some cases where *StartOdo* and *EndOdo* are equal, the starting position of the same car for its next transaction is different. Therefore, the car is supposed to have left the place where it was parked between the measurements of *StartOdo* and *EndOdo*. In that case, the Manhattan distance between the starting point of the transaction and the ending point of the transaction is used to approximate the distance traveled by the car. The Manhattan distance is more adapted to the geography of Montreal than the Euclidian distance. Given that the precision of a GPS can be affected by the presence of high buildings like those of Montreal, and that it is quite unlikely that someone would take a car for less than 500 meters, we consider that the distance traveled by the car is to be approximated with the Manhattan distance between the starting point and the ending point of the transaction considered as positive only if this distance is superior to 500 meters. Of course, this way of computing distances is very approximate, as the distance traveled between two points is often higher than the Manhattan distance betweeen those points, but a choice needed to be made in order to avoid missing values.

This process enabled to reduce by 2% the number of distances equal to zero.

Defining a coherent booking duration

Finally, a quite important data cleaning to take care of (after having applied the previously described cleaning processes), is to define the booking durations and the trip durations from the data. In particular, the fields *BookDuration* and *FirstDrive* are not always coherent. Indeed, in most cases, we had *BookDuration* $\leq$ *FirstDrive* $\leq$ *Duration* but not all the time. We need a clear value for the booking duration for each transaction. Let us take the following conventions to define it.

- If a trip occurs at the end of the booking duration (which means *Distance* > 0), and *FirstDrive*> 0, the booking duration is the minimum of *FirstDrive* and *BookDuration*. This is to avoid a booking duration which would be higher that the duration of the transaction. Indeed, we check empirically that min(*BookDuration*, *FirstDrive*)$\leq$ *Duration* is always true.

- If a trip occurs but *FirstDrive*=0, the booking duration is *BookDuration*. In that case, *FirstDrive*=0 makes no sense because it would mean that the car never leaves its zone, although the trip occurs.

- If a trip is cancelled, as all three durations are supposed to represent the time during which the car is booked in the zone and is not available, the *worst* case, which is the one when the car is booked the longest time, is considered. In that case, the booking duration is defined as max(*BookDuration*, *FirstDrive*, *Duration*). There are no overlaps problems, as the durations causing overlaps have already been removed, for all three durations.

This definition of the booking duration gives the trip duration as the difference between the total duration of the transaction (*Duration*) and the booking duration (for which a new variable *TrueBookDuration* is created).

3.2 Merging of the bookings

3.2.1 Why bookings needed to be merged

One of the first things that is standing out regarding the data is the number of cancelled trips. There are 33% of transactions with a zero distance (after cleaning and correcting the distances which are equal to zero because of a problem of the odometer). Some of them correspond to chained bookings. A **chained booking** is defined as a succession of bookings of the same car by the same user where:

- All bookings of the chain except the last one lead to a cancelled trip

- The bookings are made successively, with a small amount of time between them (less than 10 minutes).

Indeed, in a chained booking, the same user books the same car and cancels the trip, to re-book the same car after he cancelled. As the booking time is limited to 30 minutes per booking, this can be a way to *cheat* and extend the booking time.

A function is written to merge the different bookings. The chained booking stops with the first booking of the user that leads to an actual trip (positive distance). After merging the bookings, each chain of booking is merged into a single booking. The total booking duration of a merged booking corresponds to the sum of the booking durations of the different bookings in the chain, to which are added the time lapses between every two bookings (which are the same, in this case, as transactions) of the chain.

3.2.2 The distribution of the booking duration after merging

The real distribution of the booking duration after merging is shown in Figure 1. We can see that some users do cheat, by extending their booking time using chained bookings.

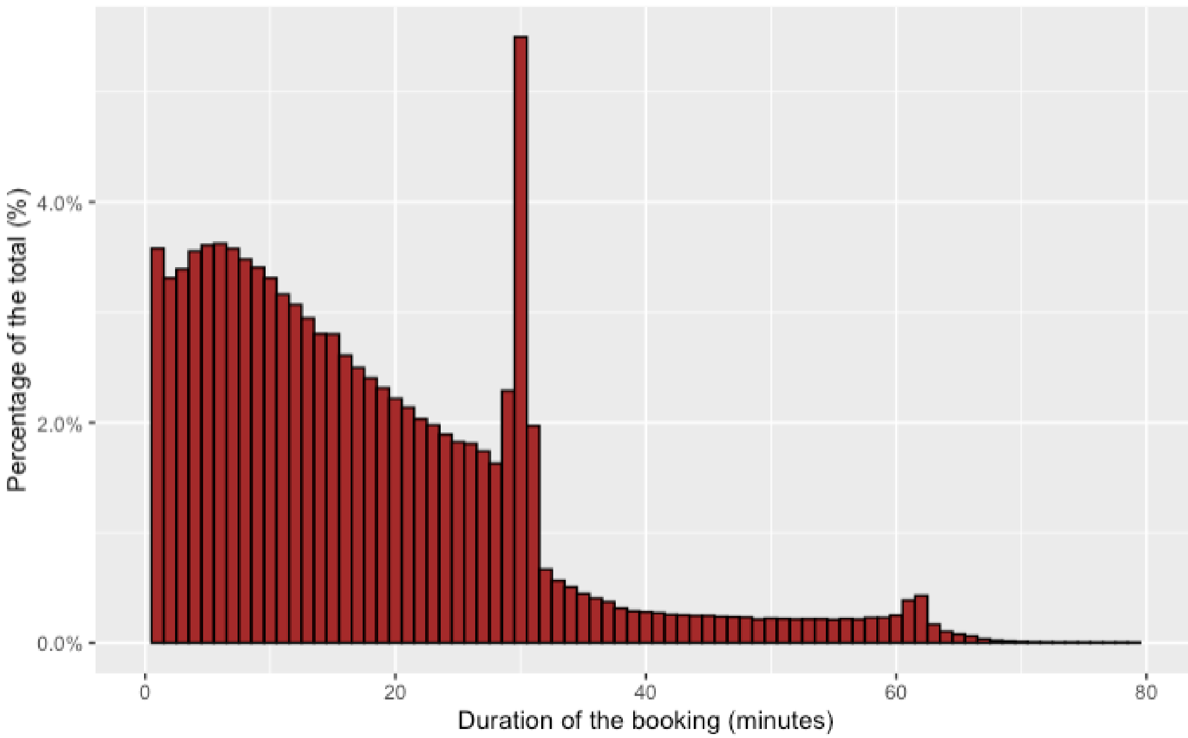

Figure 1: Distribution of the booking duration after merging

Nevertheless, only 50% of the chained bookings are used to substantially increase the booking time above 35 minutes. And even if the chained booking phenomenon exists, it remains quite limited, as only 15% of the bookings correspond to chained bookings. Moreover, merging the bookings does not change significantly the parameters of the booking duration distribution. Indeed, both mean and median of the booking duration only increase by 2 minutes.

3.3 Study of booking and trip durations

The study is done with the data obtained after merging the bookings (see Section 3.2).

3.3.1 Bookings

The distribution of the booking durations can be seen in Figure 1. The mean duration of a booking is 17 minutes. Another interesting feature is the probability of booking, which is 84.5%, and represents the proportion of transactions for which the booking duration is strictly positive.

3.3.2 Trips

Figure 2 shows the distribution of the trip duration. As there are many long trips, corresponding to users who rent a car for several days, the distribution seems heavy-tailed. A more detailed study of the distribution of the trip duration is done in Section 5.

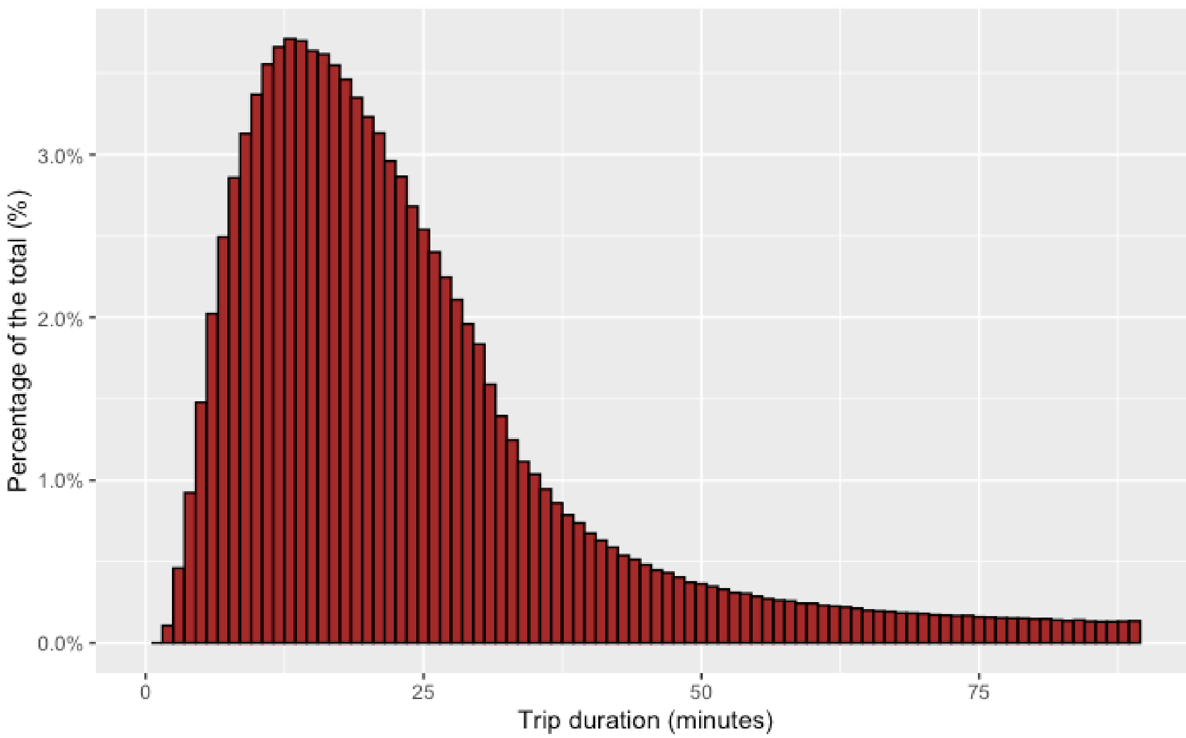

Figure 2: Distribution of the trip duration

The mean duration of a trip is 1h22min, which does not correspond to the maximum of the distribution (around 13 min). This is due to the presence of long trips over several days, which considerably increase the mean trip duration. The median of the trip duration, which is 22min, is more representative of the usual behavior of the users. The third quartile is 32 min, so 75% of the trips are less than 32min, of the same order of length as the booking durations. This shows that the booking is significant and is worth studying.

Another interesting feature is the probability of cancelling a trip, i.e. the proportion of the transactions for which the traveled distance is zero, which is 24%.

3.4 Results about user behavior

3.4.1 How the system is used throughout a day

Figure 3 shows how the Montreal car-sharing system is used in a typical weekday. We observe two peaks of booking, around 7.45 am and 5.00 pm, which are a bit earlier than the usual hours at which people go to daycare. For that, people book a car to be sure that they have a car. Each booking peak is followed by a trip peak.

The number of trips is relatively stable. There is a *plateau* between 8.30 am and 6.30 pm, with a peak at 6.30 pm. But after the morning peak hour, the high number of trips corresponds to a quite low number of bookings. From this, we can infer that those who use the Montreal car-sharing system during the day may make less bookings and plan their trips less than those who use the car-sharing system during the morning peak hour.

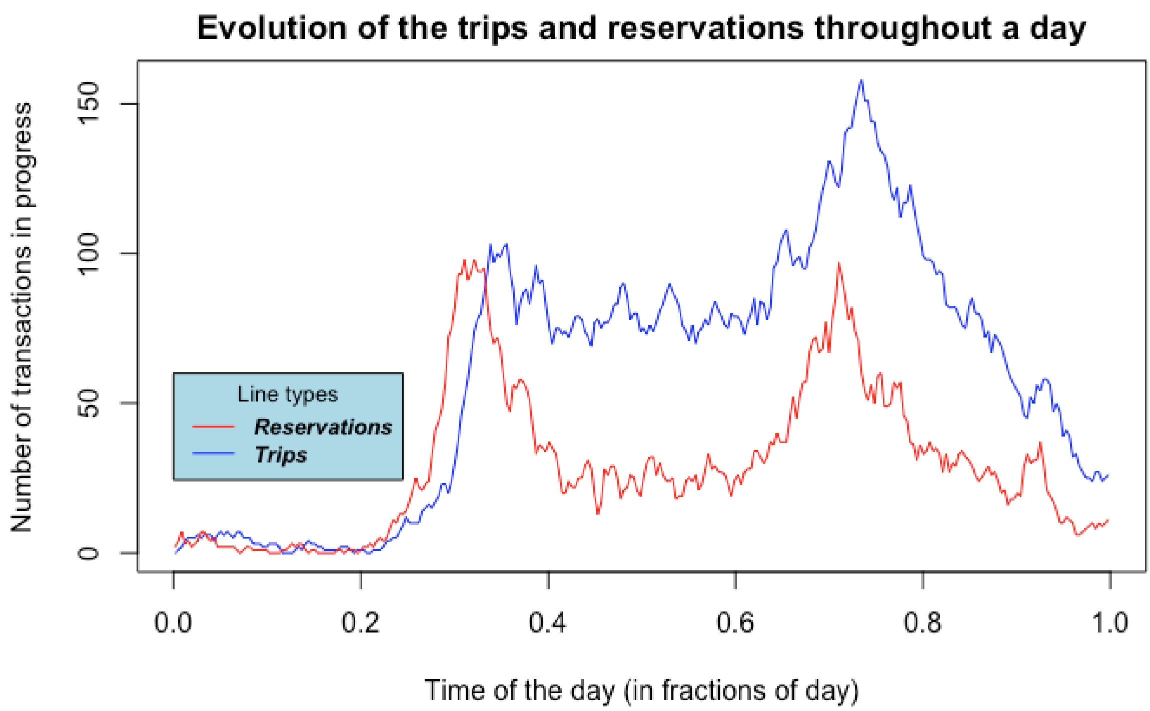

Figure 3: Evolution of the number of trips and of bookings in progress per hour on Wednesday June 7th

