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# Mathematical Formulations for Multi-Period Network Design with Modular Capacity Adjustments

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**Abstract.** Network design refers to a family of combinatorial optimization problems that are concerned with selecting a subset of typically capacitated arcs such that commodities can be routed from their origin to their destination nodes at minimal costs. Such planning problems are at the heart of several applications in domains such as transportation, telecommunication, energy and natural resources. While single-period models tailor the network to a single set of origin-destination demands, multi-period models aim at preparing the network for varying demand along time. In several applications, this implies adjusting arc capacities along time to better respond to demand changes. Some works therefore proposed models that allow for the expansion of arc capacities along the planning horizon. However, the reduction of existing capacity has mostly been ignored, even though it may be a viable and cost-beneficial option in several application contexts. This paper proposes a new multi-period network design problem variant, in which modular capacities can be added or reduced along the planning horizon in order to adapt to demand changes. The problem further allows to represent economies of scales in function of the total arc capacity, a detail that has typically been overlooked in the literature. This paper particularly emphasizes the different alternatives to formulate the problem. We propose three different mixed-integer programming formulations and analyze further modeling alternatives. We theoretically compare the strength of all formulations and evaluate their computational performance in extensive experiments. The results suggest that a recent modeling technique using more precise decision variables yields the strongest formulation, which also results in significantly faster solution times.

**Keywords**: Dynamic network design, modular capacity adjustment, mixed-integer programming

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# 1 Introduction

Network Design (ND) problems generally aim at selecting a subset of arcs such that a set of pre-defined commodities can be routed from their origin to their destination nodes. Typically, the objective is to minimize the total costs resulting from the selection of arcs and from commodity routing. Network Design problems have been found to be the underlying modeling foundation for many applications, such as the design of road networks (see, e.g. Yang and Bell, 1998), railway networks (see, e.g. Hooghiemstra et al., 1999), telecommunication networks (see, e.g. Kubat et al., 2001), energy networks and natural resource distribution networks (see, e.g. Borraz-Sánchez et al., 2016). As a consequence, they are among the most studied combinatorial optimization problems. Most of the literature in this domain assumes that the network has to be designed once, while routing can be performed either once or repeatedly on the same network, optimized for the demand of a specific moment in time. The corresponding optimization models therefore contain a single time period, referring to the moment when the network is established and the flow satisfies demands.

In most circumstances, however, demand fluctuates over time. These requirements gave rise to Multi-Period Network Design (MPND) problems, in which a set of different demands may be defined for each time period. In particular, the initially designed network does not necessarily have to remain the same throughout the entire planning horizon, but may be modified to better adapt to changing demand. For example, a highway transportation network may be gradually expanded over time when initial budget is limited, gas distribution networks may be gradually expanded to better adapt to new city demographics, and telecommunication networks may be repeatedly modified to respond to constantly changing user demands, server locations and evolving hardware.

In most of such applications, the maximum throughput of commodity flow is limited by the arc capacities. While in some applications those arc capacities may be fixed and difficult to alter, several applications (particularly in telecommunications and energy) allow to modify arc capacities: expand capacity when demand tends to increase and reduce capacity when it tends to be idle in the future (e.g., in order to eliminate maintenance costs of rarely used equipment). Several models in the literature consider the expansion of arc capacities along time. However, the reduction of arc capacities has not received much attention.

This paper considers a problem variant of multi-period multi-commodity network design which allows to choose arc capacities from a discrete set of predefined options and to modify those capacities throughout the planning horizon in order to adapt to demand changes. In contrast to continuous capacity sizes, such predefined capacity sizes are a natural choice, given that, in practice, arcs are typically not available in any arbitrary capacity size, but only in the most common standard sizes (e.g., number of lanes on a high-way, diameter of pipes in gas distribution networks, data throughput in telecommunication networks). We will refer to those capacity options as modular capacity levels. In practice, decision-makers may then have the possibility of adjusting these capacities along time to better react to demand that changes over time and location.

We propose three different mixed-integer programming (MIP) formulations for this problem variant. In particular, two models are based on decision variables that consider the current capacity level at each time period, which we will refer to as the Single Capacity-Index (SCI) formulations. The third model is based on a more recent modeling technique that has been proposed by Jena et al. (2015) in the context of facility location, where decision variables explicitly model capacity level changes between two subsequent time periods; this formulation is also referred to as the Generalized Modular Capacity (GMC) formulation, and allows for modeling a large variety of problem variants with capacity adjustments. Extensive computational experiments on instances generated based on benchmarks from the literature indicate that the GMC formulation for the here proposed Multi-period Network Design Problem provides significantly lower integrality gaps than the two formulations based on traditional techniques. Even though modern general-purpose MIP solvers are nowadays capable of dynamically adding cuts to strengthen formulations with higher integrality gaps, for most of the instances, the advantage stemming from the strength of the GMC formulation is significantly preserved throughout the optimization process, enabling the solver to prove optimality in significantly shorter computing times than for the SCI formulations.

Our contributions can therefore be summarized as follows. First, we introduce a new MPND problem variant, in which arc capacity can be expanded or reduced in modular quantities along time. To the best of our knowledge, this is the first MPND problem variant that explicitly considers the reduction/contraction of installed capacity. Furthermore, our problem allows for the representation of economies of scale in function of the total capacity installed, which has also received less attention in the literature. Second, we propose three mathematical formulations for this problem and theoretically compare their strength of the linear programming (LP) relaxation bounds. We also consider the use of different commodity-flow variables, and prove that the resulting formulations provide weaker LP relaxation bounds. Third, we empirically compare the formulations in extensive computational experiments.

# 2 Related Literature

Network design generally refers to the study of how to structure a network of arcs (or edges) such that demand can be routed from their origin to their destination nodes. These planning problems are suitable for a variety of applications, such as the design and configuration of telecommunication networks (Kubat et al., 2001), road and railway transportation networks (Hooghiemstra et al., 1999), and gas distribution networks (Borraz-Sánchez et al., 2016). While most of the planning problems focus on the decisions linked to the actual network, in certain applications, the interaction of the designed network with other operational decisions is complex. Researchers have therefore proposed more complex problem variants, such as service network design for freight transportation (see, e.g., Crainic, 2000), which may take into consideration further aspects such as freight consolidation, frequency of shipments or handling decisions at the terminal nodes.

Most of the literature focuses on the case with a single time period (see, e.g., Magnanti and Wong, 1984; Minoux, 1989; Crainic, 2000, and the references therein), i.e., the planning assumes that the network is optimized once for a set of specific node demands, and afterwards used without further modification. Given that, in most of the applications, the maximum throughput on the arcs is limited by a given capacity, a vast majority of the literature imposes arc capacities. Classical variants, such as the Capacitated Fixed-charge (multi-commodity) Network Design problem (see, e.g., Crainic et al., 2020), are among the most studied combinatorial optimization problems. Such problems are typically modeled as MIPs, known to be NP-hard and therefore difficult to solve.

When compared to single-period problem variants with fixed arc capacities, significantly less work has been done for network design problems that either consider different demands over multiple time periods or those that require to select the arc capacities from a discrete set of possible capacity expansions (which enables the representation of economies of scale). Such features add considerable complexity to the mathematical models. To the best of our knowledge, our work is the first to tackle both features at the same time. In the following, we will therefore separately review works related to either of these two features. Also note that, in order to remain within the scope of our paper, we will mostly restrict our review to (linear and non-linear) MIP models.

## 2.1 Multi-period capacitated network design problems

Single-period ND problems have received major attention in the literature, including those where capacity modules of different sizes can be installed on the arcs (see, e.g., Melián et al., 2004), allowing for the representation of economies of scale. In contrast, research on multi-period ND problems is rather recent, which may stem from the fact that those problems tend to be harder to solve than other network-based problems, such as facility location problems. In contrast to facility location problems, where capacity decisions typically concern the production level of the facilities, network design problems may consider capacities either at the nodes or at the arcs. The former may represent service stations in a transportation network (see, e.g., Kubat et al., 2001; Marín and Jaramillo, 2008). The latter may refer to traffic on roads, water or oil throughput in pipelines or data throughput in telecommunication networks. We will here focus on arc capacities, and will classify the related works into multi-period network design models that only select the arcs (with predefined fixed capacities), and those that additionally select the capacity level for each arc.

**Multi-period models with predefined arc capacities.** A work closely related to our problem is the one of Fragkos et al. (2021), who extend the classical fixed-charge multi-commodity network design problem to multiple time periods. Demands are independently defined for different commodities and may change over the different time periods. Arcs (with predefined fixed capacities) may be selected at any time period and once installed, they will be available until the end of the planning horizon. The authors propose a Benders decomposition method to solve larger instances. Our problem additionally allows to choose the arc capacities from a set of capacity levels and to expand or reduce those along the planning horizon.

A more specific model has been proposed by Papadimitriou and Fortz (2014), in which fixed-capacity arcs may be installed to route flow to meet the demand that independently occurs at different time periods. Tailored to an application in telecommunications, once established flow routes cannot be changed in later time periods in order to avoid changes in the so-called routing table. In order to avoid maintenance costs of unused arcs, the usage of such arcs can be ceased. However, the presented model does not allow to impose costs for such capacity closure, which is one of the key-features in our problem.

**Multi-period models with different arc capacity levels.** In contrast to fixed arc-capacities, several works consider the capacity levels of the installed arcs to be part of the decisions. We now review such works and pay particular attention to the ability of representing economies of scale in function of the total arc capacity installed. Lai and Shih (2013) consider a stochastic railway network problem with uncertain demand. Arc capacity can be installed only once on each arc throughout the planning horizon and will be available in subsequent time periods. Economies of scale in the total arc capacity on each arc are therefore easily accountable for. Balakrishnan and Magnanti (1995) present a model for telecommunication, allowing for modular capacity expansion. Their model allows for the representation of economies of scale in function of the total arc capacity. However, the model is tailored to tree structures instead of general networks.

Several other works allow for the installation of multiple capacity modules on the same arc over time. While this may enable the model to represent economies of scale in function of the capacity added at each time period, it does not automatically allow for representing economies of scale in function of the total capacity on that arc accumulated over time. Works in this category include the one of Lardeux et al. (2007), who focus on a MPND problem with incremental routing requirements, in which several capacity modules can be installed on arcs over time. Ukkusuri et al. (2009) present a taxonomy for network flexibility in transportation networks and propose a general model for the flexible network design problem, also expanding total arc capacity by adding modules over time. Barmann et al. (2017) handle a multi-period rail-network planning problem with similar features. Yet another stream of models assume that capacity quantities that may be added are continuous (see, e.g., Ordóñez and Jiamin, 2007; Chang and Gavish, 1993, 1995). Naturally, those models do not capture any economies of scale in function of the chosen capacity.

While the works above are closely related to the problem considered in this paper, we conclude that economies of scale in function of the total arc capacity is rarely considered, and none of them considered capacity reduction or arc closure. However, some of the models above constitute excellent starting points to model the MPND problem here considered.

Finally, while all works discussed above assume that the discretization of time is predefined in the planning problem, it is worth noting that this may limit a realistic representation of the planning in certain applications. In this context, Boland et al. (2017) propose to model a service network design problem in a continuous planning horizon, and show how to solve those models using an iterative algorithm. Their model, however, assumes that the flow of the same commodity has to take the same path, i.e., it cannot be split among different paths from its origin to its destination node, which is different from the MPND.

## 2.2 Capacity decisions in other multi-period network problems

The literature on selecting and modifying capacities in planning problems defined on graph networks is mainly rooted in the planning of production facilities (see, e.g., Luss, 1982). The literature on capacitated facility location problems is particularly rich. Alamur et al. (2012) introduce a generalized model for the multi-period reverse logistics network design problem, allowing for modular production capacities at the facilities. Similarly, Alamur et al. (2016) consider a multi-period hub location problem in which modular capacities can be added on the hubs throughout the time horizon.

Jena et al. (2015) introduce a multi-period facility location problem, in which modular capacities can be expanded or reduced along time. They propose a modeling technique using decision variables that explicitly capture the capacity changes between two subsequent time periods. The authors show how such models provide significantly lower integrality gaps when compared to classical models. The formulation is particularly appealing to represent complex cost structures in terms of economies of scale, allowing not only to represent those economies in function of the capacity expansion or reduction, but also depending on the current capacity level. Jena et al. (2016) then use this technique to solve an application in the forestry sector, where facilities may expand or reduce capacities along time, or even be temporarily closed while not in use. Again, the modeling technique used has the advantage of yielding tighter formulations, but also allowing to represent economies of scale on several layers of the cost structure. Due to these advantages, this modeling technique will serve as a building block for one of the formulations used to model the problem here considered.

**Unsplittable flows.** We conclude this section by noting that most of the works in network design optimization has assumed that the commodity flow can be split over several arcs, i.e., the commodity flow may take different paths from their origin to their destination nodes. In practice, this may not always be desirable. For example, data packages in telecommunication networks may have to be routed on the same path (see, e.g., Papadimitriou and Fortz, 2014) or transportation fleets may have to travel together. In such cases, it has to be ensured that commodity flow is not split, which, as this requires additional constraints, cannot result in a cheaper planning solution. One possibility to implement such constraints is to change the commodity flow variables from continuous to binary variables. The resulting models are acknowledged to be harder to solve (see, e.g., Atamtürk et al., 2002; Benhamiche et al., 2016; Yaghini and Kazemzadeh, 2012). While we will focus, in this paper, on the problem variant that allows for splitting commodity flows, we will still explore the ability of the proposed formulations to handle the problem variant with unsplittable flows.

# 3 Problem Definition and Mathematical Formulations

This section defines the planning problem and proposes three MIP formulations. Section 3.1 formally defines the problem and introduces the related notation. Sections 3.2 and 3.3 introduce the two formulations based on decision variables with a single index related to the capacity levels. Section 3.4 introduces the third formulation, based on decision variables using two indices related to the capacity levels. Finally, we discuss the use of less precise flow variables in Section 3.5. Throughout these sections, we also elaborate on the theoretical strength of the respective formulations.

## 3.1 Problem definition and input parameters

We consider a network of $N$ nodes defined by set $\mathcal{N} = \{1, \ldots, N\}$, as well as a set $\mathcal{A}$ of directed arcs $(i, j) \in \mathcal{A}$, which may be constructed in order to connect node $i \in \mathcal{N}$ to node $j \in \mathcal{N}$. As in classical multi-commodity network design, we require to route $P$ commodities defined on set $\mathcal{K} = \{1, \ldots, P\}$, where each commodity $k \in \mathcal{K}$ has a predefined origin $o(k) \in \mathcal{N}$ and destination $d(k) \in \mathcal{N}$. The planning horizon spans over $T$ time periods, given by set $\mathcal{T} = \{1, \ldots, T\}$. The demand of each commodity $k \in \mathcal{K}$ that has to be routed from $o(k)$ to $d(k)$ at time period $t \in \mathcal{T}$ is given by $d^{kt}$.

All models presented in this paper formulate the commodity routing as network flows, as it is typical for network design problems (see, e.g., Chouman et al., 2017). In order to account for flow supply at origin nodes $o(k)$ and flow demand at destination nodes $d(k)$, we define the residual $b_i^k$ at each node $i \in \mathcal{N}$ as:

$$b_i^k = \begin{cases} 1 & i = o(k), \\ -1 & i = d(k), \\ 0 & \text{otherwise.} \end{cases}$$

In order to route the commodities from their origin to their destination nodes at each of the time periods, directed arcs $(i, j) \in \mathcal{A}$ may be constructed at one of the $q$ capacity levels $\ell$ predefined in set $\mathcal{L} = \{0, 1, \ldots, q\}$, making available an arc capacity that allows to route up to $U_{ij}^{\ell}$ commodity units on arc $(i, j)$ per time period. A capacity level $\ell = 0$ with $U_{ij}^0 = 0$ indicates that arc $(i, j)$ is not constructed at all, while a capacity level $\ell \geq 1$ indicates that the arc is available with a total capacity of $U_{ij}^{\ell} > 0$. The costs of routing one unit of commodity $k \in \mathcal{K}$ through arc $(i, j) \in \mathcal{A}$, open at capacity level $\ell \in \mathcal{L}$, is given by $C_{ij}^{k\ell}$.

Once an arc has been constructed, maintenance cost occurs at each time period the arc is available at a capacity level greater than 0. Specifically, the costs of maintaining an arc $(i, j) \in \mathcal{A}$ open at capacity level $\ell$ throughout one time period is given by $F_{ij}^{\ell}$. Throughout the planning horizon, the capacity level at each arc may be adjusted by either increasing or by reducing the current capacity level. Increasing the capacity at an arc $(i,j)$ by $\ell$ capacity levels implies a cost of $\overline{f}_{ij}^{\ell}$, while reducing its capacity by $\ell$ capacity levels costs $\underline{f}_{ij}^{\ell}$. The construction of an arc at capacity level $\ell$ is hence represented by an increase of capacity from level 0 to level $\ell$, whereas the complete shut-down of an arc formerly at capacity level $\ell$ is represented by a capacity decrease from level $\ell$ to level 0.

Note that, for the sake of clarity and without loss of generality, we assume that the network will be designed from scratch. If one requires to optimize on an existing network, the existing arcs with their respective capacity levels can be easily integrated in the models.

We will refer this problem to Multi-period Network Design with Modular Capacity Adjustments (MPND-CA). As discussed in Section 2, this optimization problem has not yet been addressed in the literature. Its additional capacity adjustment decisions allow for modeling in more detail the flexibility available in several applications. However, this comes at the price of more complex optimization models, which are generally hard to solve. The choice of the mathematical formulation to represent the problem may have a strong impact on the difficulty of solving the problem. In the following, we propose three MIP formulations for this problem.

## 3.2 Single Capacity-Index (SCI) Formulation

The first formulation here proposed is an extension of the formulation proposed by Fragkos et al. (2021) for the multi-period capacitated network design problem. This problem allows to select the time period at which an arc is constructed. Arcs have predefined capacities, and, once constructed, they are assumed to remain open until the end of the planning horizon.

Our problem is more flexible, providing different capacity levels to choose from, and allowing for increasing or decreasing capacity along time. We use three sets of binary variables. Binary variables $y_{ij}^{\ell t}$ take value 1 if arc $(i,j) \in \mathcal{A}$ is open (i.e., available to route commodities) at capacity level $\ell \in \mathcal{L}$ during time period $t \in \mathcal{T}$, and 0 otherwise. Binary variables $s_{ij}^{\ell t}$ take value 1, if the capacity at arc $(i,j) \in \mathcal{A}$ is expanded by $\ell$ capacity levels at the beginning of time period $t \in \mathcal{T}$, and 0 otherwise. Binary variables $w_{ij}^{\ell t}$ take value 1, if the capacity at arc $(i,j) \in \mathcal{A}$ is reduced by $\ell$ capacity levels at the beginning of time period $t \in \mathcal{T}$, and 0 otherwise. Note that such capacity level adjustments are only possible, if the resulting capacity level is defined in $\mathcal{L}$.

Finally, continuous flow variables $x_{ij}^{k\ell t} \in [0,1]$ define the fraction of demand of commodity $k$ that is directed through arc $(i,j)$ at capacity level $\ell$ and at time period $t$. If the routing costs does not depend on the capacity level, one may use less precise flow variables $x_{ij}^{kt}$. However, we here choose to not use these less precise variables, as their use comes at the expense of a weaker model, which is discussed in Section 3.5.