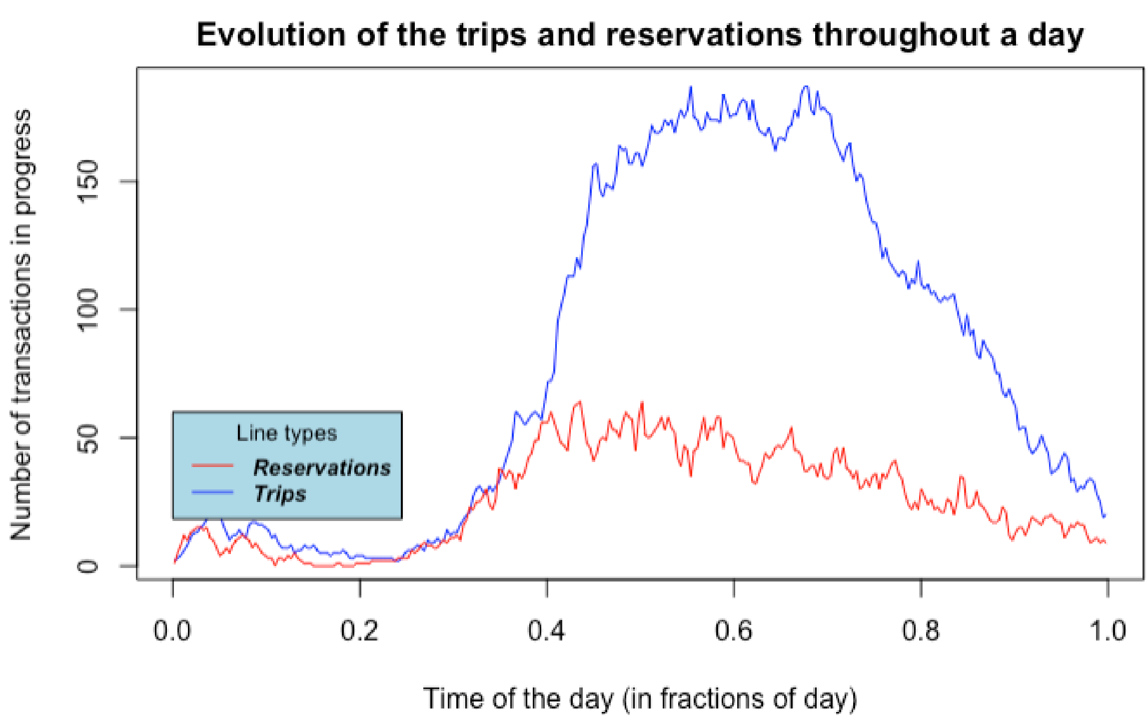

Figure 4: Typical evolution of the number of trips and of bookings in progress per hour during the weekend

Figure 4 shows the same plot during the weekend. We can see that, during the weekend, there are no peaks of bookings and trips, but a plateau. There are much more trips than bookings per hour, which may mean that most trips during the weekend have not been booked in advance.

3.4.2 Loops

The last interesting feature revealed by the data analysis is related to loop trips. A **loop trip** is defined as a trip which ends less than 1km from its starting point. Figure 5 shows the evolution of the proportion of loops among the total number of trips as a function of the trip length (distance in km).

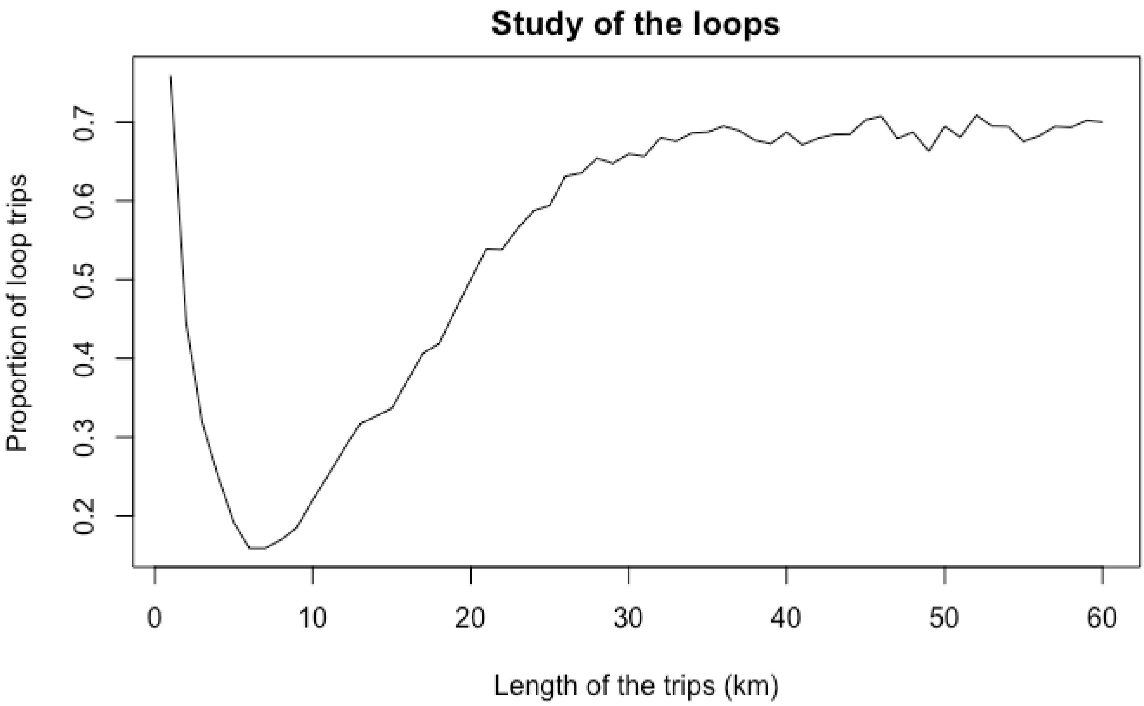

Figure 5: Proportion of users doing a loop trip depending on the trip length (distance)

On the one hand, we see that the longer the trip is, the higher the proportion of users who make loop trips is. This is quite easy to understand as users who go far away may probably be going on a holiday or visiting someone, before coming back home, especially if their trip brings them outside the service area. Therefore, they are likely to leave the car where they had taken it. Therefore, for long trips, the free-floating system is mainly used as the station-based car-sharing system.

On the other hand, we observe that the probability for short trips to end within a radius of 1km from the departure point is quite low. This shows that short trips are often one way trips. However, around 80% of the very short trips tend to come back within a radius of 1km from where the trip starts.

The trips which come back within a radius of 1km are in bijection with trips which come back to the zone of departure, called loop trips in the modelling section. A zone is a square of side 1km. More precisely, assume that the trip destination is uniform around the trip departure as the two points are close. Then this bijection is due to the following elementary property. It justifies the previous choice of radius 1km.

Proposition 3.1. *The mean Manhattan-distance between two points within a zone is equal to the mean bird's-eye distance between one point O and the disc with center O and radius 1, which is* $2/3$.

The proof, based on basic calculations, is postponed in Annex 7.2.

4 Mathematical models

4.1 A simplified homogeneous model for large-scale systems: a mean-field approach

4.1.1 Model with compulsory booking and no trip cancellation

Model description

The first model we consider is similar to the one of [3], but with significant differences. In our model, the service area is divided in smaller zones, which can be assimilated to stations. The context is a system with a total number of zones tending to infinity. The first simple model we took is an homogeneous model, which means that all the zones have the same properties. In particular, the behavior of users is the same regardless of the zone from which they start their trip and to which they end. The dynamics in the model are the following:

- Users arrive according to a Poisson process in a zone. It means that the inter-arrival times in a zone are independent and have an exponential distribution.
- When a user arrives in a zone, he immediately books a car, if there is an available car. The booking time has an exponential distribution. Otherwise he leaves the system.
- When the booking time ends, the user takes the car to start his trip towards a destination zone, which is chosen uniformly among all the zones of the service area (including the zone from which he comes). The trip time has an exponential distribution.
- At the end of his trip, the user returns the car in the destination zone previously chosen, if there is a parking space available for him in that zone. Otherwise, he chooses another zone at random, makes a trip and parks the car in the zone if a parking space is available, and repeats this procedure until he can park the car.

Note that all the inter-arrival, booking and trip times are independent with exponential distribution. It allows to obtain discrete-space Markov processes, which are easy to handle in the analysis. This model is still far from the real car-sharing system of Montreal, as the user always books his car before a trip, there is no probability of cancellation a trip after the booking. Moreover, the number of zones is large. But it is the first step before introducing a more complex model based on the real booking scheme. As far as we know it has never been studied in the literature.

Notations:

- N is the number of zones in the service area. In this particular model, $N \to \infty$, which approximate the case that the number of zones is large, which is the case in practice.
- M is the number of cars in the system.
- K is the capacity (maximum number of parking spaces available) of the zones.
- λ is the arrival rate of the users in a zone. In other words, the inter-arrival times of users in a zone have an exponential distribution of parameter λ.
- ν is the parameter for the exponential distribution of the booking duration. The mean booking duration of a booking is thus $1/\nu$.
- μ is the parameter for the exponential distribution of the trip times. The mean trip time is $1/\mu$.
- $V_i^N(t)$ is the number of available cars in zone i in a system with N zones at time t.
- $R_i^N(t)$ is the number of booked cars in zone i in a system with N zones at time t.
- $Y_{k,l}^N(t)$ is the proportion of zones with k available cars and l booked cars in a system with N zones at time t.

Similarly to the notations introduced in [3], we define

$$\chi = \{(k,l) \in \mathbb{N}^2, k+l \leq K\}$$
$$R^N(t) = (R_1^N(t), ..., R_N^N(t))$$
$$V^N(t) = (V_1^N(t), ..., V_N^N(t))$$

and

$$Y^N(t) = (Y_{k,l}^N(t), (k,l) \in \chi).$$

Therefore, we have

$$Y_{k,l}^N(t) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{(V_i^N(t), R_i^N(t))=(k,l)}$$

$(Y^N(t))$ is a Markov process, because all the duration distributions are exponential. The state space for the Markov process Y^N is

$$\mathcal{Y} = \left\{ y = (y_{k,l})_{(k,l)\in\chi}, y_{k,l} \in \frac{\mathbb{N}}{N}, \textstyle\sum_{(k,l)\in\chi} y_{k,l} = \frac{M}{N} \right\} \subset \mathcal{P}(\chi)$$

where $\mathcal{P}(\chi)$ is the set of probability measures on χ.

Remark. *Inferring the model parameters from data* The numerical values of the parameters M, λ, μ, η can be obtained from data.

In the system the total number of cars is increasing from 2014 to 2017. M is the number of cars in the system per year. As most of the studies here are made using the data in 2017, M is here the number of cars in 2017.

Given the distribution of the booking duration (see Figure 1), we can approximate it with an exponential distribution of parameter η, even if this does not take into account the peak at 30 min, it should be a good approximation, as the peak is not that high, just 4% of the total data. Therefore $1/\eta$ is the mean booking duration and we obtained $1/\eta$ =17min in Section 3 so we can calculate η.

The distribution of μ is quite different from an exponential distribution, especially because it seems heavy-tailed, and its mean and median are very different (see Section 5). We can approximate it with an exponential distribution of same mean, but this may not reflect the real behavior of the system. This will be discussed in Section 5 dealing with the simulation results.

λ represents the rate of arrival of the users, which is the number of users arriving per zone per unit of time. Figure 6 shows the evolution of the number of users arriving in the system per hour. It has been calculated with a one-hour sliding window.

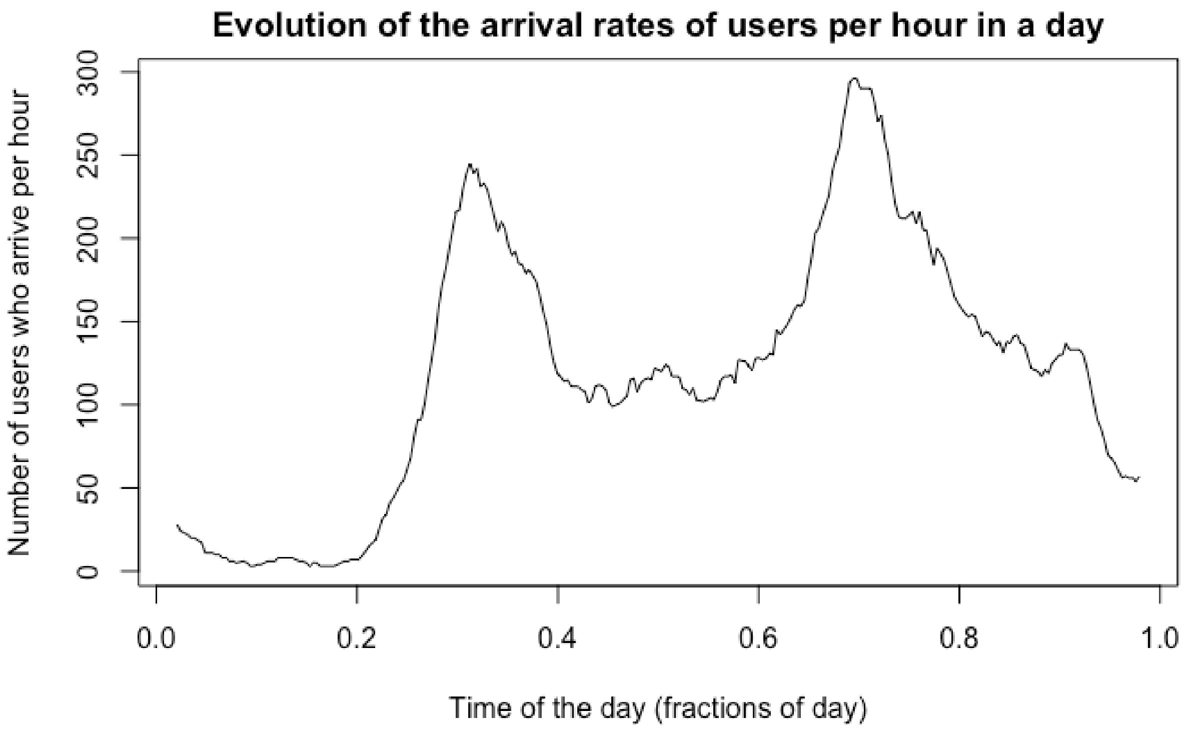

Figure 6: Evolution of the global λ throughout a day

However, from the data we only infer the global rate of arrival in the system, and not the rate of arrival per zone. Moreover we need to estimate the number of zones. For that, the service area is estimated, using the app of Communauto to visualize it. By identifying the boroughs of Montreal which are parts of the service area, and using their own area, the total service area is approximately 100 km^2. Then, considering zones of 1 km^2, the approximate number of zones is 100. Using Figure 6 we obtain

$$0.03 \leq \lambda \leq 3.08.$$

In particular, the mean of the service rate is 1.24 user per hour per zone.