Given that the here used $y_{ij}^{\ell t}$ variables have a single capacity-index $\ell$, we refer this formulation to as the single capacity-index formulation (SCI), given by:

$$\text{(SCI)} \quad \min \sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} C_{ij}^{k\ell} d^{kt} x_{ij}^{k\ell t} + \sum_{(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} F_{ij}^{\ell} y_{ij}^{\ell t} +$$

$$\sum_{(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} \overline{f}_{ij}^{\ell} s_{ij}^{\ell t} + \sum_{(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} \underline{f}_{ij}^{\ell} w_{ij}^{\ell t} \tag{1}$$

$$s.t. \sum_{j:(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} x_{ij}^{k\ell t} - \sum_{j:(j,i)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} x_{ji}^{k\ell t} = b_i^k \ \ \forall i \in \mathcal{N}, \ \ \forall k \in \mathcal{K}, \ \ \forall t \in \mathcal{T} \tag{2}$$

$$\sum_{k\in\mathcal{K}} d^{kt} x_{ij}^{k\ell t} \leq U_{ij}^{\ell} y_{ij}^{\ell t} \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall \ell \in \mathcal{L}, \ \ \forall t \in \mathcal{T} \tag{3}$$

$$\sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} = \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell(t-1)} + \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell t} - \sum_{\ell\in\mathcal{L}} \ell w_{ij}^{\ell t} \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall t \in \mathcal{T}^{\setminus\{1\}} \tag{4}$$

$$\sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} = \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell t} \ \ \forall (i,j) \in \mathcal{A}, \ \ t = 1 \tag{5}$$

$$\sum_{\ell\in\mathcal{L}} y_{ij}^{\ell t} = 1 \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall t \in \mathcal{T} \tag{6}$$

$$\sum_{\ell\in\mathcal{L}} s_{ij}^{\ell t} = 1 \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall t \in \mathcal{T} \tag{7}$$

$$\sum_{\ell\in\mathcal{L}} w_{ij}^{\ell t} = 1 \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall t \in \mathcal{T} \tag{8}$$

$$0 \leq x_{ij}^{k\ell t} \leq 1 \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall k \in \mathcal{K}, \ \ \forall \ell \in \mathcal{L}, \ \ \forall t \in \mathcal{T}$$

$$y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t} \in \{0,1\} \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall \ell \in \mathcal{L}, \ \ \forall t \in \mathcal{T}.$$

The objective function (1) minimizes the total costs that account for commodity routing, maintaining open arcs, as well as capacity expansions and reductions. Constraints (2) are the commodity flow conservation constraints that assure that commodities are routed from their origins to their destinations. Constraints (3) are capacity constraints, ensuring that the routed commodity flow does not exceed the capacity available at the arc. Constraints (4) and (5) are the (capacity) flow conservation constraints that describe the capacity available at each arc and time period. In particular, constraints (5) ensure that the initial capacity flow at the first time period must be added by means of a capacity expansion, while constraints (4) for the subsequent time periods ensure that the available capacity level takes into consideration the capacity expansions and reductions performed at the beginning of the corresponding time period. Constraints (6), (7) and (8) make sure that, at each arc, exactly one capacity level is selected, and that not more than one capacity expansion and reduction is performed at each time period. Note, again, that 0 is part of $\mathcal{L}$, the set of possible capacity levels.

In order to strengthen the formulation above, we may also adapt the well-known *strong linking constraints* (see, e.g., Chouman et al., 2017) as follows:

$$x_{ij}^{k\ell t} \leq y_{ij}^{\ell t} \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall k \in \mathcal{K}, \ \ \forall \ell \in \mathcal{L}, \ \ \forall t \in \mathcal{T}. \tag{9}$$

These inequalities are redundant to the MIP formulation (1)-(8), but are generally acknowledged to significantly strengthen network-flow based formulations, such as those for facility location and network design problems.

## 3.3 Second Single Capacity-Index (SCI2) Formulation

Constraints (4) in the formulation above define the capacity level of an arc at a given time period in function of the the capacity level of the previous time period and the current capacity expansion or reduction. Alternatively, the new capacity level at the current time period can also be computed as the sum of all expansions and reductions conducted since the beginning of the planning horizon. The corresponding constraints, replacing constraints (4), can be written as:

$$\sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} = \sum_{\ell\in\mathcal{L}}\sum_{t'=1}^{t'=t} \ell s_{ij}^{\ell t'} - \sum_{\ell\in\mathcal{L}}\sum_{t'=1}^{t'=t} \ell w_{ij}^{\ell t'} \quad \forall (i,j)\in\mathcal{A},\ \ \forall t\in\mathcal{T}^{\setminus\{1\}} \tag{10}$$

Given the potential theoretical and practical implications of using constraints (10) instead of constraints (4), we treat the resulting formulation as a new model. Therefore, we define the second single capacity-index (SCI2) formulation as (1)-(3), (5)-(8) and (10), and may also include the strong linking constraints (9). Even though the SCI and SCI2 formulations use different constraint sets to describe the capacity flow conservation, it can be shown that both formulations provide the same LP relaxation bound as stated in the following theorem.

**Theorem 1 (Equivalence relationship)** *Let $\overline{SCI}$ be the LP relaxation of formulation $SCI$, and $\overline{SCI2}$ be the LP relaxation of formulation $SCI2$. Let $v(\overline{SCI})$ and $v(\overline{SCI2})$ be the optimal values of the LP relaxations of SCI and SCI2, respectively. It holds that $v(\overline{SCI}) = v(\overline{SCI2})$, i.e., both formulations provide the same LP relaxation bounds.*

Proof. See Appendix A.

## 3.4 Generalized Modular Capacities (GMC) Formulation

We now present a more general formulation, inspired by a modeling technique proposed by Jena et al. (2015) for multi-period facility location problems. In contrast to the SCI formulation, which uses one $y$ variable for each capacity level $\ell$, this technique uses more complex binary variables that explicitly model the change of capacity level between two subsequent time periods. Specifically, we use binary variables $y_{ij}^{\ell_1\ell_2 t}$ that take value 1, if arc $(i,j)\in\mathcal{A}$ changes its capacity level from level $\ell_1\in\mathcal{L}$ to $\ell_2\in\mathcal{L}$ at the beginning of time period $t\in\mathcal{T}$. We further use the continuous commodity routing variables $x_{ij}^{k\ell t}$ as previously defined for the SCI formulation.

For each of the $y_{ij}^{\ell_1\ell_2 t}$ variables, the corresponding costs $f_{ij}^{\ell_1\ell_2 t}$ to be considered in the objective function are composed of the costs to expand capacity, reduce capacity and to maintain the arc at the current capacity level $\ell_2$. These costs are computed as:

$$f_{ij}^{\ell_1\ell_2 t} = \begin{cases} F_{ij}^{\ell_2} + \overline{f}_{ij}^{(\ell_2-\ell_1)} & \text{if } \ell_2-\ell_1>0, \text{ i.e., the capacity is expanded,}\\ F_{ij}^{\ell_2} + \underline{f}_{ij}^{(\ell_1-\ell_2)} & \text{if } \ell_2-\ell_1<0, \text{ i.e., the capacity is reduced,}\\ F_{ij}^{\ell_2} & \text{if } \ell_2=\ell_1, \text{ i.e., the capacity remains the same.}\end{cases}$$

The MIP formulation, referred to as the Generalized Modular Capacity (GMC) formulation, is given by:

$$\text{(GMC)} \quad \min \sum_{(i,j)\in\mathcal{A}}\sum_{k\in\mathcal{K}}\sum_{\ell\in\mathcal{L}}\sum_{t\in\mathcal{T}} C_{ij}^{k\ell} d^{kt} x_{ij}^{k\ell t} + \sum_{(i,j)\in\mathcal{A}}\sum_{\ell_1\in\mathcal{L}}\sum_{\ell_2\in\mathcal{L}}\sum_{t\in\mathcal{T}} f_{ij}^{\ell_1\ell_2 t} y_{ij}^{\ell_1\ell_2 t} \tag{11}$$

$$s.t. \quad \sum_{j:(i,j)\in\mathcal{A}}\sum_{\ell\in\mathcal{L}} x_{ij}^{k\ell t} - \sum_{j:(j,i)\in\mathcal{A}}\sum_{\ell\in\mathcal{L}} x_{ji}^{k\ell t} = b_i^k \quad \forall i\in\mathcal{N},\ \ \forall k\in\mathcal{K},\ \ \forall t\in\mathcal{T} \tag{12}$$

$$\sum_{k\in\mathcal{K}} d^{kt} x_{ij}^{k\ell t} \le \sum_{\ell_1\in\mathcal{L}} U_{ij}^{\ell} y_{ij}^{\ell_1\ell t} \quad \forall (i,j)\in\mathcal{A},\ \ \forall \ell\in\mathcal{L},\ \ \forall t\in\mathcal{T} \tag{13}$$

$$\sum_{\ell_1\in\mathcal{L}} y_{ij}^{\ell_1\ell(t-1)} = \sum_{\ell_2\in\mathcal{L}} y_{ij}^{l\ell_2 t} \quad \forall (i,j)\in\mathcal{A},\ \ \forall \ell\in\mathcal{L},\ \ \forall t\in\mathcal{T}^{\setminus\{1\}} \tag{14}$$

$$\sum_{\ell_2\in\mathcal{L}} y_{ij}^{0\ell_2 t} = 1, \quad \forall (i,j)\in\mathcal{A},\ \ t=1 \tag{15}$$

$$\sum_{\ell_2\in\mathcal{L}}\sum_{\ell_1\in\mathcal{L}} y_{ij}^{\ell_1\ell_2 t} = 1 \quad \forall (i,j)\in\mathcal{A},\ \ t=1 \tag{16}$$

$$0 \le x_{ij}^{k\ell t} \le 1 \quad \forall (i,j)\in\mathcal{A},\ \ \forall k\in\mathcal{K},\ \ \forall \ell\in\mathcal{L},\ \ \forall t\in\mathcal{T}$$

$$y_{ij}^{\ell_1\ell_2 t} \in \{0,1\} \quad \forall (i,j)\in\mathcal{A},\ \ \forall \ell_1\in\mathcal{L},\ \ \forall \ell_2\in\mathcal{L},\ \ \forall t\in\mathcal{T}.$$

The objective function (11) minimizes the total costs that account for commodity routing and maintaining open arcs, as well as capacity expansions and reductions. Constraints (12) are the commodity flow conservation constraints. Constraints (13) are the capacity constraints. Constraints (14) are the capacity flow conservation constraints. Finally, constraints (15) impose that all facilities are at capacity level 0 at the beginning of the planning horizon, while constraints (16) ensure that only one capacity level is selected at the beginning of the planning.

We further adapt the previous strong linking constraints (9) to the GMC formulation as follows:

$$x_{ij}^{k\ell t} \leq \sum_{\ell_1 \in \mathcal{L}} y_{ij}^{\ell_1 \ell t} \;\; \forall (i,j) \in \mathcal{A}, \;\; \forall k \in \mathcal{K}, \;\; \forall \ell \in \mathcal{L}, \;\; \forall t \in \mathcal{T}. \tag{17}$$

The GMC formulation is more general than the SCI formulations. In particular, the GMC formulation naturally allows to model different problems than here considered, for example, a problem variant in which arcs may be constructed and then temporarily closed and reopened (for details, see Jena et al., 2015). In addition, the GMC formulation also allows to explicitly represent more complicated cost structures for the capacity changes, e.g., it would allow to represent the fact that expanding the capacity from level 1 to level 3 does not imply the same costs as expanding from level 2 to level 4, even though, in both cases, the capacity has been expanded by a total of 2 levels. Such a cost structure cannot be represented by the SCI formulation. Interestingly, even though the GMC formulation is more general than the SCI formulation, it provides, in fact, a stronger LP relaxation bound, which is stated in the following theorem.

**Theorem 2 (Dominance relationship)** *Let $\overline{SCI}$ be the LP relaxation of formulation $SCI$, and $\overline{GMC}$ be the LP relaxation of formulation $GMC$. Let $v(\overline{SCI})$ and $v(\overline{GMC})$ be the optimal values of the LP relaxations of SCI and GMC, respectively. It holds that $v(\overline{GMC}) \geq v(\overline{SCI})$, i.e., the GMC formulation is stronger (strictly stronger for some instances) than the SCI formulation.*

Proof. See Appendix B.

The strength of the GMC formulation comes from constraints (14), using one constraint for each capacity level. This is different in the corresponding capacity flow constraints (4) in the SCI formulation, which do not allow for such a separation by capacity level.

## 3.5 A note on alternative formulations

In our problem definition, the operational costs $C_{ij}^{k\ell}$ to route the commodities through the network may be dependant on the current capacity level $\ell$. This requires commodity flow variables $x_{ij}^{k\ell t}$ that explicitly depend on $\ell$.

In many practical applications, however, the operational costs may remain the same, no matter the underlying arc capacity. In such cases, one may use flow variables $x_{ij}^{kt}$, without an index $\ell$, and therefore less decision variables. Such alternative formulations can be found in Appendix C. Nevertheless, it may still be beneficial to use the more detailed variables $x_{ij}^{k\ell t}$ given that they allow us to use (a) one capacity constraint per capacity level as presented in constraints (3) and (13), and (b) one strong linking constraint per capacity level as presented in constraints (9) and (17). The combination of these two sets of more precise constraints make the GMC, SCI and SCI2 formulations tighter than their respective alternative formulations with variables $x_{ij}^{kt}$ (here called GMC-A, SCI-A and SCI2-A, presented in Appendix C). The following theorem formalizes their relationship in terms of their LP relaxation strengths.

**Theorem 3 (Dominance relationship over alternative formulations)** *Let $\overline{FML}$ be the LP relaxation of formulation FML. Let also $v(\overline{FML})$ be the optimal value of the LP relaxation of formulation FML. The following dominance relationships between the main and the alternative formulations hold:*

*(A) $v(\overline{SCI}) \geq v(\overline{SCI\text{-}A})$, i.e., the SCI formulation with its respective strong linking constraints is stronger (strictly stronger for some instances) than the SCI-A formulation with its respective strong linking constraints;*

*(B)* $v(\overline{SCI2}) \geq v(\overline{SCI2\text{-}A})$*, i.e., the SCI2 formulation with its respective strong linking constraints is stronger (strictly stronger for some instances) than the SCI2-A formulation with its respective strong linking constraints;*

*(C)* $v(\overline{GMC}) \geq v(\overline{GMC\text{-}A})$*, i.e., the GMC formulation with its respective strong linking constraints is stronger (strictly stronger for some instances) than the GMC-A formulation with its respective strong linking constraints.*

Proof. See Appendix D.

Even though the GMC, SCI and SCI2 formulations are stronger than their alternative formulations, the presence of index $\ell$ in the flow variables $x_{ij}^{k\ell t}$ increases considerably the number of decision variables of the problem, particularly for larger values of capacity levels. Empirically, however, experiments showed that the main formulations provide a considerably smaller integrality gap than the alternative formulations. In particular, the GMC formulation provided an integrality gap about eight times smaller than the GMC-A formulation, on average, which, overall, resulted in much faster solution times.

**A note on unsplittable commodity flows.** We conclude this section by noting that the above formulations assume that commodity flow can be split over several arcs. As mentioned before, specific applications may require that flow is not split, i.e., the commodity flow associated to an origin-destination pair for a given time-period follows a single path. In this case, the three formulations above can easily be adapted by changing the variable domain of flow variables $x_{ij}^{k\ell t}$ from continuous $[0, 1]$ to binary $\{0, 1\}$. As we discuss the results of our computational results in the next section, we will also briefly report on the results of the problem variants with unsplittable flows. Detailed results can be found in Appendix F.

# 4 Computational experiments

In this section, we will computationally assess the strength of the GMC, SCI and SCI2 formulations, as well as the difficulty of solving them. We first elaborate on the problem instances used to evaluate the formulations, as well as on the computing environment in Section 4.1. We then analyze the solution difficulty of the LP relaxations, as well as the integrality gaps of the formulations in Section 4.2. Note that, for the sake of brevity, we focus on a planning horizon with 10 time periods. Section 4.3 presents the comparison of the three formulations when optimizing on the original MIP formulations. In particular, Section 4.3.1 assumes that CPLEX default settings are used. Such advanced commercial solvers may not always be accessible in practice. We therefore also emulate a basic solver and compare the formulations in a basic branch-and-cut environment in Section 4.3.2. Finally, we explore the impact of different lengths of the planning horizon on the difficulty of solving the formulations in Section 4.4.

## 4.1 Problem Instances and Computing Environment

**Problem instances.** Due to the lack of instances containing all parameters relevant for the MPND-CA, we extend existing benchmark instances from similar problems. In particular, following the proposal of Fragkos et al. (2021), we have used instances from Pazour et al. (2010), studying a real-world rail network for freight distribution. Those instances are then extended following methodologies proposed in the network design literature. In particular, we consider Crainic et al. (2001) to generate different levels of fixed costs and arc capacities, as well as Fragkos et al. (2021) to account, among others, for multiple time periods. In order to use those instances for the MPND-CA, we then extend those instances to multiple capacity levels. The detailed procedure generating our problem instances can be found in Appendix E.

We consider four sizes of the planning horizon $T$, comprising 5, 10, 15, or 20 time periods. For each size $T$ of the planning horizon, we have a total of 384 problem instances, with the following combinations of characteristics:

- Network: we consider three different networks from Pazour et al. (2010): the *JBH50red* network has 50 nodes, 198 arcs, and 626 commodities; the *USC30red* network has 30 nodes, 126 arcs, and 87 commodities; finally, the *USC53red* network has 53 nodes, 278 arcs, and 245 commodities.

- Fixed costs: arc construction fixed costs can be either *low* or *high* relative to the routing costs.
- Capacities: arc capacities can be *loose* or *tight* when compared to the total commodity routing demand.
- Routing costs: the commodity per-unit routing costs on arcs can be either *random* or based on the *Euclidean* distance between the corresponding nodes.
- Demand evolution: commodity demand may either be *random* or *increase* throughout the planning horizon.
- Number of capacity levels $q$: arcs may have either 3 or 5 capacity levels (not including level 0).
- Capacity adjustment costs: the costs to expand one capacity level are either set to exactly 100% of the costs of maintaining capacity for one time-period ($P^E_C = 1$) or 5 times higher ($P^E_C = 5$). The costs to reduce one capacity level is set to 10% of the costs to expand one capacity level.
- Economies of scale: the costs to maintain capacity at level $\ell \geq 2$, or to reduce or expand capacity by $\ell \geq 2$ levels are computed based on the corresponding costs at the previous level $\ell - 1$ and reflect economies of scale of two types: positive economies of scale, where maintenance, capacity reduction and expansion is 15% cheaper than at the previous level; and negative economies of scale, where maintenance is 15% more expensive than at the previous level, while capacity reduction and expansion is 15% cheaper.

**Computing environment.** All formulations have been implemented in Python version 3.8, using the general-purpose MIP solver CPLEX version 12.9. The CPLEX solver has been constrained in all executions to a single thread (i.e., `parameters.threads` = 1) in order to avoid bias related to computational resources. Each problem instance has been separately executed on the Beluga cluster[1] on the Compute Canada network with a memory limitation of 24Gb of RAM and a limit of 12 hours of computing time (`parameters.timelimit` = 43200). Each node of the server contains 2 CPUs (Intel Gold 6148 Skylake, 2.4 GHz).

## 4.2 Linear Programming Relaxation and Integrality Gaps

We now empirically compare the strength of the three formulations, as well as their difficulty to be solved by an MIP solver (specifically, CPLEX with standard parameters). We here focus on the 384 problem instances with 10 time periods

| Description | | | GMC | | SCI | | SCI2 | |
|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | Avg. LPR time (min.) | Avg. int. gap | Avg. LPR time (min.) | Avg. int. gap | Avg. LPR time (min.) | Avg. int. gap |
| Network | JBH50red | 81 | 12.8 | 0.01% | 9.6 | 0.39% | 7.6 | 0.39% |
| | USC30red | 97 | 0.2 | 0.06% | 0.1 | 0.55% | 0.1 | 0.55% |
| | USC53red | 76 | 9.5 | 0.07% | 9.4 | 0.62% | 4.9 | 0.62% |
| $T$ | 10 | 254 | 7.0 | 0.04% | 5.9 | 0.52% | 3.9 | 0.52% |
| $q$ | 3 | 133 | 8.5 | 0.05% | 7.3 | 0.42% | 4.8 | 0.42% |
| | 5 | 121 | 5.3 | 0.04% | 4.4 | 0.63% | 2.9 | 0.63% |
| Fixed costs | Low | 173 | 1.6 | 0.04% | 1.0 | 0.24% | 1.0 | 0.24% |
| | High | 81 | 18.5 | 0.06% | 16.4 | 1.10% | 10.3 | 1.10% |
| Capacities | Loose | 176 | 9.0 | 0.02% | 8.0 | 0.59% | 5.2 | 0.59% |
| | Tight | 78 | 2.4 | 0.09% | 1.4 | 0.35% | 1.2 | 0.35% |
| Routing costs | Euclidean | 125 | 9.0 | 0.02% | 6.3 | 0.47% | 4.6 | 0.47% |
| | Random | 129 | 5.0 | 0.07% | 5.6 | 0.57% | 3.3 | 0.57% |
| Demand behavior | Increasing | 126 | 9.4 | 0.04% | 7.8 | 0.51% | 4.7 | 0.51% |
| | Random | 128 | 4.6 | 0.05% | 4.1 | 0.53% | 3.2 | 0.53% |
| $P^C_E$ | 1 | 122 | 4.9 | 0.03% | 3.8 | 0.22% | 3.4 | 0.22% |
| | 5 | 132 | 8.9 | 0.06% | 7.9 | 0.79% | 4.4 | 0.79% |
| Economies of scale | Inverse | 124 | 6.6 | 0.04% | 5.7 | 0.52% | 4.2 | 0.52% |
| | Positive | 130 | 7.3 | 0.05% | 6.2 | 0.52% | 3.7 | 0.52% |
| All | | 254 | 7.0 | 0.04% | 5.9 | 0.52% | 3.9 | 0.52% |

Table 1: Average LPR solution time and integrality gaps of the formulations for instances for which the optimal integer solution is known.