4.1.2 Asymptotic behavior of the simple model and limiting invariant measure

Proposition 4.1 (Mean-field limit)**.** *As the system gets large, process $(Y^N(t))_{[0,T]}$ tends to a deterministic process $(y(t))_{[0,T]}$ solution of the ODE*

$$\begin{aligned}\dot{y}(t) = \sum_{(k,l)\in\chi} y_{k,l}(t)[&(e_{k-1,l+1} - e_{k,l})\lambda \mathbb{1}_{k>0} \\ &+ (e_{k,l-1} - e_{k,l})\eta l \\ &+ (e_{k+1,l} - e_{k,l})\mu(s - \sum_{(i,j)\in\chi}(i+j)y_{i,j}(t))\mathbb{1}_{i+j<K})].\end{aligned}$$

The number of available and reserved cars behave respectively as the number of customers in the tandem of two queues, the first one a $M/M/1$ queue and the second one a $M/M/\infty$ as in Figure 7.

Proof. Denoting by $(e_{k,l})_{(k,l)\in\chi}$ the vectors of the canonical basis of $\mathbb{R}^{|\chi|}$, the transitions for the Markov process Y^N are the following.

When a user arrives in a zone with k available cars and l booked cars,

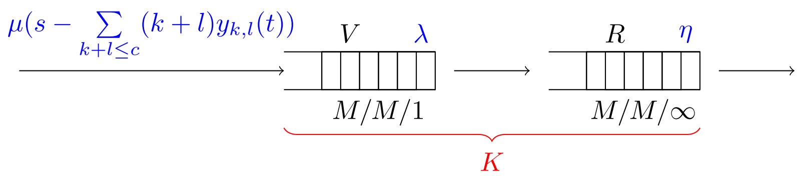

Figure 7: Mean-field limit of the simple model: the state of a zone as a tandem of queues of capacity K.

$$y \to y + \tfrac{1}{N}(e_{k-1,l+1} - e_{k,l}) \text{ at rate } \lambda N y_{k,l} \mathbb{1}_{\{k>0\}}.$$

Indeed, in that case, the number of available cars decreases by one (a car is booked by the user who arrives in the zone) and the number of reserved cars increases by one. This causes $y_{k,l}$ to decrease by $1/N$ and $y_{k-1,l+1}$ to increase by $1/N$. The rate at which this transition occurs corresponds to the number of zones with k available cars and l booked cars where a user can arrive ($Ny_{k,l}$ because the probability that a zone has k available cars and l booked cars is $y_{k,l}$) multiplied by the arrival rate of the users at a zone.
Of course, the transition occurs only if there are cars available, i.e. if $k > 0$. If there is no car available in the zone, no transition occurs (the user leaves the system).

When a user picks up the car he has previously booked in a zone with k available cars and l booked cars,

$$y \to y + \tfrac{1}{N}(e_{k,l-1} - e_{k,l}) \text{ at rate } \eta l N y_{k,l}.$$

The argument is similar to the previous case. Here, the rate at which the transition occurs corresponds to the number of cars which can be picked up in a zone with k available cars and l booked cars (i.e. $lNy_{k,l}$) multiplied by the rate at which bookings finish (i.e. η)

When a user returns his car after a trip in a zone with k available cars and l booked cars,

$$y \to y + \tfrac{1}{N}(e_{k+1,l} - e_{k,l}) \text{ at rate } \mu(M - \textstyle\sum_{(k,l)\in\chi}(i+j)y_{i,j})y_{k,l}N\mathbb{1}_{k+l<K}.$$

Here, the rate at which the transition occurs corresponds to the rate at which trips finish (i.e. μ) multiplied by the number of cars which can be parked in a zone with k available cars and l booked cars. The number of such cars is the number of cars in circulation, which corresponds to the number of cars which are not parked ($\sum_{(i,j)\in\chi}(i+j)y_{i,j}N$ is the number of cars which are parked) multiplied by the probability of a zone to have k available cars and l booked cars, which is $y_{k,l}$.
Of course, the transition occurs only if the destination zone has not reached its capacity of occupied parking spaces, i.e. if $k + l < K$. If there is no parking space available at the destination zone, no transition occurs, because the user cannot put back his car: he continues driving.

With the same arguments as in [3], we obtain that for $T > 0$, $(Y^N(t))_{t\in[0,T]}$ converges to $(y(t))_{t\in[0,T]}$ unique solution with $y(0)$ fixed of the differential equation

$$\begin{aligned}\dot{y}(t) = \sum_{(k,l)\in\chi} y_{k,l}(t)[&(e_{k-1,l+1} - e_{k,l})\lambda\mathbb{1}_{k>0}\\ &+ (e_{k,l-1} - e_{k,l})\eta l\\ &+ (e_{k+1,l} - e_{k,l})\mu(s - \sum_{(i,j)\in\chi}(i+j)y_{i,j}(t))\mathbb{1}_{i+j<K})].\end{aligned}$$

Here, we recognize the form $\dot{y}(t) = y(t)L_y(t)$. From this, we can conclude, with the same arguments as [3], that the empirical distribution of the stations converges to an invariant measure whose generator is $L_y(t)$. Thus, we can identify the large-scale behavior of our system with the behavior of the system generated by $L_y(t)$, which is the tandem of two queues shown in Figure 7. It is analogous to the mean-field limit obtained in [3].

□

Limiting invariant measure of the system.

The result obtained here is very similar to the one in [3]. The difference is that in the tandem of two queues obtained here, there is first an $M/M/1$ queue and then an $M/M/\infty$ queue, whereas it is the opposite for [3]. However, the invariant measure is the same.

Proposition 4.2 (Invariant measure)**.** *The equilibrium point of the mean-field limit in Proposition 4.1, i.e. the invariant measure of the system in Figure 7, is*

$$\pi_{k,l} = \frac{1}{Z(\rho_R, \rho_V)} \rho_V^k \frac{\rho_R^l}{l!} \tag{1}$$

where $Z(\rho_R, \rho_V) = \sum_{(k,l)\in\chi} \rho_V^k \rho_R^l / l!$ *is the normalization constant,* $\rho_R = \rho_V \lambda/\eta$ *and* ρ_V *is uniquely determined by*

$$\rho_V = \frac{\mu}{\lambda}\left(s - \sum_{(k,l)\in\chi} (k+l)\pi_{k,l}\right). \tag{2}$$

Proof. This result is generalized in the following. The proof is a special case of the proof of Proposition 4.4 with $\alpha = 1$ and $\beta = 0$. □

4.1.3 Model with possibility of booking and possibility of cancelling the trip

The previous model is too simple to handle the real free-floating car-sharing system in Montreal, because it does not consider the possibility of cancelling a trip after the booking or starting a trip without a previous booking.

Let us introduce the following notations.

- α is the probability of booking a car when arriving at a zone
- β is the probability of cancelling a trip when the booking expires.

Let us assume in the following that $\alpha > 0$ and $\beta < 1$. We keep the same notations as before. In this section, the Markov process $(Y^N(t))_{[0,T]}$ describes the evolution of this new model for the Montreal car-sharing system, where it is possible to start a trip without booking the car, and to cancel a trip at the end of the booking. Let us state the following results.

Proposition 4.3 (Mean-field limit)**.** *The Markov process* $(Y^N(t))_{[0,T]}$ *tends to a deterministic process* $(y(t))_{[0,T]}$ *as the system gets large solution of the ODE*

$$\begin{aligned}\dot{y}(t) = \sum_{(k,l)\in\chi} y_{k,l}(t)[&(e_{k-1,l+1} - e_{k,l})\alpha\lambda \mathbb{1}_{\{k>0\}}\\ &+ (e_{k-1,l} - e_{k,l})(1-\alpha)\lambda \mathbb{1}_{\{k>0\}}\\ &+ (e_{k,l-1} - e_{k,l})(1-\beta)\eta l\\ &+ (e_{k+1,l-1} - e_{k,l})\beta\eta l\\ &+ (e_{k+1,l} - e_{k,l})\mu(s - \sum_{(i,j)\in\chi}(i+j)y_{i,j}(t))\mathbb{1}_{k+l<K}]\end{aligned}$$

The numbers of available and reserved cars at a given station behave respectively as the numbers of customers in the tandem of two queues, the first one a $M/M/1$ *queue and the second one a* $M/M/\infty$ *as in Figure 8.*

Proof. The arguments are similar to those of Proposition 4.2. See Annex 7.3 for more details. □

Proposition 4.4 (Invariant measure)**.** *The invariant measure of this tandem of queues is a product measure of the form*

$$\pi_{k,l} = \frac{1}{Z(\rho_R, \rho_V)} \rho_V^k \frac{\rho_R^l}{l!} \tag{3}$$

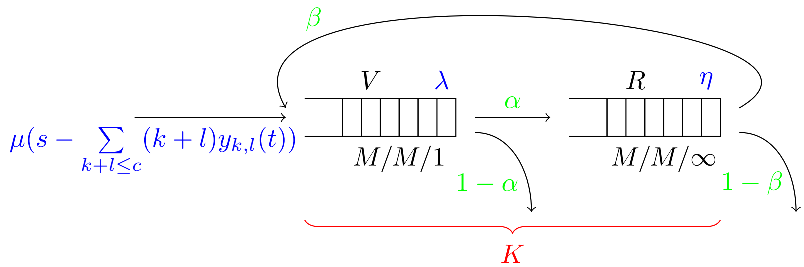

Figure 8: Mean-field approach of the model with possibility of booking and cancelling the trip: A typical zone as an open Jackson network of two queues with overall capacity K.

where $Z(\rho_R, \rho_V) = \sum_{(k,l)\in\chi} \rho_V^k \rho_R^l / l!$ *is the normalization constant,* $\rho_R = \rho_V \alpha\lambda/\eta$ *and* ρ_V *is uniquely determined by*

$$\rho_V = \frac{\alpha\mu}{\lambda(1-\alpha\beta)} \left(s - \sum_{(k,l)\in\chi} (k+l)\pi_{k,l} \right). \tag{4}$$

Recall that $\alpha > 0$ and $\beta < 1$, and thus $\alpha\beta < 1$. For the proof, see Annex 7.4.

Equivalence with the previous model.
Note that the model with possibility to book and to cancel the booking is equivalent to the previous model (simple model with a compulsory booking of a car for each user who arrives in a zone with available cars and no trip cancellation) if we change μ to $\widetilde{\mu}$ and η to $\widetilde{\eta}$ well defined, as $\alpha > 0$ and $\beta < 1$, by

$$\widetilde{\mu} = \frac{\mu}{1-\alpha\beta},$$
$$\widetilde{\eta} = \frac{\eta}{\alpha}.$$

This shows that the model with possibility of booking and of cancellation has the same stationary behavior, when $N \to \infty$, as the simple model of 4.1.1 with modified parameters. More precisely, the invariant measure of the model with possibility of booking and of cancellation is the same as the invariant measure of a simple model with compulsory booking and no cancellation. Therefore we can keep a simple model but its parameters need to be modified so that it fits with the reality. Being able to remain within the framework of the simple model is convenient, as it makes the analysis easier. Moreover in the second model with possibility of taking the car without booking and of cancelling the trip, reservation and trip times do not have exponential distribution. The equivalence means that the large-scale stationary invariant measure is the same as if they have exponential distribution with the same mean.

4.1.4 Performance study

Thanks to the equivalence between the two models, that the model with possibility of booking and of cancelling is equivalent to a simple model with no probability of cancelling and where everyone books his car before his trip starts, we can use the simple model to investigate the optimal size of the system. By noting that $E(V+R) = \sum_{(k,l)\in\chi} (k+l)\pi_{k,l}$, where $\pi_{k,l}$ is given by equation (1) from Proposition 4.2, equation (2) is rewritten

$$s = \frac{\lambda}{\mu}\rho + E(V+R) \tag{5}$$

which is a fixed point equation in $\rho_V = \rho$ because Proposition 4.2 gives also a linear dependence between ρ_R and ρ_V. Therefore, the invariant measure of the system can be expressed only in terms of ρ solution of (5).

The optimal size of the system is the one which minimizes the proportion of problematic zones defined by

$$P_b = \pi_{0,.} + \pi_S - \pi_{0,S} \tag{6}$$

where $\pi_{0,.}$ is the proportion of zones with no cars available, π_S is the proportion of zones which are saturated (no parking spaces available) and $\pi_{0,S}$ is the proportion of zones which have no cars available and are saturated.