[1] More information about the Beluga cluster in https://docs.computecanada.ca/wiki/Beluga/en.

Table 1 summarizes the computing time required to solve the linear programming relaxations (LPR) of the three formulations, as well as their integrality gaps[2] for problem instances for which the optimal integer solution (which is required to compute the integrality gap) is known (in total 254 out of 384 instances). The results are separated by the different attributes of the problem instances and averaged over the respective subset of instances. As per Theorem 1, the SCI and SCI2 formulations provide the same integrality gaps. However, the SCI2 formulation is solved in substantially shorter computing times, and that consistently for all instance attributes.

Theorem 2 suggests that the GMC formulation may theoretically provide stronger LP relaxation bounds than the two other formulations. On our problem instances, the dominance of the GMC formulation is clearly pronounced, providing an integrality that is about 13 times smaller, on average, than those of the other two formulations (0.04% vs 0.52%). The GMC formulation takes about 17% more time, on average, to solve the corresponding LP relaxation. This is not surprising, given that this formulation has significantly more $y$ variables. However, given the significantly lower integrality gaps provided by this formulation, this additional computational time may be well invested, as the strength of the formulation may speed up the solution time of the original MIP.

Throughout all three formulations, some problem instance characteristics induce higher integrality gaps than others. In particular, instances with tight capacities, random routing costs, and high capacity expansion costs ($P_E^C = 5$) present higher integrality gaps, and can be expected to be more difficult to solve. The same holds true for networks USC30red and USC53red, which have generally higher integrality gaps than JBH50red networks. This may be due to the fact that the first two networks include fewer commodities (87 and 245, respectively) than the latter (626).

## 4.3 CPLEX Optimization

We will now explore the difficulty of solving the different formulations by means of a general purpose MIP solver.

| Description | | | GMC | SCI | SCI2 |
|---|---|---|---|---|---|
| Attribute | Value | # | Avg. MIP time (min.) | Avg. MIP time (min.) | Avg. MIP time (min.) |
| Network | JBH50red | 77 | 26.3 | 43.1 | 39.5 |
| | USC30red | 87 | 4.1 | 19.7 | 26.6 |
| | USC53red | 69 | 41.2 | 75.7 | 69.3 |
| $T$ | 10 | 233 | 22.4 | 44.0 | 43.5 |
| $q$ | 3 | 125 | 21.9 | 42.6 | 44.8 |
| | 5 | 108 | 23.1 | 45.6 | 42.0 |
| Fixed costs | Low | 155 | 19.1 | 38.6 | 41.9 |
| | High | 78 | 29.1 | 54.6 | 46.7 |
| Capacities | Loose | 174 | 14.0 | 25.6 | 22.0 |
| | Tight | 59 | 47.2 | 98.3 | 107.0 |
| Routing costs | Euclidean | 121 | 22.8 | 40.9 | 38.5 |
| | Random | 112 | 22.0 | 47.3 | 48.9 |
| Demand behavior | Increasing | 113 | 24.9 | 54.3 | 46.5 |
| | Random | 120 | 20.1 | 34.3 | 40.7 |
| $P_E^C$ | 1 | 114 | 21.7 | 38.9 | 35.6 |
| | 5 | 119 | 23.1 | 48.9 | 51.1 |
| Economies of scale | Inverse | 114 | 20.7 | 36.6 | 40.6 |
| | Positive | 119 | 24.1 | 51.1 | 46.2 |
| All | | 233 | 22.4 | 44.0 | 43.5 |

Table 2: Average MIP solution time (CPLEX default settings) for instances that have been solved by all three formulations.

### 4.3.1 Optimization with CPLEX Default Settings

We now explore the difficulty of solving the different formulations, i.e., finding the optimal integer solutions and proving optimality, by means of CPLEX with default settings. Table 2 presents the average MIP solution times of the three formulations as reported by CPLEX. To ensure a fair comparison among the different problem instances, the results are averaged only over instances that have been solved by all three formulations

[2]The integrality gap of a formulation for a given problem instance is defined as $\frac{v^* - v_{LP}}{v^*}$, where $v_{LP}$ is the objective function of the optimal LP relaxation solution and $v^*$ is the optimal integer solution.

within the given time limit (a total 233 instances). As suspected, the small integrality gaps of the GMC formulations significantly accelerate the solution of the problem. On average, problems are solved within 22 minutes using the GMC formulation, while it takes about twice the time with either the SCI or SCI2 formulation. This improvement is remarkably preserved throughout the different problem characteristics, which suggests that the GMC formulation is preferable in practice. Even though the LP relaxation of the SCI2 formulation was solved much faster than the one of the SCI formulation, both formulations solve the original problem in about the same time on average (44 minutes). Instance characteristics seem, however, to play a role: depending on the network, as well as the different attributes for fixed costs, capacities and routing costs, one formulation seems to be faster for one attribute value, while the other is faster for the other attribute value.

| Description | | | GMC | | | | SCI | | | | SCI2 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | # nfs | opt gap | # os | # nos | # nfs | opt gap | # os | # nos | # nfs | opt gap | # os | # nos |
| Network | JBH50red | 51 | 43 | 0.03% | 4 | 4 | 37 | 0.03% | 0 | 14 | 39 | 0.03% | 3 | 9 |
| | USC30red | 41 | 0 | 0.38% | 10 | 31 | 0 | 0.55% | 1 | 40 | 3 | 0.60% | 0 | 38 |
| | USC53red | 59 | 21 | 5.17% | 7 | 31 | 30 | 0.50% | 0 | 29 | 30 | 0.56% | 3 | 26 |
| $T$ | 10 | 151 | 64 | 2.44% | 21 | 66 | 67 | 0.45% | 1 | 83 | 72 | 0.50% | 6 | 73 |
| $q$ | 3 | 67 | 23 | 3.16% | 8 | 36 | 30 | 0.55% | 0 | 37 | 30 | 0.58% | 5 | 32 |
| | 5 | 84 | 41 | 1.69% | 13 | 30 | 37 | 0.37% | 1 | 46 | 42 | 0.43% | 1 | 41 |
| Fixed costs | Low | 37 | 11 | 0.02% | 18 | 8 | 5 | 0.03% | 1 | 31 | 7 | 0.03% | 4 | 26 |
| | High | 114 | 53 | 3.47% | 3 | 58 | 62 | 0.71% | 0 | 52 | 65 | 0.79% | 2 | 47 |
| Capacities | Loose | 18 | 0 | 0.36% | 2 | 16 | 1 | 0.32% | 0 | 17 | 0 | 0.79% | 2 | 16 |
| | Tight | 133 | 64 | 2.98% | 19 | 50 | 66 | 0.48% | 1 | 66 | 72 | 0.41% | 4 | 57 |
| Routing costs | Euclidean | 71 | 37 | 4.02% | 4 | 30 | 36 | 0.53% | 0 | 35 | 39 | 0.85% | 1 | 31 |
| | Random | 80 | 27 | 1.42% | 17 | 36 | 31 | 0.39% | 1 | 48 | 33 | 0.26% | 5 | 42 |
| Demand behavior | Increasing | 79 | 32 | 3.02% | 13 | 34 | 36 | 0.36% | 0 | 43 | 38 | 0.48% | 4 | 37 |
| | Random | 72 | 32 | 1.75% | 8 | 32 | 31 | 0.54% | 1 | 40 | 34 | 0.52% | 2 | 36 |
| $P_E^C$ | 1 | 78 | 33 | 2.90% | 8 | 37 | 34 | 0.45% | 1 | 43 | 38 | 0.50% | 1 | 39 |
| | 5 | 73 | 31 | 1.94% | 13 | 29 | 33 | 0.45% | 0 | 40 | 34 | 0.50% | 5 | 34 |
| Economies of scale | Inverse | 78 | 33 | 0.37% | 10 | 35 | 33 | 0.43% | 0 | 45 | 34 | 0.35% | 3 | 41 |
| | Positive | 73 | 31 | 4.66% | 11 | 31 | 34 | 0.47% | 1 | 38 | 38 | 0.69% | 3 | 32 |
| All | | 151 | 64 | 2.44% | 21 | 66 | 67 | 0.45% | 1 | 83 | 72 | 0.50% | 6 | 73 |

Table 3: Number of instances without feasible solution, with optimal solution and non-optimal solution, as well as average optimality gap (CPLEX default settings) for instances which have not been solved to optimality by at least one of the formulations.

We now analyze the 151 (out of 384) problem instances that have not been solved by all three formulations within the given time limit. Table 3 presents, for each of the three formulations, the number of instances for which no feasible solution has been found within the given time limit (# nfs), either due to hitting the time limit or by exceeding the available memory. Among those instances for which the respective formulation has found a feasible solution, the table reports the average optimality gap reported by CPLEX (opt gap), the number of instances which have been solved to optimality (# os) and the number of instances which have not been solved to optimality (# nos).

The summary results in this table have to be interpreted carefully, however, due to several reasons. First, column "# nfs" may count instances that run out of memory, even though they have already found instances of high quality. For example, we further analyzed the results of instances in network JBH50red, which reported more instances in this column for the GMC formulation than it did for the SCI formulations. Here, for several instances, the solver had already found instances with optimality gaps around 0.05%; however, the solver later ran out of memory. Second, the average optimality gap over all instances has been reported as 2.44% for the GMC formulation, compared to 0.45% and 0.50% for the two other formulations. The average for the GMC formulation does, however, consider 3 more instances than the average for the SCI formulation (and 8 more instances than for the SCI2 formulation), which do not at all find feasible solutions for these problem instances. Excluding the 3 instances from the GMC average reduces the average optimality gap to 0.39%, which quickly changes the conclusions in favor of the GMC formulation.

In conclusion, based on the results presented in the previous three tables, the GMC formulation seems to have substantial advantages in practice when compared to the other two formulations, most likely due to its strong LP relaxation bound.