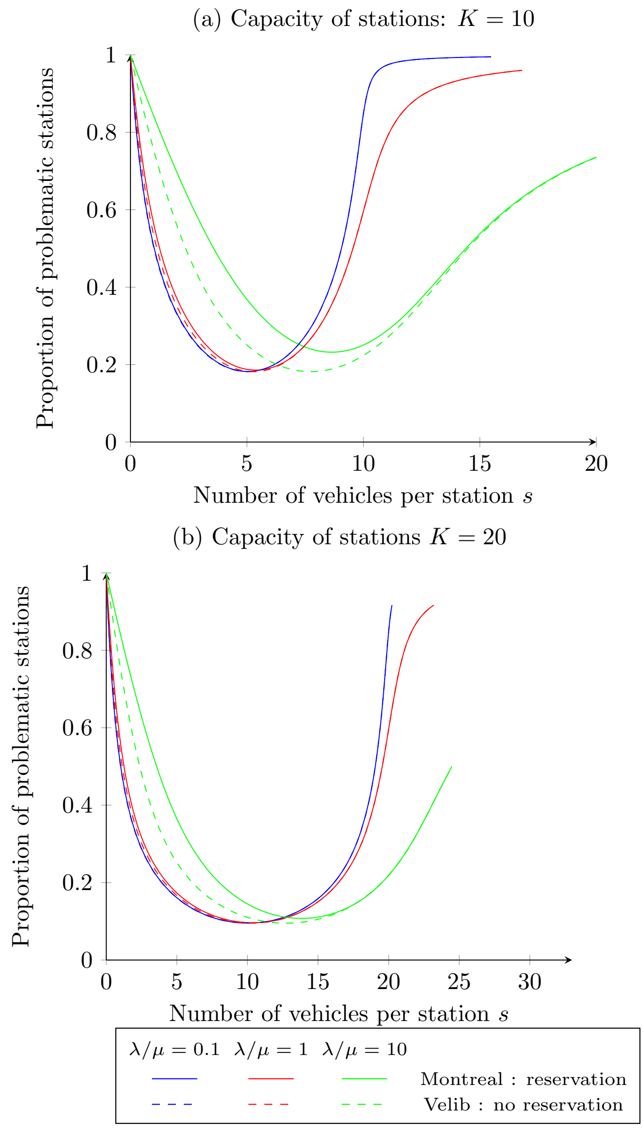

Figure 9: Parametric curve $\rho_V \mapsto (s, P_b)$ for $K = 10$ **(a)** and $K = 20$ **(b)** for the car-sharing model with car reservation (Flex Montreal) and without reservation (bike-sharing). In both cases $\eta/\mu = 4$.

Proposition 4.5 (Optimal fleet size)**.** *The optimal size of the system in the case of a small reservation time ($1/\eta << 1$) is*

$$s^* = \frac{K}{2} + \frac{\lambda}{\mu} + \frac{\lambda}{2\eta}\frac{K^2 - 3K/2 - 1}{K^2 - 1} + O\left(\frac{1}{\eta^2}\right)$$

and at this optimum s^*,

$$P_b^* = \frac{2}{K+1} + \frac{2}{K^2 + K + 1}\frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right).$$

Proof. See Annex 7.5. □

Figure 9 is a plot of (s, P_b) which clearly shows a minimum. We also plot on this figure the plot of $s \mapsto P_b$ for the bike-sharing system Velib of Paris, based on the results of [4], as in [3] and [5]. Recall that, for the bike-sharing system, $s^* = K/2 + \lambda/\mu$ and the corresponding value $P_b^* = 2/(K+1)$. We can see that the optimal size of the system is larger for the Montreal car-sharing system than for the Paris bike-sharing system. Moreover, as in the Paris bike-sharing system (Velib'), the optimal size of the system gets higher with the load of the system. A significant difference from the bike-sharing system of Paris (which has no booking system) appears at a high load. At a high load, we can see that the booking has a negative impact on the system, as the proportion of problematic stations for the optimal size of the system in the Montreal car-sharing system is larger (with an additional term $2/((K^2+K+1)\nu)$ for small mean reservation time) than the one in the Paris Velib system (which is $2/(K+1)$). This difference is decreasing with K as observed on Figure 9, comparing $K = 10$ and 20.

4.1.5 Strengths and weaknesses of the simple homogeneous model

The simple homogeneous model is quite convenient for an analysis like the one performed in Section 4.1.4 to find the optimal fleet size of the system. However, this model is only valid for an infinite number of zones, which may not be realistic as the service area is not infinite. Moreover, it is a homogeneous model, which means that all the zones must be similar (same rate of arrival of users, etc.). This is also unrealistic as some zones may be more popular than other and then have a higher rate of arrival of users for instance. That is why we further propose a inhomogeneous model with fixed size.

4.2 The inhomogeneous model with fixed size as a Jackson network

4.2.1 Additional notations and model with infinite capacity

In this part, we study a model similar to the one described in [6] for bike-sharing systems (without booking). The car-sharing system of Montreal is represented as a closed queuing network where customers (cars) move from one queue to the other. This model is an inhomogeneous model. Notations are similar to the previous ones, except that the parameters now depend on the zones they refer to. We use mainly notations defined in [6].

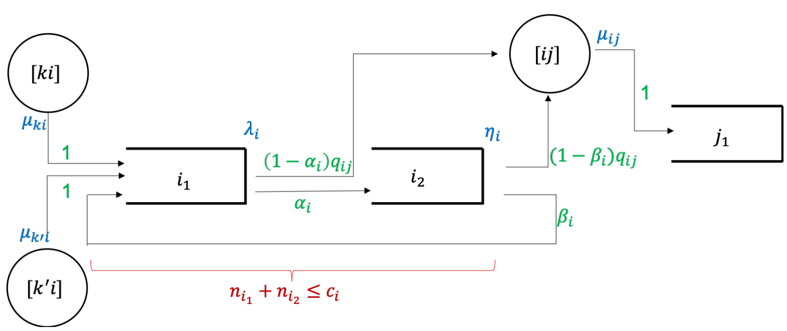

Figure 10: Montreal free-floating car-sharing system as a closed Jackson network.

We keep on terms as cars and zones instead of classical customers and nodes in queuing theory.
Types of queues. There are three types of queues in the closed Jackson network.

- Customers in $./M/1$ queue i are the available cars in zone i. Those queues are $./M/1$ because, when arriving in a zone, users are served (i.e. they manage to book or take a car) successively, by order of arrival. There are N such queues, numbered from 1 to N, queue i with an exponential rate of service λ_i, which is the arrival rate of users in zone i.
- Customers in $./M/\infty$ queue i are the reserved cars in zone i. A user is served in such a queue when the booking is finished. Being served means either taking the car to start a trip or cancelling the booking.

The service time has an exponential distribution with mean $1/\eta_i$, corresponding to the mean booking duration in zone i. There are N such queues, queue i of service rate η_i. These queues are $./M/\infty$ because all bookings evolve independently one from the other, so there is a server for every car which enters the queue.

- Customers of the $./M/\infty$ queue indexed by $[ij]$ are the cars which are going from zone i to zone j. There are N^2 such queues. The service times in these queues are the trip times in the system. The service time for queue $[ij]$ has an exponential distribution of parameter μ_{ij}, where $1/\mu_{ij}$ is the mean trip time from zone i to zone j. Those queues are $./M/\infty$ because all trips evolve independently one from the other as if there is a server for each car in the queue. As a first step, we assume that the zones have infinite capacity. This means that a car is always parked at a trip completion time, because there is always an available parking space available at the destination. This makes the analysis easier. In the next section, the results obtained here will be extended for finite capacity.

Notations for indexes.

- We index the queues with available cars in a zone with i_1, where $1 \leq i \leq N$ corresponds to the number of the zone.
- We index the queues with booked cars in a zone with i_2, where $1 \leq i \leq N$ corresponds to the number of the zone.
- We index the queues with cars driving from zone i to zone j with $[ij]$, where $1 \leq i, j \leq N$, the pair origine-destination.

State of the system: We describe a state of the system with

- $n_{i_1} \in \mathbb{N}$ the number of available cars in zone i,
- $n_{i_2} \in \mathbb{N}$ the number of booked cars in zone i,
- $n_{[i,j]} \in \mathbb{N}$ the number of cars driving from zone i to zone j.

A state of the system is therefore $n = (n_{i_1}, n_{i_2}, 1 \leq i \leq N, n_{[i,j]}, 1 \leq ij \leq N)$.

Additional notations and definitions. We also define the following probabilities.

- α_i is the probability to book a car when arriving in zone i,
- β_i is the probability to cancel a trip after having booked it in zone i,
- q_{ij} is the probability to start a trip from zone i to zone j

Here, the probabilities to go from one queue to another are the following, for all i_1, i_2 and $[ij]$:

- $p_{i_1,i_2} = \alpha_i$ is the probability to book a car when arriving at zone i,
- $p_{i_1,[ij]} = (1 - \alpha_i)q_{ij}$ is the probability to directly start a trip to zone j when arriving at zone i,
- $p_{i_2,[ij]} = (1 - \beta_i)q_{ij}$ is the probability to start a trip to zone j after the booking at zone i
- $pi_2, i_1 = \beta_i$ is the probability to cancel the trip after the booking at zone i
- $p_{[ij],j_1} = 1$ is the probability to park the car in zone j at the end of the trip (which is 1 as the capacity of the zones is supposed infinite in this model).

Let us assume that routing matrix $(q_{ij})_{1\leq i,j\leq N}$ has a unique invariant measure $(\nu_i)_{1\leq i\leq N}$. By definition,

$$\nu_j = \sum_{i=1}^{N} \nu_i q_{ij}. \tag{7}$$

4.2.2 Product form invariant measure

Proposition 4.6 (Product-form invariant measure)**.** *The invariant measure π of the inhomogeneous model, as a closed Jackson network described in Section 4.2.1, has the following product form*

$$\pi(n_{i_1}, n_{i_2}, 1 \le i \le N, n_{[ij]}, 1 \le i, j \le N) = C \prod_{1 \le i,j \le N} \rho_{i_1}^{n_{i_1}} \frac{\rho_{i_2}^{n_{i_2}}}{n_{i_2}!} \frac{\rho_{[ij]}^{n_{[ij]}}}{n_{[ij]}!} \tag{8}$$

where C is the normalization constant and

$$\begin{cases} \rho_{i_1} = \dfrac{\theta_{i_1}}{\lambda_i}, & \theta_{i_1} = \dfrac{\delta \nu_i}{1 - \alpha_i \beta_i}, \\ \rho_{i_2} = \dfrac{\theta_{i_2}}{\eta_i}, & \theta_{i_2} = \dfrac{\delta \alpha_i \nu_i}{1 - \alpha_i \beta_i}, \\ \rho_{[ij]} = \dfrac{\theta_{[ij]}}{\mu_{[ij]}}, & \theta_{[ij]} = \delta \nu_i q_{ij} \end{cases} \tag{9}$$

where δ depending on (α_i), (β_i) and (ν_i) is

$$\delta = \left(1 + \sum_{j=1}^{N} \frac{(1 + \alpha_i)\nu_i}{1 - \alpha_i \beta_i} \right)^{-1}. \tag{10}$$

In particular, in the homogeneous case where all the stations are similar, i.e. if for all i, $\alpha_i = \alpha$, $\beta_i = \beta$,

$$\delta = \frac{1 - \alpha\beta}{2 + \alpha(1 - \beta)}. \tag{11}$$

Proof. Applying the well-known result on closed networks of [1], as in [7], we know that the invariant measure π of the network is a product measure given by

$$\pi(n_{i_1}, n_{i_2}, 1 \le i \le N, n_{[ij]}, 1 \le i, j \le N) = C \prod_{1 \le i,j \le N} \rho_{i_1}^{n_{i_1}} \frac{\rho_{i_2}^{n_{i_2}}}{n_{i_2}!} \frac{\rho_{[ij]}^{n_{[i,j]}}}{n_{[ij]}!}$$

where

$$\rho_{i_1} = \frac{\theta_{i_1}}{\lambda_i}, \qquad \rho_{i_2} = \frac{\theta_{i_2}}{\eta_i}, \qquad \text{and} \qquad \rho_{[ij]} = \frac{\theta_{[ij]}}{\mu_{[ij]}}.$$

We know until [1] (see for example the general theory of closed queuing networks in [11]), that θ is the invariant measure of $P = (p_{kl})$. This means that θ_{i_1}, θ_{i_2} and $\theta_{[ij]}$ are solutions of

$$\begin{cases} \theta_{i_1} = \sum_k \theta_k p_{k,i_1} \\ \theta_{i_2} = \sum_k \theta_k p_{k,i_2} \\ \theta_{[ij]} = \sum_k \theta_k p_{k,[ij]}. \end{cases}$$

Here, this can be rewritten

$$\begin{cases} \theta_{i_1} = \theta_{i_2} p_{i_2,i_1} + \sum_{k=1}^{N} \theta_{[ki]} \\ \theta_{i_2} = \theta_{i_1} p_{i_1,i_2} \\ \theta_{[ij]} = \theta_{i_1} p_{i_1,[ij]} + \theta_{i_2} p_{i_2,[ij]}. \end{cases} \tag{12}$$

The second equation of (12) gives

$$\theta_{i_2} = \alpha_i \theta_{i_1} \tag{13}$$

Plugging (13) in the third equation of (12), we further obtain

$$\theta_{[ij]} = (1 - \alpha_i \beta_i) q_{ij} \theta_{i_1}. \tag{14}$$

Using this expression of $\theta_{[ij]}$ for all i, j, the first equation of (12) finally gives

$$(1 - \alpha_i \beta_i)\theta_{i_1} = \sum_{k=1}^{N} (1 - \alpha_k \beta_k) q_{ki} \theta_{k_1}. \tag{15}$$

Similarly to the study by [6], remembering equation (7), we note that equation (12) is equivalent to, for any constant δ,

$$\begin{aligned}
\theta_{i_1} &= \frac{\delta \nu_i}{1 - \alpha_i \beta_i} \\
\theta_{i_2} &= \frac{\delta \alpha_i \nu_i}{1 - \alpha_i \beta_i} \\
\theta_{[ij]} &= \delta \nu_i q_{ij}.
\end{aligned}$$

But, by definition, (θ_k) is a probability measure. It yields that

$$\sum_{i=1}^{N} \theta_{i_1} + \sum_{i=1}^{N} \theta_{i_2} + \sum_{i,j=1}^{N} \theta_{[ij]} = 1$$

which gives

$$\sum_{i=1}^{N} \frac{\nu_i}{1 - \alpha_i \beta_i} + \sum_{i=1}^{N} \frac{\alpha_i \nu_i}{1 - \alpha_i \beta_i} + \sum_{i,j=1}^{N} \nu_i q_{ij} = \frac{1}{\delta}.$$

Using the fact that ν_i is a probability measure and the invariant measure of q_{ij},

$$\delta = \frac{1}{1 + \sum_{i=1}^{N} \frac{(1+\alpha_i)\nu_i}{1-\alpha_i \beta_i}}.$$

Then, if for all i, $\alpha_i = \alpha$ and $\beta_i = \beta$, straightforwardly,

$$\delta = \frac{1 - \alpha\beta}{2 + \alpha(1 - \beta)}.$$

□

4.2.3 Equivalence with a simpler model

As in Section 4.1, we can show that the model with probabilities α_i of booking a car and probabilities β_i of cancelling a trip at zone i, $1 \leq i \leq N$ is equivalent to a simpler model with modified booking and trip rates of the zones. By equivalent, we mean that the invariant measures of both models are equal.

Proposition 4.7. *The model with probability of booking a car α_i and probability of cancelling a trip β_i is equivalent to the simple model where the trip and the booking durations have exponential distribution with the same mean as in the original model. It amounts to modifying the booking and trip rates of the zones as follows. For all i and j,*

$$\begin{aligned}
\widetilde{\eta}_i &= \alpha_i \eta_i, \\
\widetilde{\mu}_{[ij]} &= \frac{\mu_{ij}}{1 - \alpha_i \beta_i}.
\end{aligned}$$

Proof. We apply the same argument as in Proposition 4.6 to find the invariant measure of this system. See Annex 7.6 for details. □

The equivalent model is represented in Figure 11.

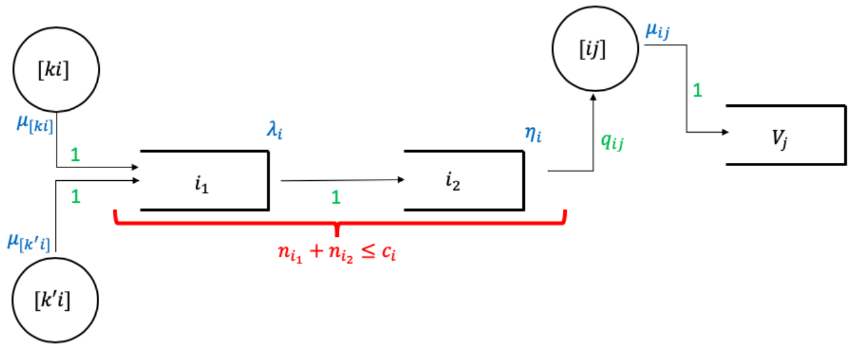

Figure 11: Equivalent model for the inhomogeneous model.

4.2.4 Extension to a model with finite capacity

We now extend the previous results to a finite capacity system in which each zone i has a capacity c_i. This means, at state n, $n_{i_1} + n_{i_2} \leq c_i < \infty$.

The dynamics are the same as previously, except in the case where a car is parked. If a car is in the queue corresponding to the cars going from zone i to zone j there are two possibilities at the end of the trip: if there is a parking space available in zone j, the car is parked, otherwise, it starts another trip towards a zone k chosen among all the zones with probability (q_{jk}). This model does not take into account the locality of the dynamics. It would be more realistic if, when a user does not find an available parking space in his targeted zone, he goes towards a zone close to the initial destination zone. This is not taken account in this model.

Proposition 4.8. *For a fixed number N of zones, when considering probabilities α_i of booking and β_i of cancelling a trip at zone i, the invariant measure of the system is a product measure given by equation* (8) *with equation* (9).

Proof. We show that, given a state n, the above measure satisfies the balance equation

$$\sum_{n' \neq n} \frac{\pi(n')}{\pi(n)} Q(n', n) = \sum_{n' \neq n} Q(n, n')$$

where $Q(n', n)$ is the jump rate from state n to n'.

To compute $Q(n, n')$ and $Q(n', n)$, we need all the transitions and the rates at which they occur. We denote a state by $n = (n_{i_1}, n_{i_2}, 1 \leq i \leq N, n_{[ij]}, 1 \leq i, j \leq N)$. We note $(e_{i_1}, e_{i_2}, 1 \leq i \leq N, e_{[ij]}, 1 \leq i, j \leq N)$ the canonical basis of the state space. We give the transition rates as matrix jump Q components, with obvious notation for n'.

The transitions from a given state n are the following.

$$\begin{aligned}
&n \to n - e_{i_1} + e_{i_2} \quad Q(n, n') = \alpha_i \lambda_i \mathbb{1}_{n_{i_1} > 0} \\
&n \to n - e_{i_1} + e_{[ij]} \quad Q(n, n') = (1 - \alpha_i) \lambda_i q_{ij} \mathbb{1}_{n_{i_1} > 0} \\
&n \to n - e_{i_2} + e_{[ij]} \quad Q(n, n') = (1 - \beta_i) \eta_i n_{i_2} q_{ij} \\
&n \to n - e_{i_2} + e_{i_1} \quad Q(n, n') = \beta_i \eta_i n_{i_2} \\
&n \to n - e_{[ij]} + e_{j_1} \quad Q(n, n') = \mu_{[ij]} n_{[ij]} \mathbb{1}_{n_{j_1} + n_{j_2} < c_j} \\
&n \to n - e_{[ki]} + e_{[ij]} \quad Q(n, n') = \mu_{[ki]} n_{[ki]} q_{ij} \mathbb{1}_{n_{i_1} + n_{i_2} = c_i}
\end{aligned}$$

The transitions to a given state n are the following.

$$\begin{aligned}
n - e_{i_2} + e_{i_1} \to n \quad & Q(n', n) = \alpha_i \lambda_i \\
n - e_{[ij]} + e_{i_1} \to n \quad & Q(n', n) = (1-\alpha_i)\lambda_i q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}<c_i, n_{[ij]}>0} \\
n - e_{[ij]} + e_{i_2} \to n \quad & Q(n', n) = (1-\beta_i)\eta_i (n_{i_2}+1) q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}<c_i, n_{[ij]}>0} \\
n - e_{i_1} + e_{i_2} \to n \quad & Q(n', n) = \beta_i \eta_i (n_{i_2}+1) \mathbb{1}_{n_{i_1}>0} \\
n - e_{j_1} + e_{[ij]} \to n \quad & Q(n', n) = \mu_{[ij]} (n_{[ij]}+1) \mathbb{1}_{n_{j_1}>0} \\
n + e_{[ki]} - e_{[ij]} \to n \quad & Q(n', n) = \mu_{[ki]} (n_{[ki]}+1) q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}=c_i}
\end{aligned}$$

We now need to check that the balance equation is satisfied.

For each queue k, let us denote by $g_k(n_k)$ the function on the set of possible states for each queue. Similarly to what is done in [6], let us define g_k for each type of queue.

$$\begin{aligned}
g_{i_1}(n_{i_1}) &= \lambda_i \mathbb{1}_{n_{i_1}>0}, \\
g_{i_2}(n_{i_2}) &= \eta_i n_{i_2}, \\
g_{[ij]}(n_{[ij]} &= \mu_{[ij]} n_{[ij]}.
\end{aligned}$$

Let us also define $g_k!(n_k)$ by

$$g_k!(0) = 1, \quad g_k!(n_k) = g_k(n_k) g_k!(n_k - 1) \text{ for } n_k \in \mathbb{N} \setminus \{0\}.$$

Equation (8) we want to check can be rewritten

$$\pi(n_{i_1}, n_{i_2}, 1 \le i \le N, n_{[ij]}, 1 \le i, j \le N) = C \prod_{i_1, i_2, [ij]} \frac{\theta_{i_1}^{n_{i_1}}}{g_{i_1}(n_{i_1})!} \frac{\theta_{i_2}^{n_{i_2}}}{g_{n_{i_2}!}(n_{i_2})!} \frac{\theta_{[ij]}^{n_{[i,j]}}}{g_{n_{[ij]}}(n_{[ij]})!}.$$

In particular, as in [6], straightforwardly,

$$\frac{\pi(n - e_k + e'_k)}{\pi(n)} = \frac{\theta_{k'} g_k(n_k)}{\theta_k g_{k'}(n_{k'}+1)}.$$

Thus, the left-hand side of the balance equation can be rewritten, with algebra,

$$\begin{aligned}
\sum_{n' \neq n} \frac{\pi(n')}{\pi(n)} Q(n', n) = & \sum_{i=1}^{N} \frac{\theta_{i_1}}{\theta_{i_2}} \alpha_i \eta_i n_{i_2} \\
& + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\theta_{i_1}}{\theta_{[ij]}} \mu_{[ij]} n_{[ij]} (1-\alpha_i) q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}<c_i} \\
& + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\theta_{i_2}}{\theta_{[ij]}} \mu_{[ij]} n_{[ij]} (1-\beta_i) q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}<c_i} \\
& + \sum_{i=1}^{N} \frac{\theta_{i_2}}{\theta_{i_1}} \lambda_i \mathbb{1}_{n_{i_1}>0} \beta_i \\
& + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\theta_{[ij]}}{\theta_{j_1}} \lambda_j \mathbb{1}_{n_{j_1}>0} \\
& + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\theta_{[ki]}}{\theta_{[ij]}} \mu_{[ij]} n_{[ij]} q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}=c_i}
\end{aligned}$$

Moreover from the proof of Proposition, by equations (13), (14) and (15), $\theta_{i_1}, \theta_{i_2}, \theta_{[ij]}$ satisfy

$$\begin{cases} \theta_{i_1} = \sum_{k=1}^{N} (1-\alpha_k\beta_k) q_{ki}\theta_{k_1}, \\ \theta_{i_2} = \alpha_i \theta_{i_1}, \\ \theta_{[ij]} = (1-\alpha_i\beta_i) q_{ij}\theta_{i_1}. \end{cases}$$

Using those equations and simple algebra, the left-hand side of the balance equation can be rewritten

$$\begin{aligned} \sum_{n' \neq n} \frac{\pi(n')}{\pi(n)} Q(n', n) &= \sum_{i=1}^{N} \eta_i n_{i_2} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\theta_{i_1}}{\theta_{[ij]}} \mu_{[ij]} n_{[ij]} (1-\alpha_i) q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}<c_i} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\alpha_i\theta_{i_1}}{\theta_{[ij]}} \mu_{[ij]} n_{[ij]} (1-\beta_i) q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}<c_i} \\ &+ \sum_{i=1}^{N} \alpha_i \lambda_i \beta_i \mathbb{1}_{n_{i_1}>0} \\ &+ \sum_{j=1}^{N} \lambda_j (1-\alpha_j\beta_j) \mathbb{1}_{n_{j_1}>0} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{\theta_{[ij]}} (1-\alpha_i\beta_i) \theta_{i_1} \mu_{[ij]} n_{[ij]} q_{ij} \mathbb{1}_{n_{i_1}+n_{i_2}=c_i}. \end{aligned}$$

The sum of the second, third and sixth term is

$$\sum_{j=1}^{N} \frac{1}{\theta_{[ij]}} \mu_{[ij]} n_{[ij]} \sum_{i=1}^{N} (1-\alpha_i\beta_i) \theta_{i_1} q_{ij} = \sum_{i,j=1}^{N} \mu_{[ij]} n_{[ij]}$$

using also equation (14) on $\theta_{[ij]}$. The sum of the fourth and fifth term is $\sum_{i=1}^{N} \lambda_i \mathbb{1}_{n_{i_1}>0}$. Therefore, the left-hand side of the balance equation can be rewritten

$$\sum_{n' \neq n} \frac{\pi(n')}{\pi(n)} Q(n', n) = \sum_{i=1}^{N} \lambda_i \mathbb{1}_{n_{i_1}>0} + \sum_{i=1}^{N} \eta_i n_{i_2} + \sum_{i=1}^{N}\sum_{j=1}^{N} \mu_{[ij]} n_{[ij]} \tag{16}$$

The right-hand side of the balance equation is

$$\begin{aligned}\sum_{n'\neq n} Q(n,n') = &\sum_{i=1}^{N}\alpha_i\lambda_i \mathbb{1}_{n_{i_1}>0}\\ &+\sum_{i=1}^{N}\beta_i\eta_i n_{i_2}\\ &+\sum_{i=1}^{N}(1-\alpha_i)\lambda_i\mathbb{1}_{n_{i_1}>0}\sum_{j=1}^{N}q_{ij}\\ &+\sum_{i=1}^{N}(1-\beta_i)\eta_i n_{i_2}\sum_{j=1}^{N}q_{ij}\\ &+\sum_{j=1}^{N}\mu_{[ij]}n_{[ij]}\mathbb{1}_{n_{j_1}+n_{j_2}<c_j}\\ &+\sum_{i=1}^{N}\sum_{k=1}^{N}\mu_{[ki]}n_{[ki]}\mathbb{1}_{n_{i_1}+n_{i_2}=c_i}\sum_{j=1}^{N}q_{ij}\end{aligned}$$

By using the fact that $\sum_{j=1}^{N} q_{ij} = 1$ i.e. that (q_{ij}) is a stochastic matrix, re-indexing the last term by changing i to j and k to i and regrouping the terms with the indicator function,

$$\sum_{n'\neq n} Q(n,n') = \sum_{i=1}^{N}\lambda_i\mathbb{1}_{n_{i_1}>0} + \sum_{i=1}^{N}\eta_i n_{i_2} + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_{[ij]}n_{[ij]}$$

which by equation (16) is equal to the left-hand side of the balance equation. It ends the proof. □

4.2.5 Large-scale approximation

When $N \to \infty$, the same result as [6, Corollary 5.2] holds, using similar arguments. Indeed, with appropriate assumptions (see [6, Corollary 5.2]), the invariant measure on a finite set of zones i and routes $[ij]$ tends, when $N \to \infty$, to

$$\prod_i \frac{1}{Z_i}(\gamma_N\rho_{i_1})^{n_{i_1}}\frac{(\gamma_N\rho_{i_2})^{n_{i_2}}}{n_{i_2}!}\prod_{[ij]}\frac{(\gamma_N\rho_{[ij]})^{n_{[i,j]}}}{n_{[ij]}!}e^{-\gamma_N\rho_{[ij]}}$$

where

$$Z_i = \sum_{n_{i_1},n_{i_2},n_{i_1}+n_{i_2}\leq c_i}(\gamma_N\rho_{i_1})^{n_{i_1}}\frac{(\gamma_N\rho_{i_2})^{n_{i_2}}}{n_{i_2}!}$$

and γ_N solves the equation

$$M_N = \sum_{i=1}^{N}\frac{\sum\limits_{k+l\leq c_i}(k+l)\frac{(\gamma_N\rho_{i_2})^l}{l!}(\gamma_N\rho_{i_1})^k}{\sum\limits_{k+l\leq c_i}\frac{(\gamma_N\rho_{i_2})^l}{l!}(\gamma_N\rho_{i_1})^k} + \sum_{1\leq i,j\leq N}\gamma_N\rho_{[ij]}.$$

4.2.6 Loops

In the data analysis, the probability p for a user to make a loop trip is quite significant (around 25%). However, in the homogeneous model which is easy to handle, the probability to go to each zone is uniform. The aim will be to prove an equivalence between the model with loops and the homogeneous model. It is given in a special case by the following result.