### 4.3.2 Optimization in a basic branch-and-cut environment

The previous experiments have used CPLEX (with default parameters) to solve the problem to optimality. The commercial solver is acknowledged to be among the most powerful ones, embedding preprocessing techniques, heuristics to find upper bounds and elaborate cut generation. We now compare the formulations' abilities to be solved in a branch-and-cut environment, but assuming that only a basic branch-and-cut solver is available (e.g., due to budget limitations). To emulate such a solver, we use CPLEX, but turn off all advanced preprocessing (`parameters.preprocessing.presolve` = 0), heuristics (`parameters.mip.strategy.heuristicfreq` = -1) and cut generation (`parameters.mip.limits.cutpasses` = -1). Since we solely focus on the formulations' impact on the enumeration tree in the branch-and-cut environment, we further pass to the solver (as cut-off value) the optimal upper bound for the respective instance (i.e., its optimal integer solution value). These experiments therefore only consider the 254 instances with 10 time periods for which the optimal integer solution is known. Note that the availability of the cut-off value does not guarantee that the solver will actually find the corresponding optimal solution.

| Description | | | GMC | | | | SCI | | | | SCI2 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | # nfs | # os | opt gap | time (min.) | # nfs | # os | opt gap | time (min.) | # nfs | # os | opt gap | time (min.) |
| Network | JBH50red | 81 | 6 | 75 | 0.00% | 128.0 | 70 | 3 | 0.05% | 677.5 | 75 | 3 | 0.06% | 686.5 |
| | USC30red | 97 | 20 | 77 | 0.00% | 17.7 | 36 | 59 | 0.01% | 56.5 | 38 | 58 | 0.01% | 49.7 |
| | USC53red | 76 | 18 | 58 | 0.00% | 56.2 | 70 | 2 | 0.11% | 585.3 | 71 | 0 | 0.07% | 720.0 |
| $T$ | 10 | 254 | 44 | 210 | 0.00% | 67.7 | 176 | 64 | 0.03% | 184.8 | 184 | 61 | 0.02% | 152.1 |
| $q$ | 3 | 133 | 18 | 115 | 0.00% | 83.7 | 85 | 37 | 0.03% | 216.6 | 89 | 35 | 0.02% | 197.0 |
| | 5 | 121 | 26 | 95 | 0.00% | 48.3 | 91 | 27 | 0.02% | 133.9 | 95 | 26 | 0.01% | 76.2 |
| Fixed costs | Low | 173 | 36 | 137 | 0.00% | 38.4 | 126 | 36 | 0.03% | 205.1 | 131 | 35 | 0.02% | 168.5 |
| | High | 81 | 8 | 73 | 0.00% | 122.8 | 50 | 28 | 0.03% | 154.1 | 53 | 26 | 0.02% | 127.6 |
| Capacities | Loose | 176 | 7 | 169 | 0.00% | 54.1 | 99 | 64 | 0.03% | 177.8 | 107 | 61 | 0.02% | 143.9 |
| | Tight | 78 | 37 | 41 | 0.01% | 123.9 | 77 | 0 | 0.07% | 720.0 | 77 | 0 | 0.03% | 720.0 |
| Routing costs | Euclidean | 125 | 15 | 110 | 0.00% | 75.6 | 88 | 31 | 0.03% | 157.0 | 91 | 31 | 0.01% | 128.1 |
| | Random | 129 | 29 | 100 | 0.00% | 59.1 | 88 | 33 | 0.02% | 209.9 | 93 | 30 | 0.02% | 174.8 |
| Demand behavior | Increasing | 126 | 25 | 101 | 0.00% | 60.6 | 94 | 28 | 0.02% | 102.6 | 95 | 28 | 0.01% | 96.5 |
| | Random | 128 | 19 | 109 | 0.00% | 74.3 | 82 | 36 | 0.03% | 242.0 | 89 | 33 | 0.02% | 196.4 |
| $P_E^C$ | 1 | 122 | 23 | 99 | 0.00% | 71.2 | 76 | 37 | 0.02% | 200.7 | 79 | 35 | 0.02% | 201.3 |
| | 5 | 132 | 21 | 111 | 0.00% | 64.6 | 100 | 27 | 0.04% | 161.9 | 105 | 26 | 0.02% | 73.8 |
| Economies of scale | Inverse | 124 | 23 | 101 | 0.00% | 65.4 | 84 | 34 | 0.02% | 190.8 | 90 | 31 | 0.01% | 131.0 |
| | Positive | 130 | 21 | 109 | 0.00% | 69.9 | 92 | 30 | 0.03% | 178.4 | 94 | 30 | 0.02% | 172.1 |
| All | | 254 | 44 | 210 | 0.00% | 67.7 | 176 | 64 | 0.03% | 184.8 | 184 | 61 | 0.02% | 152.1 |

Table 4: Number of instances without feasible solution and with optimal solution, as well as average optimality gap and solution time (basic branch-and-cut environment).

Table 4 presents, for each of the three formulations, the number of instances for which no feasible solution has been found (# nfs), the number of instances which have been solved to optimality (# os), the average optimality gap among the instances for which a feasible solution has been found (opt gap) and the average computing time as reported by CPLEX.

These experiments, emulating a basic branch-and-cut solver, highlight the advantage of using GMC formulation over the other two formulations. The GMC formulation does not find a feasible solution for 44 of the 254 instances, while the SCI and SCI2 formulations do not provide a feasible solution for about four times more instances. All other 210 instances are solved to optimality by the GMC formulation within the given time limit. In contrast, the other two formulations do not solve to optimality the majority of the instances. The strong performance of the GMC formulation is also in line with the lower computing times it requires to find solutions and to prove optimality, which is significantly lower than for the SCI formulations.

These results suggest that, for our problem instances, the GMC formulation remains preferable over the other two formulations. Note, however, that the dominance of the GMC formulation may not necessarily be preserved on other problem instances. For example, if the number of capacity levels is much higher than 5, the quadratic number of $y$ decision variables may result in a prohibitively large optimization model. More than 5 capacity levels, however, appear to be rarely the case in practice.

**Results for the Problem Variant with Unsplittable Commodity Flows.** As previously mentioned, we have also explored the problem variant that does not allow for splitting the commodity flows of origin-destination pairs within the same time-period. The resulting formulations simply use binary instead of continuous flow variables. In summary, the general tendencies remain the same. The performance difference among the formulation is, however, less pronounced. The integrality gap of the GMC formulation is, on average, 0.45%, while it is 1.0% for the other two formulations. Intuitively, this would suggest that the corresponding mixed-integer formulations are harder to solve. Solving the mixed-integer formulations to optimality takes about 23 minutes with the GMC (i.e., about the same amount of time as for the splittable variant) vs. 32 minutes with the other two formulations when using CPLEX default parameters (which is less than expected). We suspect that CPLEX generates a variety of cuts to strengthen the SCI formulations, which ultimately speeds up the solution process. This suspicion is also in line with the results for the experiments when emulating the basic branch-and-cut solver. Here, the SCI formulations do not find any feasible solution for over 200 of the remaining 267 instances, while the GMC formulation does not find a feasible solution only for 82 instances. The original strength of the formulation therefore seems to make a major difference in solving the original problem. Detailed results can be found in Appendix F.

## 4.4 Impact of Planning Horizon Length

All previous experiments have been carried on problem instances with a planning horizon comprising 10 time periods. Intuitively, larger lengths of the planning horizon should increase the difficulty of the problem. To this end, we now study the performance of the three formulations with varying lengths of the planning horizon. Specifically, we will consider problem instances from the USC30red network with 5, 10, 15, and 20 time periods.

| Description | | | GMC | | SCI | | SCI2 | |
|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | Avg. LPR time (min.) | Avg. int. gap | Avg. LPR time (min.) | Avg. int. gap | Avg. LPR time (min.) | Avg. int. gap |
| Network | USC30red | 391 | 0.2 | 0.07% | 0.1 | 0.57% | 0.1 | 0.57% |
| $T$ | 5 | 96 | 0.1 | 0.10% | 0.1 | 1.18% | 0.0 | 1.18% |
| | 10 | 97 | 0.2 | 0.06% | 0.1 | 0.55% | 0.1 | 0.55% |
| | 15 | 99 | 0.2 | 0.06% | 0.1 | 0.33% | 0.1 | 0.33% |
| | 20 | 99 | 0.3 | 0.05% | 0.2 | 0.24% | 0.2 | 0.24% |

Table 5: Average LPR solution time and integrality gaps (CPLEX default settings) for USC30red network instances with different lengths of the planning horizon for which the optimal integer solution is known.

As in the previous sections, Table 5 summarizes the results of the LPR solution, as well as the integrality gaps of the formulations, but separated for the different lengths of the planning horizon. For all three formulations, the average time required to solve the LP relaxation increases as the planning horizon contains more time periods. In contrast, the integrality gap tends to reduce with large planning horizons. This is most likely linked to the fact that the relative importance of each commodity demand diminishes as the number of time periods, and therefore the number of commodity demands increases. Comparing the three formulations, the GMC formulation offers the lowest integrality gaps, no matter the size of the planning horizon.

| Description | | | GMC | SCI | SCI2 |
|---|---|---|---|---|---|
| Attribute | Value | # | Avg. MIP time (min.) | Avg. MIP time (min.) | Avg. MIP time (min.) |
| Network | USC30red | 352 | 3.3 | 19.8 | 21.7 |
| $T$ | 5 | 92 | 3.4 | 21.8 | 19.4 |
| | 10 | 87 | 4.1 | 19.7 | 26.6 |
| | 15 | 87 | 2.4 | 12.3 | 9.6 |
| | 20 | 86 | 3.2 | 25.4 | 31.2 |

Table 6: Average MIP solution time (CPLEX default settings) for USC30red network instances with different lengths of the planning horizon that have been solved by all three formulations.