Proposition 4.9 (Invariant measure in a homogeneous model with loops)**.** *The invariant measure ν_i of the routing matrix in a model with loops (q_{ij}) defined by*

$$q_{ii} = p, \qquad 1 \le i \le N$$
$$q_{ij} = \frac{1-p}{N-1}, \qquad 1 \le i \ne j \le N$$

is the uniform probability measure as in the homogeneous model.

Proof. In the homogeneous model, for all i, j, $q_{ij} = 1/N$, $\nu_i = 1/N$ is obtained as the solution of

$$\sum_{i=1}^{N} \nu_i q_{ij} = \nu_j$$

using the fact that $\sum_{i=1}^{N} \nu_i = 1$ Similarly, for the model with loops, the equation satisfied by the invariant measure can be rewritten

$$p\nu_j + \frac{1-p}{N-1}(1-\nu_j) = \nu_j$$

which gives also $\nu_j = 1/N$. □

5 Experiments

5.1 Python simulator

Different Python functions are written to simulate the system dynamics. The variables updated by the Python functions represent the state of the zones (number of available cars and of booked cars at each zone), the sets of cars which are available and booked at each station, and the number of circulating cars.

The output of the simulator consists in graphs which can show either the time evolution of the system (evolution of the number of available cars per zone, of booked cars per zone, and of the total number of cars in circulation divided by the number of zones) or the time evolution of the proportion problematic zones. Problematic zones are zones which have no available cars (because the cars in the zone are all booked or because the zone has no car) or which have no parking spaces available (they are saturated).

The input of the simulator consists in the parameters of the system: λ, η, μ, the number of zones N and the fleet size parameter s of the system (which is equal to the total number of cars in the system divided by the number of zones). Another input parameter is the maximum number of iterations of the simulator. Indeed, the simulator is a loop function, every loop corresponding to an event (jump of the Markov state process), which can be a user arrival, a booking completion time or a trip completion time. The number of iterations determines the number of successive events simulated.

First, two simulators are built, one to represent the model with a probability of booking and a probability of cancellation, and the other one to represent the simple model where the users always book their car before traveling and where they cannot cancel their trips. This enables to check the equivalence between the two models, with the modified parameters for the simple model determined by the analysis (see Proposition 4.7).

5.2 Input parameters

Simulations are performed for the model with η and λ determined from the data (see Section 4.1.2).

One wonder how to choose parameter μ, as the mean trip time is too large compared to the median trip time for assuming exponential distribution of the trip times because the distribution of the trip times seems heavy-tailed from the data analysis. The first attempt to fix this problem is to remove the largest trip times. Indeed, the largest trip times are of several days, not necessarily with a huge distance traveled. Some other trips are longer than the majority of trips (for instance, longer than 2 hours) and correspond to a very short distance. The users making such trips may rent the car for some amount of time and then use it for several trips, parking at each trip completion time. Even if there are several trips, there is only one transaction (and therefore one trip duration) for the operator. Such a use of the car can be spotted by looking at the ratio between the traveled distance and the trip duration. Those trips are not taken into account in our model, because we assume that, when a car starts a trip, it drives during the whole trip duration, without parking at intermediate points before its destination zone. Therefore, we remove those long trips. The condition we propose for removing is that, if the trip is longer than 2 hours, the average speed of the car (defined by the ratio between the traveled distance and the trip duration) should be larger than 5 km/h, otherwise the trip is removed. The underlying idea is that it is quite unlikely that a user constantly driving has a speed lower than 5 km/h during 2 hours, even in traffic jams in the city.

This leads to the elimination of some trips, and the mean moves from 1 h 22 min to 38 min and the median from 22 min to 16 min. However, the mean is still larger than the mean of a variable which has an exponential distribution of median 16 min. Indeed, we know that for a variable X with exponential distribution, $E(X) = m(X)/ln(2)$. Nevertheless the simulations are carried out with this value 38 min for $1/\mu$.

5.3 First results

Figures 12 and 13 show respectively the time evolution of the system and the time evolution of the proportion of problematic zones.

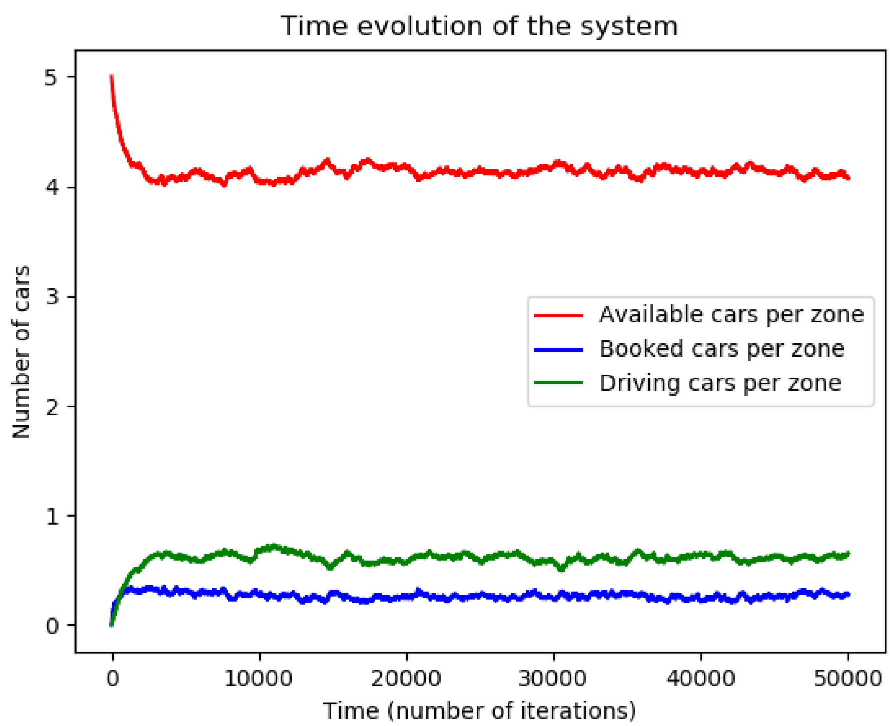

Figure 12: Evolution of the state of the system for capacity $K = 10$, $\lambda = 1.24$, $s = 5$.

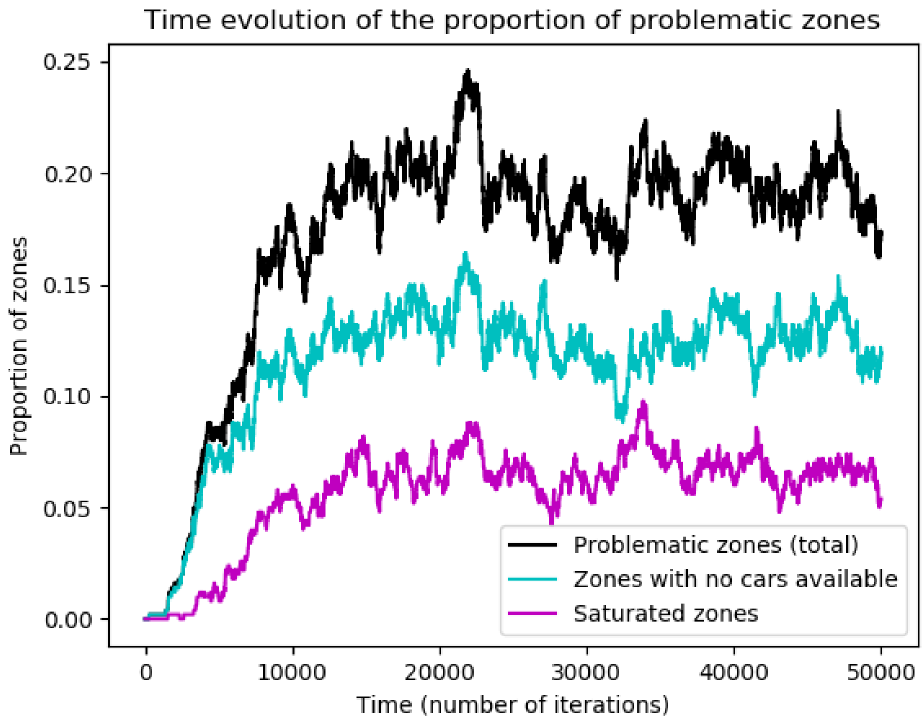

Figure 13: Evolution of the proportions of problematic zones for capacity $K = 10$, $\lambda = 1.24$, $s = 5$.

We can see that those functions of time oscillate quite a lot, which is due to the fact that they are obtained from one trajectory of a random process. However, they reach a stationary state, which is coherent with the model analysis, where the target quantities are invariant measures. Despite the oscillations, we can obtain quite precise values to describe the stationary state of the system, by calculating means on a large number of iterations after equilibrium has been reached. It allows us to compare the model with possibility of booking and of cancelling with the model where everyone books and no one cancels his trip: the number of available cars per zone, booked cars per zone and circulating cars per zone were derived for both models. The results obtained are close up to 2 significant digits.

5.4 Alternative simulation with the real distribution of the trips

5.4.1 Real trip time distribution

As previously noticed, the distribution of the trips seems heavy-tailed. As we know its mean and its median, we can wonder what is the real trip time distribution, based on the relation between mean and median for different heavy-tailed distributions.

A rough method for that to use *Maple* to plot different heavy-tailed distributions with the same mean and median as the Montreal trip duration distribution. The only plot which had a plot shape close to the one of the trip distribution as shown by R is a log-normal distribution. R is then used to confirm that this is a log-normal distribution. Figure 12 shows the distribution of the logarithm of the trip duration, which fits a normal distribution.

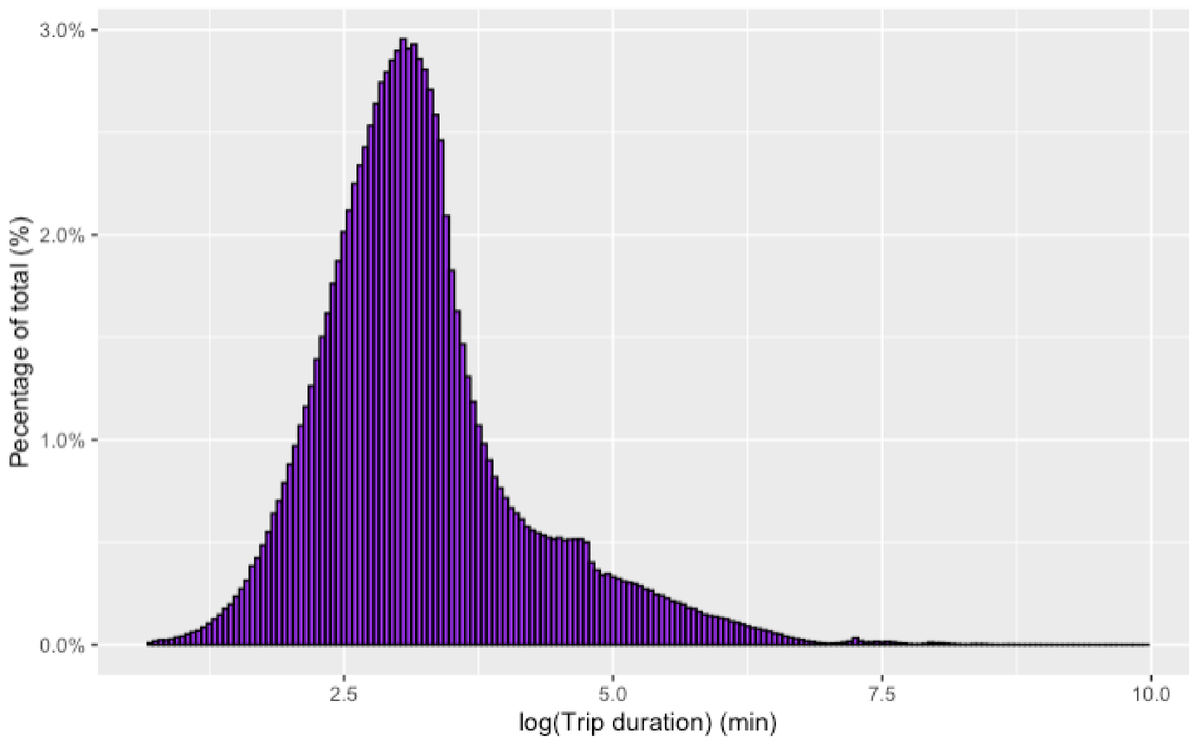

Figure 14: Distribution of the logarithm of the trip duration

A further check is the Q-Q plot of this distribution. It is quite close to a straight line, so we can consider that the logarithm of the trip duration has a normal distribution, which shows that the trip duration has a log-normal distribution.