Table 6 summarizes the optimization results (using CPLEX default parameters) for problem instances which have been solved to optimality by all three formulations within the given time limit. Surprisingly, the length of the planning horizon does not seem to impact much the difficulty of solving the problem, and that throughout all three formulations. We suspect, however, that this would change once the number of time periods considered in the planning horizon is sufficiently large.

| Description | | | GMC | | | | SCI | | | | SCI2 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | # nfs | opt gap | # os | # nos | # nfs | opt gap | # os | # nos | # nfs | opt gap | # os | # nos |
| Network | USC30red | 160 | 1 | 0.42% | 39 | 120 | 4 | 0.66% | 4 | 152 | 26 | 0.87% | 3 | 131 |
| $T$ | 5 | 36 | 0 | 0.69% | 4 | 32 | 1 | 0.65% | 0 | 35 | 0 | 0.74% | 0 | 36 |
| | 10 | 41 | 0 | 0.38% | 10 | 31 | 0 | 0.55% | 1 | 40 | 3 | 0.60% | 0 | 38 |
| | 15 | 41 | 1 | 0.31% | 12 | 28 | 0 | 0.70% | 3 | 38 | 10 | 0.85% | 3 | 28 |
| | 20 | 42 | 0 | 0.32% | 13 | 29 | 3 | 0.75% | 0 | 39 | 13 | 1.42% | 0 | 29 |

Table 7: Number of instances without feasible solution, with optimal solution and non-optimal solution, as well as average optimality gap (CPLEX default settings) for USC30red network instances with different lengths of the planning horizon which have not been solved to optimality by at least one of the formulations.

Finally, Table 7 focuses on the remaining problem instances, that have not been solved for at least one of the formulations. Considering the number of instances for which no feasible solutions have been found, in particular for the SCI and SCI2 formulations, it appears that the problem becomes more difficult to solve (given its size) as it comprises more time periods. Among the three formulations, the last two tables confirm the advantage of using the GMC formulation as opposed to the SCI and SCI2 formulations. No matter the length of the planning horizon, the GMC formulation solves the same instances in significantly shorter computing times (approximately 2-4 minutes vs. approximately 12-31 minutes), finds feasible solutions for all but one problem instance, and presents lower optimality gaps for the remaining instances.

# 5 Conclusions

We have introduced a new multi-period multi-commodity network design problem, in which arc capacities can be gradually increased or decreased along the planning horizon. With respect to the existing literature, our problem additionally allows for the possibility of choosing the arc capacity from a set of modular capacity instead of imposing a predefined arc capacity, as well as for the reduction of arc capacity (which has not been considered before). Arc capacities can therefore be adjusted as the demand changes over time, which is particularly important in domains such as telecommunications.

We have proposed three mixed-integer programming formulations for this problem: two formulations based on classical modeling techniques, using one decision variable for each capacity level; the third formulation (called the GMC formulation) stems from the recent facility location literature, using more precise decision variables to represent the exact capacity changes performed at each time period. We have shown that the latter formulation theoretically provides stronger LP relaxation bounds, while the former two formulations are equally strong. We have also discussed the use of alternative, simpler flow variables, and have shown how the resulting models provide weaker LP relaxation bounds.

To empricially compare the formulations, computational experiments have then been carried out on problem instances that extend benchmark instances from the literature. The results generally indicate that the GMC formulation outperforms the other two formulations in all relevant criteria: first, it provides integrality gaps that are, on average, 13 time smaller; second, using CPLEX default parameters, it enables the solver to solve the problem instances, on average, twice as fast; and, third, emulating a more basic branch-and-cut environment, it proves optimality 2-3 times faster than the other formulations. Further experiments have shown that the number of time periods used in the planning horizon does, surprisingly, not seem to have a strong impact on the difficulty of solving the problem. Finally, experiments on a different problem variant, in which the commodity flow cannot be split over several arcs, have shown tendencies similar to those described above, even though the performance difference of the different formulations is less pronounced.

While our results may guide practitioners to select the most beneficial formulation for their respective application, they also offer a good starting point to develop more advanced solution methods to solve large-scale problem instances of this problem. In either of those cases, our results suggest the GMC formulation to be the formulation of choice.

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# A Proof of Theorem 1: Equivalence Relationship

We here prove Theorem 1, i.e., the equivalence relationship between the SCI and SCI2 formulations. Since both formulations only differ on how to write the capacity flow conservation constraints, we will prove more specifically that equations (4)-(5) and (10) are equivalent.

To show how an equation (4) for a given time period $t$ is transformed into an equivalent equation (10) for the same time period, we can recursively substitute the term $\sum_{\ell\in\mathcal{L}} y_{ij}^{\ell(t-1)}$ in the RHS of constraint (4) by the entire RHS of the same constraint at $t-1$. Recursively replacing the occurrences of $\sum_{\ell\in\mathcal{L}} y_{ij}^{\ell(t-1)}$ will result in the following series of equations:

$$\begin{aligned}\sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} &= \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell(t-1)} + \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell t} - \sum_{\ell\in\mathcal{L}} \ell w_{ij}^{\ell t} = \\ &= \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell(t-2)} + \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell(t-1)} - \sum_{\ell\in\mathcal{L}} \ell w_{ij}^{\ell(t-1)} + \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell t} - \sum_{\ell\in\mathcal{L}} \ell w_{ij}^{\ell t} = \\ &\dots \\ &= \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell 1} + \sum_{\ell\in\mathcal{L}}\sum_{t'=2}^{t'=t} \ell s_{ij}^{\ell t'} - \sum_{\ell\in\mathcal{L}}\sum_{t'=2}^{t'=t} \ell w_{ij}^{\ell t'} \quad \forall (i,j)\in\mathcal{A}.\end{aligned}$$

In the above, we may now use equation (5) to substitute the term $\sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell 1}$ by $\sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell 1} - \sum_{\ell\in\mathcal{L}} \ell w_{ij}^{\ell 1}$, which leads to the following equation, equivalent to equation (10):

$$\sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} = \sum_{\ell\in\mathcal{L}}\sum_{t'=1}^{t'=t} \ell s_{ij}^{\ell t'} - \sum_{\ell\in\mathcal{L}}\sum_{t'=1}^{t'=t} \ell w_{ij}^{\ell t'} \quad \forall (i,j)\in\mathcal{A}.$$

In order to show how an equation (10) for a given time period $t$ is transformed into an equivalent equation (4) for the same time period, we can reverse the procedure used above. It follows that both constraints, and therefore both formulations are equivalent. As a consequence, both formulations also provide the same LP relaxation bound.

# B Proof of Theorem 2: Dominance Relationship

We prove here Theorem 2, i.e., the dominance relationship between the GMC and SCI formulations. To be precise, we will prove that GMC formulation is stronger (strictly stronger for some instances) than the SCI formulations. The proof consists of two main steps. In the first step, we prove that the GMC formulation is at least as strong as the SCI formulation by showing that from any solution $\{x_{ij}^{k\ell t}, y_{ij}^{\ell_1\ell_2 t}\}$ feasible in $\overline{\text{GMC}}$ (i.e., the LP relaxation of GMC), we can construct a solution $\{x_{ij}^{k\ell t}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ that is feasible in $\overline{\text{SCI}}$ (i.e., the LP relaxation of SCI) and has the same objective function value. Given that any solution found for the $\overline{\text{GMC}}$ can also be found for $\overline{\text{SCI}}$, the solution space of the latter is potentially larger and may include solutions with lower objective function values (i.e., therefore providing bounds of inferior quality). In the second step, we show that for a specific problem instance, the optimal solution provided by $\overline{\text{GMC}}$ is strictly superior to the one provided by $\overline{\text{SCI}}$. This proves that the GMC formulation is strictly stronger than the SCI formulation. Since the SCI formulations are equivalent by Theorem 1, it also proves that the GMC formulation is strictly stronger than the SCI2 formulation.

### (a) Construction of feasible $\overline{\text{SCI}}$ solution from a $\overline{\text{GMC}}$ solution

Let $\{x_{ij}^{k\ell t}, y_{ij}^{\ell_1\ell_2 t}\}$ be any solution feasible in $\overline{\text{GMC}}$. We now construct an equivalent solution $\{x_{ij}^{k\ell t}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ that is feasible in $\overline{\text{SCI}}$ (i.e., it satisfies all of its constraints) and has the same objective function value.

We first set its variables $y_{ij}^{\ell t}$, $s_{ij}^{\ell t}$ and $w_{ij}^{\ell t}$ in function of the $\overline{\text{GMC}}$ variables as follows:

$$y_{ij}^{\ell t} = \sum_{\ell_1\in\mathcal{L}} y_{ij}^{\ell_1 \ell t} \qquad \forall (i,j)\in\mathcal{A}, \forall \ell\in\mathcal{L}, \forall t\in\mathcal{T} \tag{18}$$

$$s_{ij}^{\ell t} = \sum_{\ell_1 \in \mathcal{L}} y_{ij}^{\ell_1(\ell_1+\ell)t} \qquad \forall (i,j) \in \mathcal{A}, \forall \ell \in \mathcal{L}, \forall t \in \mathcal{T} \tag{19}$$

$$w_{ij}^{\ell t} = \sum_{\ell_1 \in \mathcal{L}} y_{ij}^{\ell_1(\ell_1-\ell)t} \qquad \forall (i,j) \in \mathcal{A}, \forall \ell \in \mathcal{L}, \forall t \in \mathcal{T} \tag{20}$$

For any time period $t \in \mathcal{T}$ and arc $(i,j) \in \mathcal{A}$, all $y_{ij}^{\ell_1 \ell_2 t}$ variables from the $\overline{\text{GMC}}$ solution sum to 1. This has two implications. First, the variables $y_{ij}^{\ell t}$, $s_{ij}^{\ell t}$ and $w_{ij}^{\ell t}$ (as defined above) will take values between 0 and 1 (therefore, their domain constraints are satisfied). Second, for any time period $t \in \mathcal{T}$ and arc $(i,j) \in \mathcal{A}$, $\sum_{\ell \in \mathcal{L}} y_{ij}^{\ell t} = 1$, $\sum_{\ell \in \mathcal{L}} s_{ij}^{\ell t} = 1$ and $\sum_{\ell \in \mathcal{L}} w_{ij}^{\ell t} = 1$. Constraints (6), (7) and (8) are therefore satisfied.

We then set the $x_{ij}^{k\ell t}$ variables in the $\overline{\text{SCI}}$ solution to take the same values as the $x_{ij}^{k\ell t}$ variables in the $\overline{\text{GMC}}$ formulation. Given that the flow conservation constraints (12) are satisfied in the $\overline{\text{GMC}}$ formulation, the equivalent constraints (2) therefore also hold.

Replacing the $y_{ij}^{\ell t}$ variables in constraints (3) in the $\overline{\text{SCI}}$ formulation using equations (18) yields the corresponding constraints (13) in the $\overline{\text{GMC}}$ formulation. Both constraints are therefore equivalent. Therefore, constraints (3) are satisfied.