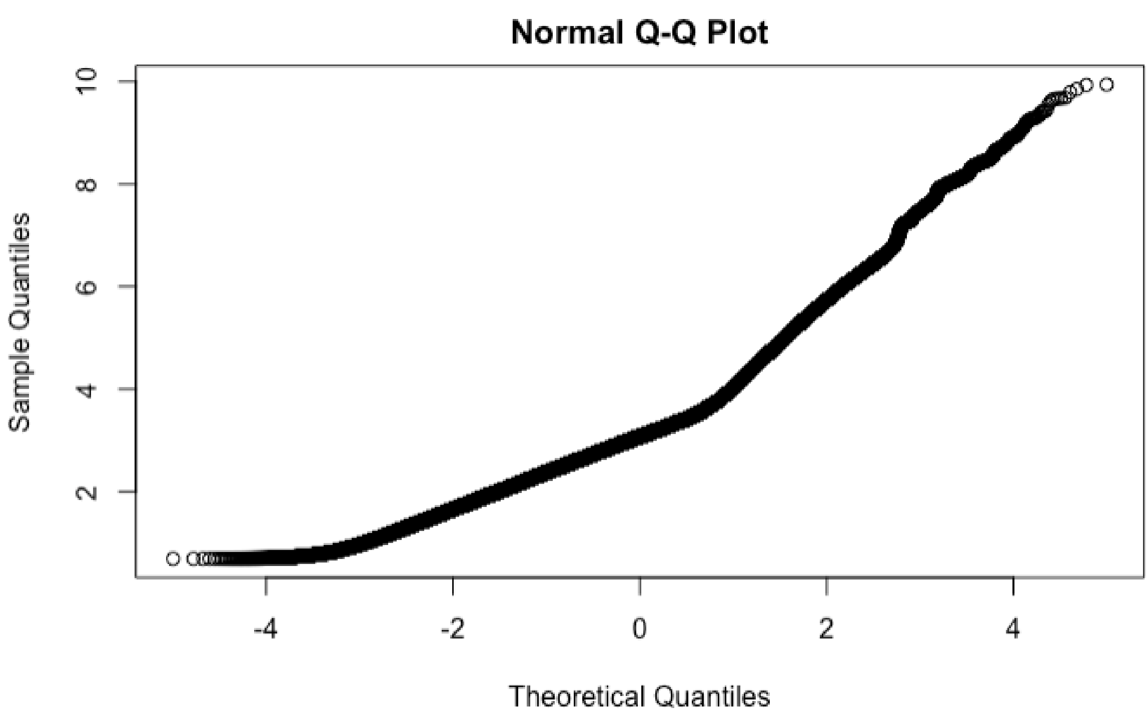

Figure 15: Q-Q plot of the logarithm of the trip duration

5.4.2 Results

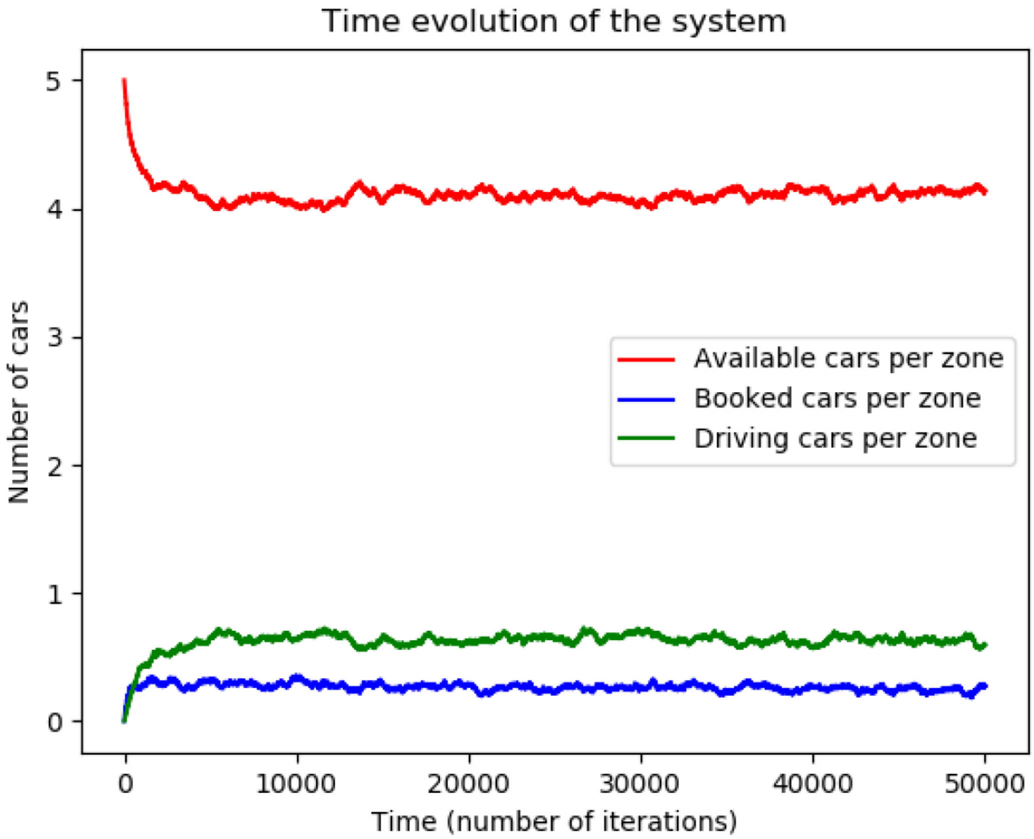

Figure 16: Evolution of the system with log-normal distribution with same parameter values as in Figure 12.

The results obtained with log-normal distribution in Figure 16 are quite similar to those with exponential distribution in Figure 12. The evolution of the states of the system has a shape similar to the one of the evolution of the system with exponential distributions and the same limiting values. As $s = 5 = K/2$, s is close to the optimal size of the system which is calculated in Section 4.1.5. The simulator allows us to investigate the behavior with different traffic loads. See Figure 17.

6 Conclusion and future work

The first part of the paper highlights some interesting features of the Montreal free-floating car-sharing system: the usage throughout a day, the booking time and trip time distributions, and the probability for a user to do a loop-trip depending on his trip length.

This data analysis allows us to propose an original mathematical model for it. Thanks to a mean-field approach, it is possible to compute (or to plot and read on a graph) the optimal size of the system in different scenarios, corresponding to different regimes of the system. The mean-field approach is convenient for the mathematical analysis but best suited to homogeneous systems. We further study an heterogeneous model. The main result is that this model at fixed size is a closed Jackson network and reaches an equilibrium for which the invariant measure has a product measure, but on a constrained state space. Furthermore the invariant measure could be used to prove that, in large-scale systems, the state of a finite set of zones and routes are independent with closed form distributions. It allows to investigate the optimal fleet size of the system and address the dimensioning problem.

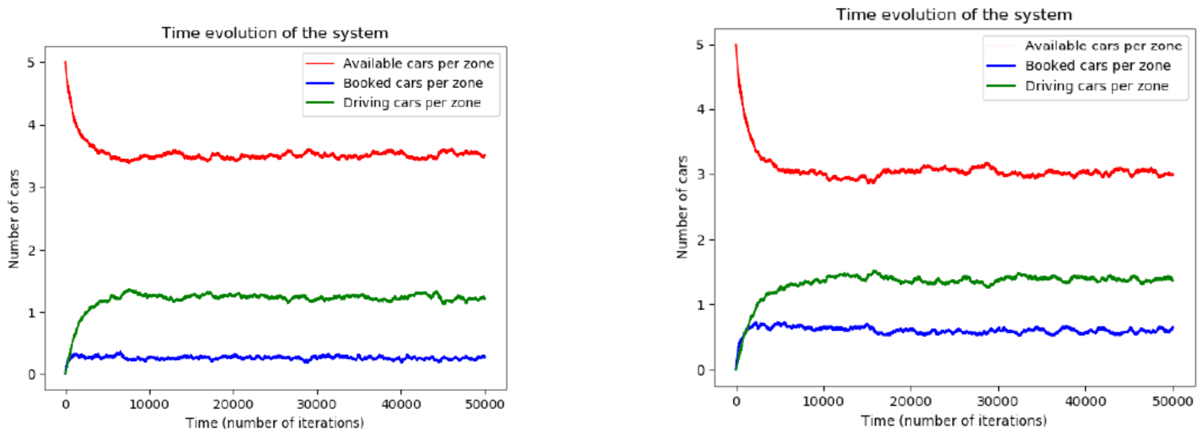

Figure 17: Evolution of the system with log-normal distribution with different parameter values

A Python simulator of the system validates the mathematical results and provides insights on performance for non exponential distributions.

The paper points to the problem of the trip time distribution, which is quite different from an exponential distribution, certainly heavy-tailed. The study of models with general distributions is much more difficult.

Finally, for the Montreal car-sharing system, it would be interesting to have a model which aggregates the free-floating system and the station-based system (as both are owned by Communauto with the same users). Indeed, the issue is to understand how the two systems interact, for instance how often the free-floating system is used as a station-based system and why. The issue for the aggregation of both models would be to optimize the aggregated system, as usual by addressing the dimensioning problem (the best fleet size) of the aggregated model, the location problem (where the stations should be located within the service area) and the relocation problem.
As a conclusion, modelling free-floating car-sharing systems allows stochastic analysis as a tool to understand the system behavior. It is still very fertile for research, especially the free-floating scheme. This paper is a first step in their analysis and should provide a basis for further work.

Acknowledgements

The authors thank Communauto for providing its data of Montreal free-floating car-sharing system, for their interest for the work and for helpful explanations.

7 Annex

7.1 Function to merge the bookings

```
#Fusion des réservations
MergeRes<-function(df){
  df_prov<-df
  result<-c(0)
  while(nrow(subset(df_prov, ResSucc==1))>0) {
    df_prov<-select(df_prov, CarID, CustomerID, TripDate, TripDateStr, DurationNumDay, DurationNumMin,
TrueBookDuration, TrueBookDurationMin, ResSucc, Distance, Merged)
    next_res_succ<-select(slice(df_prov, 2:nrow(df_prov)), ResSucc)
    next_res_succ<-rename(next_res_succ, NextResSucc=ResSucc)
    next_res_succ<-rbind(next_res_succ, 0)
    previous_res_succ<-select(slice(df_prov, 1:(nrow(df_prov)-1)), ResSucc)
    previous_res_succ<-rename(previous_res_succ, PreviousResSucc=ResSucc)
    previous_res_succ<-rbind(0, previous_res_succ)
    next_tbd<-select(slice(df_prov, 2:nrow(df_prov)), TrueBookDuration, TrueBookDurationMin);
    next_tbd<-rename(next_tbd, NextTrueBookDuration=TrueBookDuration, NextTrueBookDurationMin=TrueBookDurationMin);
    next_tbd<-rbind(next_tbd, c(0,0));
    next_d<-select(slice(df_prov, 2:nrow(df_prov)), DurationNumDay, DurationNumMin);
    next_d<-rename(next_d, NextDurationNumDay=DurationNumDay, NextDurationNumMin=DurationNumMin);
    next_d<-rbind(next_d, c(0,0));
    next_dist<-select(slice(df_prov, 2:nrow(df_prov)), Distance);
    next_dist<-rename(next_dist, NextDistance=Distance);
    next_dist<-rbind(next_dist, 0);
    next_tripdate<-select(slice(df_prov, 2:nrow(df_prov)), TripDate);
    next_tripdate<-rename(next_tripdate, NextTripDate=TripDate);
    next_tripdate<-rbind(next_tripdate, 0);
    df_prov<-cbind(df_prov, next_res_succ, previous_res_succ, next_tbd, next_d, next_dist, next_tripdate);
    df_prov<-mutate(df_prov, keepRow=if_else(PreviousResSucc==1&ResSucc==0, -1, 1),
TrueBookDuration=if_else(ResSucc==1&NextResSucc==0,
TrueBookDuration+NextTrueBookDuration+NextTripDate-DurationNumDay-TripDate, TrueBookDuration),
DurationNumDay=if_else(ResSucc==1&NextResSucc==0, NextDurationNumDay+NextTripDate-TripDate, DurationNumDay),
Distance=if_else(ResSucc==1&NextResSucc==0, Distance+NextDistance, Distance),
Merged=if_else(ResSucc==1&NextResSucc==0&Merged==0, 1, Merged), ResSucc=if_else(ResSucc==1&NextResSucc==0, 0,
ResSucc));
    df_prov<-subset(df_prov, keepRow==1)
    result<-c(result, nrow(df_prov), nrow(subset(df_prov, ResSucc==1)))
  }
  return(df_prov)
}
```

Figure 18: Function to merge the bookings

CarID	UserID	Dist	Dur	Succ ?
1	11	20	10	0
2	31	0	20	1
2	31	10	15	0
3	25	0	25	1
3	25	0	30	1
3	25	7	10	0

CarID	UserID	Dist	Dur	Succ ?
1	11	20	10	0
2	31	10	30	0
3	25	0	25	1
3	25	7	40	0

CarID	UserID	Dist	Dur	Succ ?
1	11	20	10	0
2	31	10	30	0
3	25	7	65	0

Figure 19: How the function of Figure 18 works

7.2 Proof of Proposition 3.1

Proof. Let us replace usual notations $\|\|_1$ and $\|\|_2$ by M for *Manhattan* and V for *Vol d'oiseau*. Let us prove that the mean M-distance between two points of a square with square side length 1 is equal to the mean V-distance between O and a point of $D(0, 1)$.

The first quantity is

$$\begin{aligned}
&\int_{[0,1]^2} dxdy \int_{[0,1]^2} dx'dy'(|x-x'|+|y-y'|) \\
&= 4\int_{0<x'<x<1, 0<y'<y<1} dxdy \int dx'dy'(x-x'+y-y') \\
&= 4\int_{[0,1]^2} dxdy \int_0^x dx' \int_0^y dy'(x-x'+y-y') \\
&= 4\int_{[0,1]^2} dxdy \int_0^x dx' \left[(x-x'+y)y' - \frac{y'^2}{2}\right]_0^y \\
&= 4\int_{[0,1]^2} dxdy \int_0^x dx' \left(x-x'+\frac{y}{2}\right) y = 4\int_{[0,1]^2} ydxdy \left[\left(x+\frac{y}{2}\right)x' - \frac{x'^2}{2}\right]_0^x \\
&= 4\int_{[0,1]^2} ydxdy \left(\frac{x}{2}+\frac{y}{2}\right) x = 4\int_{[0,1]^2} yx^2 dxdy = 4\left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 = \frac{2}{3}.
\end{aligned}$$

The second quantity is the ratio of

$$\iint_D \sqrt{x^2+y^2}dxdy = \int_0^{2\pi} d\theta \int_0^1 r^2 dr = \frac{2\pi}{3}$$

and

$$\iint_D dxdy = \int_0^{2\pi} d\theta \int_0^1 rdr = \pi$$

which is 2/3. It ends the proof. □

7.3 Proof of Proposition 4.3

Proof of the mean-field limit of the model with probability of booking and of cancelling. With the notations of Section 4.1, the transitions of the process Y^N are the following.

When a user arrives in a zone with k available cars and l available parking spaces and books a car:

$$y \to y + \tfrac{1}{N}(e_{k-1,l+1} - e_{k,l}) \text{ at rate } \alpha\lambda N y_{k,l} \mathbb{1}_{\{k>0\}}.$$

Here, the α term corresponds to the probability for a user to book a car when he arrives at a station.

When a user arrives in a zone with k available cars and l available parking spaces and takes a car directly without booking it:

$$y \to y + \tfrac{1}{N}(e_{k-1,l} - e_{k,l}) \text{ at rate } (1-\alpha)\lambda N y_{k,l} \mathbb{1}_{\{k>0\}}.$$

Here, the $1-\alpha$ term corresponds to the probability for a user to directly start a trip with a car, without booking it.

When a user picks up a car which he has previously booked, in a zone with k available cars and l available parking spaces:

$$y \to y + \tfrac{1}{N}(e_{k,l-1} - e_{k,l}) \text{ at rate } (1-\beta)\eta l N y_{k,l}.$$

Here, the $1-\beta$ term corresponds to the probability for a user not to cancel his trip (the said user is therefore able to pick up his car when the booking finishes).

When a user cancels his trip after having booked a car, in a zone with k available cars and l available parking spaces:

$$y \to y + \tfrac{1}{N}(e_{k+1,l-1} - e_{k,l}) \text{ at rate } \beta\eta l N y_{k,l}.$$

Here, the β term corresponds to the probability for a user to cancel his trip when the booking finishes.

When a user returns his car after a trip in a zone with k available cars and l booked cars:

$$y \to y + \tfrac{1}{N}(e_{k+1,l} - e_{k,l}) \text{ at rate } \mu(M - \textstyle\sum_{(i,j)\in\chi}(i+j)y_{i,j})y_{k,l}N\mathbb{1}_{k+l<K}$$

With the same arguments as before, we obtain the differential equation:

$$\begin{aligned}\dot{y}(t) = \sum_{(k,l)\in\chi} y_{k,l}[&(e_{k-1,l+1} - e_{k,l})\alpha\lambda\mathbb{1}_{\{k>0\}]} \\ &+ (e_{k-1,l} - e_{k,l})(1-\alpha)\lambda\mathbb{1}_{\{k>0\}} \\ &+ (e_{k,l-1} - e_{k,l})(1-\beta)\eta l \\ &+ (e_{k+1,l-1} - e_{k,l})\beta\eta l \\ &+ (e_{k+1,l} - e_{k,l})\mu(\frac{M}{N} - \sum_{(i,j)\in\chi}(i+j)y_{i,j})\mathbb{1}_{k+l<K}]\end{aligned}$$

From this, we can conclude that the systems has the asymptotic behavior of the queuing network presented before. □

7.4 Proof of Proposition 4.4

Proof. The idea of the proof is that, if external arrival rate $\mu(s - \sum_{k+l\leq K}(k+l)\pi_{k,l})$ is replaced by a fixed Λ, we have to check that the invariant measure, which is the invariant measure of the closed Jackson network, restricted to the state space, the set $(k,l), k+l \leq K$, holds for the system. It amounts to checking that the measure satisfies the balance equation

$$\sum_{n'\neq n} \frac{\pi(n')}{\pi(n)} Q(n', n) = \sum_{n'\neq n} Q(n, n'). \tag{17}$$

Then we have to prove that, if $\Lambda = \mu(s - \sum\limits_{k+l\leq K}(k+l)\pi_{k,l})$ then such a measure exists and is unique.

For the first step, to compute the balance equation, we need the transitions from states n' to a given state $n = (k,l)$. For transition $n' \to n$, jump rate $Q(n', n)$ and $\pi(n')/\pi(n)$ are the following.

If a user books a car,

$$\begin{aligned}&(k+1, l-1) \to (k,l) \\ &Q(n',n) = \alpha\lambda\mathbb{1}_{l>0} \\ &\frac{\pi(n')}{\pi(n)} = \frac{\rho_V l}{\rho_R} = \frac{\eta l}{\alpha\lambda}.\end{aligned}$$

If a user starts a trip without booking the car,

$$\begin{aligned}&(k+1, l) \to (k,l) \\ &Q(n',n) = (1-\alpha)\lambda\mathbb{1}_{k+l<K} \\ &\frac{\pi(n')}{\pi(n)} = \rho_V = \frac{\Lambda\lambda}{1-\alpha\beta}.\end{aligned}$$

If a user starts a trip after having booked the car,

$$\begin{aligned}&(k, l+1) \to (k,l) \\ &Q(n',n) = (1-\beta)\eta(l+1)\mathbb{1}_{k+l<K} \\ &\frac{\pi(n')}{\pi(n)} = \frac{\rho_R}{(l+1)} = \frac{\alpha\Lambda}{\eta(1-\alpha\beta)(l+1)}.\end{aligned}$$

If a user cancels a trip at the end of the booking,

$$\begin{aligned}
&(k-1,l) \to (k,l)\\
&Q(n',n) = \beta\eta(l+1)\\
&\frac{\pi(n')}{\pi(n)} = \frac{\rho_R}{\rho_V(l+1)} = \frac{\alpha\lambda}{\eta(l+1)}.
\end{aligned}$$

If a user parks his car (he only manages to do so if there are available parking spaces at the destination zone),

$$\begin{aligned}
&(k+1,l-1) \to (k,l)\\
&Q(n',n) = \Lambda \mathbb{1}_{k>0}\\
&\frac{\pi(n')}{\pi(n)} = \frac{1}{\rho_V} = \frac{\lambda(1-\alpha\beta)}{\Lambda}.
\end{aligned}$$

From this, checking balance equation (17) is straightforward.

For the second step, checking that π exists and is unique amounts to checking that equation in ρ_V

$$\rho_V = \frac{\mu}{\lambda}\left(s - \sum_{k+l\le K}(k+l)\pi_{k,l}\right)$$

where π is given as a function of ρ_V by equation (3) and $\rho_R = \rho_V\alpha\lambda/\eta$ has a unique solution. First the previous equation is rewritten

$$s = \frac{\lambda}{\mu}\rho_V + \sum_{k+l\le K}(k+l)\pi_{k,l}.$$

Then the right-hand side of this equation is straightforwardly a non-decreasing and continuous function of ρ_V, from 0 to $+\infty$ when ρ_V goes from 0 to $+\infty$. By intermediate value theorem, $\rho_V > 0$ solution of this fixed point equation is unique. It ends the proof. □

7.5 Proof of Proposition 4.5

Proof. Recall that, by definition,

$$P_b = \pi_{0,.} + \pi_S - \pi_{0,S}. \tag{18}$$

By denoting ρ_V as ρ, we have that ρ is solution of

$$s = \frac{\lambda}{\mu}\rho + \mathbb{E}(V+R). \tag{19}$$

The aim is to find ρ such that P_b is minimal. It can be checked that this ρ differs from 1.

We assume in the following that $1/\eta$ tends to 0. By definition of P_b and of π, $P_b = N/Z$ where

$$N = \frac{(\rho\lambda/\eta)^K}{K!} + \sum_{l=0}^{K-1}\frac{(\lambda/\eta)^l}{l!}(\rho^l + \rho^K) \quad \text{and } Z = \sum_{n=0}^{K}\rho^n\sum_{l=0}^{n}\frac{(\lambda/\eta)^l}{l!}.$$

Let K and λ be fixed and let us define $x = \lambda/\eta$. Thus P_b is a function of x and ρ. To find the minimum of P_b, let us introduce function φ

$$(x,\rho) \mapsto Z(x,\rho)\frac{\partial N}{\partial\rho}(x,\rho) - N(x,\rho)\frac{\partial Z}{\partial\rho}(x,\rho)$$

which is $Z^2\partial P_b/\partial\rho$. As previously recalled, without reservation i.e. $x = 0$, there is a unique minimum for P_b at $\rho = 1$ which yields $\varphi(0,1) = 0$. For x enough small, existence and uniqueness of an extremum for P_b is given by implicit function theorem for φ. Indeed, simple derivations give

$$\frac{\partial\varphi}{\partial\rho}(0,1) = \frac{K(K^2-1)}{3} \neq 0,$$

there exists a unique ψ defined and C^1 for x small enough such that $\varphi(x, \psi(x)) = 0$. Moreover,

$$\psi(x) = 1 + \psi'(0)x + O(x^2) \quad \text{with } \psi'(0) = -\frac{\frac{\partial \varphi}{\partial x}(0,1)}{\frac{\partial \varphi}{\partial \rho}(0,1)} = -\frac{3}{K(K^2-1)}$$

as $\partial\varphi/\partial x$ at $(0,1)$ is 1. Furthermore, as N and Z are C^∞, $\rho \mapsto \varphi(x,\rho)$ is C^1 and non-decreasing for $x = 0$ (see [4, Theorem 1] for details). It remains true for x small enough by continuity of $\partial\varphi/\partial\rho$. Thus the extremum of P_b is a minimum. In conclusion, P_b has a unique minimum for $1/\eta$ small enough, which corresponds to

$$\rho^* = 1 - \frac{3\lambda}{K(K^2-1)\eta} + O((\lambda/\eta)^2).$$

Plugging this in equations (18) and (19) give the expansion at order 1 of P_b^* and s^* in λ/η at 0

$$s^* = \frac{K}{2} + \frac{\lambda}{\mu} + \frac{K^2 - 3K/2 - 1}{2(K^2-1)}\frac{\lambda}{\eta} + O\left(\frac{1}{\eta^2}\right)$$

which, for large K, can be written

$$s^* \simeq \frac{K}{2} + \frac{\lambda}{\mu} + \frac{\lambda}{2\eta} + O\left(\frac{1}{\eta^2}\right).$$

□

7.6 Proof of Proposition 4.7

Proof. The invariant measure of the simple model is a product measure $\widetilde{\pi}$ with

$$\widetilde{\rho}_{i_1} = \frac{\widetilde{\theta}_{i_1}}{\widetilde{\lambda}_i}, \quad \widetilde{\rho}_{i_2} = \frac{\widetilde{\theta}_{i_2}}{\widetilde{\eta}_i} \quad \text{and} \quad \widetilde{\rho}_{[ij]} = \frac{\widetilde{\theta}_{[ij]}}{\widetilde{\mu}_{[ij]}}.$$

This system is the same as the previous one, with $\widetilde{\alpha}_i = 1$ and $\widetilde{\beta}_i = 0$ for all i. We obtain $\widetilde{\delta} = 3$ and then

$$\widetilde{\rho}_{i_1} = \frac{\nu_i}{3\widetilde{\lambda}_i}, \quad \widetilde{\rho}_{i_2} = \frac{\nu_i}{3\widetilde{\eta}_i} \quad \text{and} \quad \widetilde{\rho}_{[ij]} = \frac{\nu_i q_{ij}}{3\widetilde{\mu}_{[ij]}}.$$

Assume that this model is equivalent to the previous one. Thus

$$\widetilde{\rho}_{i_1} = \rho_{i_1}, \quad \widetilde{\rho}_{i_2} = \rho_{i_2} \quad \text{and} \quad \widetilde{\rho}_{[ij]} = \rho_{[ij]}$$

which gives

$$\begin{aligned}
\widetilde{\lambda}_i &= \frac{(1-\alpha_i\beta_i)}{3\delta}\lambda_i, \\
\widetilde{\eta}_i &= \frac{(1-\alpha_i\beta_i)}{3\delta(\alpha_i,\beta_i)}\frac{\alpha_i\eta_i}{\delta}, \\
\widetilde{\mu}_{[ij]} &= \frac{\mu_{ij}}{3\delta} = \frac{(1-\alpha_i\beta_i)}{3\delta}\frac{\mu_{ij}}{1-\alpha_i\beta_i}.
\end{aligned}$$

Let us denote $\delta' = (1-\alpha_i\beta_i)/(3\delta(\alpha_i,\beta_i))$. It holds that

$$\begin{aligned}
\widetilde{\lambda}_i &= \delta'\lambda_i, \\
\widetilde{\eta}_i &= \delta'\alpha_i\eta_i, \\
\widetilde{\mu}_{[ij]} &= \delta'\frac{\mu_{ij}}{1-\alpha_i\beta_i}.
\end{aligned}$$

This shows that by taking the following parameters, the invariant measures will be the same

$$\widetilde{\lambda}_i = \lambda_i, \quad \widetilde{\eta}_i = \alpha_i\eta_i \quad \text{and} \quad \widetilde{\mu}_{[ij]} = \frac{\mu_{ij}}{1-\alpha_i\beta_i}.$$

Indeed, the fact that $\delta' \neq \delta$ will compensate when computing the normalization constant. □

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