We now show that constraints (4) and (5) in the $\overline{\text{SCI}}$ formulation hold. Using the equations (18), (19) and (20) to replace variable $y_{ij}^{\ell t}$, $s_{ij}^{\ell t}$ and $w_{ij}^{\ell t}$, respectively, in constraints (4) yields the following equation (note that the term corresponding to the $w$ variables has been brought to the LHS of the equation), for all $t \in \mathcal{T}^{\setminus\{1\}}$ and $(i,j) \in \mathcal{A}$:

$$\sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1 \ell t} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1(\ell_1-\ell)t} = \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1 \ell (t-1)} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1(\ell_1+\ell)t}$$

Using constraints (14), we can replace the occurrences of the terms $\sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1 \ell (t-1)}$ with the terms $\sum_{\ell \in \mathcal{L}} \sum_{\ell_2 \in \mathcal{L}} \ell y_{ij}^{\ell \ell_2 t}$ in the previous equations, obtaining the following equations:

$$\sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1 \ell t} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1(\ell_1-\ell)t} = \sum_{\ell \in \mathcal{L}} \sum_{\ell_2 \in \mathcal{L}} \ell y_{ij}^{\ell \ell_2 t} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y_{ij}^{\ell_1(\ell_1+\ell)t} \tag{21}$$

We will now show that this equation holds for all values of $q \in \mathbb{N}, q \geq 1$, hence proving that constraints (4) are also satisfied. Note that these equations are given for all $t \in \mathcal{T}^{\setminus\{1\}}$ and $(i,j) \in \mathcal{A}$. For the sake of clarity, we therefore omit the variable indices $i$, $j$ and $t$ in the following developments, leading to the following equation:

$$\sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y^{\ell_1 \ell} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y^{\ell_1(\ell_1-\ell)} = \sum_{\ell \in \mathcal{L}} \sum_{\ell_2 \in \mathcal{L}} \ell y^{\ell \ell_2} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y^{\ell_1(\ell_1+\ell)} \tag{22}$$

**Proposition**: Equation (22) is true for all sizes of $\mathcal{L} = \{0, 1, 2, \ldots, q\}$, i.e. for any $q \geq 1, q \in \mathbb{N}$.

**Proof via induction**

**Basic step:** Equation (22) holds for $q = 1$, i.e. $\mathcal{L} = \{0, 1\}$.

LHS:

$$\sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y^{\ell_1 \ell} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y^{\ell_1(\ell_1 - \ell)} = y^{01} + y^{11} + \boldsymbol{y^{0,-1}} + y^{10}$$

RHS:

$$\sum_{\ell \in \mathcal{L}} \sum_{\ell_2 \in \mathcal{L}} \ell y^{\ell \ell_2} + \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} \ell y^{\ell_1(\ell_1 + \ell)} = y^{10} + y^{11} + y^{01} + \boldsymbol{y^{12}}$$

Note that the variables $\boldsymbol{y^{0,-1}}$ and $\boldsymbol{y^{12}}$ indicated in bold above do not exist for $\mathcal{L} = \{0, 1\}$. They are therefore omitted. Both the LHS and RHS of the equation are therefore equal, which proves that equation (22) holds for $q = 1$.

**Induction step:** Assuming equation (22) holds for a specific $\mathcal{L} = \{0, 1, 2, \ldots, q\}$, $q \geq 1$, we now show that equation (22) also holds for $\mathcal{L} = \{0, 1, 2, \ldots, q + 1\}$ with $q + 1$ capacity levels.

Equation (22) for $q+1$ capacity levels has the same terms as with $q$ capacity levels. In addition, it contains the following terms on the LHS and RHS.
LHS:

$$\sum_{\ell_1=0}^{q} (q+1) y^{\ell_1(q+1)} + \sum_{\ell=0}^{q+1} \ell y^{(q+1)\ell} + \sum_{\ell_2=0}^{q+1} \ell y^{(q+1)(q+1-\ell)}$$

RHS:

$$\sum_{\ell_2=0}^{q} (q+1) y^{(q+1)\ell_2} + \sum_{\ell=0}^{q+1} \ell y^{\ell(q+1)} + \sum_{\ell_1=0}^{q} (q+1) y^{\ell_1(\ell_1+q+1)} + \sum_{\ell_1=1}^{q} (q+1-\ell_1) y^{\ell_1(q+1)}$$

Variables that corresponds to invalid transitions (i.e., those that are not part of $\mathcal{L} = \{0, 1, 2, ...q, q + 1\}$) have been omitted in the terms above. Note that the first two terms in LHS stem from the first term of the left-hand-side of equation (22), while the third term in LHS stem from the second term of the left-hand-side of equation (22). Further, the first two terms in RHS stem from the first term of the right-hand-side of equation (22), while the third term in RHS stems from the second term of the right-hand-side of equation (22). Finally, that same term also yields the the last term in RHS represents the terms where both indices $\ell_1 \leq q$ and $\ell_2 \leq q$, but its sum yields (q+1). By opening both equations' remaining terms, one can easily see that both sides are equal. Therefore, equation (22) holds for $q+1$ and by induction is true for any $q \geq 1, q \in \mathbb{N}$.

Constraints (5) are also satisfied, which can be validated by replacing its variables with the corresponding GMC variables using (18) and (19). Both sides of the resulting equation are equivalent once observed that the constraint is given only for $t = 1$, and $\ell_1 = 0$ at $t = 1$ (as implied by constraints (15)).

Finally, it is easily verified that strong linking constraints (9) are also satisfied, given that they are equivalent to the corresponding strong linking constraints (17) once the variables are replaced using equaitions (18).

To finalize the first step of the proof, we now show that both solutions have the same objective function value. All $x_{ij}^{k\ell t}$ variables have the same coefficients and therefore equally contribute to their respective objective function. Furthermore, based on the definition of $f_{ij}^{\ell_1 \ell_2 t}$ (see Section 3.4), the objective function coefficients of the $y_{ij}^{\ell_1 \ell_2 t}$ variables, it can be verified that the contribution of those variables to the objective function is the same as those of variable $y_{ij}^{\ell t}$, $s_{ij}^{\ell t}$ and $w_{ij}^{\ell t}$. Both solutions therefore have the same objective function, which concludes the proof that any solution feasible in $\overline{\text{GMC}}$ may be written as an equivalent solution feasible in $\overline{\text{SCI}}$ with the same objective function value.

**(b) Example of problem instance where $\overline{\text{GMC}}$ is strictly stronger than $\overline{\text{SCI}}$**

As a consequence of the the previous step, the solution space of $\overline{\text{SCI}}$ is at least as large as the solution space of $\overline{\text{GMC}}$. $\overline{\text{GMC}}$ is therefore at least as strong as $\overline{\text{SCI}}$. However, for certain problem instances, $\overline{\text{SCI}}$ may therefore find solutions that are not part of the $\overline{\text{GMC}}$ solution space, and have a lower objective function value than any of the feasible $\overline{\text{GMC}}$ solutions, therefore providing bounds that are less tight in relation to the optimal integer solution of the problem. We now provide such a problem instance, which proves that the bound provided by $\overline{\text{GMC}}$ is strictly stronger than the bound provided by $\overline{\text{SCI}}$.

Consider a problem instance with a single time period $t_1$ and a single commodity $k_1$. The network has two nodes $n_1$ and $n_2$ and a single arc $(n_1, n_2)$, on which 10 units of commodity $k_1$ must be routed from $n_1$ to $n_2$. The arc has two capacity levels, i.e., therefore $\mathcal{L} = \{0, 1, 2\}$.

Capacity levels 1 and 2 have capacities of $U_{ij}^1 = 10$ and $U_{ij}^2 = 20$, respectively. The expansion costs to construct such capacities are $\overline{f}_{n_1 n_2}^{\ell=1}$ =10\$ and $\overline{f}_{n_1 n_2}^{\ell=2}$ =15\$ for capacity levels 1 and 2, respectively. They therefore follow typical economies of scale. The maintenance costs to operate the arc throughout the time period are $F_{n_1 n_2}^1$ =10\$ and $F_{n_1 n_2}^2$ =30\$, respectively, for the two capacity levels. They therefore involve inverse economies scales (i.e., the maintenance cost per unit is larger as the capacity level increases). Routing costs and reduction costs are all set to 0.

For this problem instance, the optimal solution of $\overline{\text{SCI}}$ partially constructs the arc on capacity level 2 ($s_{n_1 n_2}^{\ell=2 t_1} = 0.5$ and $s_{n_1 n_2}^{\ell=0, t_1} = 0.5$), but maintains on capacity level 1 ($y_{n_1 n_2}^{\ell=1, t_1} = 1.0$). The flow is routed via $x_{n_1 n_2}^{k_1, \ell=1, t_1} = 1.0$. Constructing on one capacity level and maintaining on another allows the formulation to find a solution of low objective function value 17.50\$.

In contrast, the optimal solution of $\overline{\text{GMC}}$ is forced to both construct and maintain on the same capacity level. It chooses capacity level 1 ($y_{n_1 n_2}^{\ell_1=0, \ell_2=1, t_1} = 1.0$) and also routes via $x_{n_1 n_2}^{k_1, \ell=1, t_1} = 1.0$. Its associated objective function value is 20.00\$, which is also equivalent to the optimal integer solution.

The two steps (a) and (b) above prove Theorem 2, validating that the GMC formulation dominates the SCI formulation in terms of LP relaxation strength. Since the SCI formulations are equivalent by Theorem 1, it also proves that the GMC formulation dominates the SCI2 formulation in terms of LP relaxation strength.

# C Alternative formulations

This appendix presents the alternative SCI, SCI2 and GMC formulations, which we will refer to SCI-A, SCI2-A and GMC-A, respectively.

## C.1 SCI-A: Alternative SCI formulation

Binary variables $y_{ij}^{\ell t}$, $s_{ij}^{\ell t}$, and $w_{ij}^{\ell t}$ are defined as in Section 3.2, whereas now continuous variables $x_{ij}^{kt} \in [0, 1]$ define the fraction of demand of commodity $k$ that is directed through arc $(i, j)$ at time period $t$. Note that here index $\ell$ has been removed from the original continuous variables $x_{ij}^{k\ell t}$. The new SCI-A formulation writes as follows:

$$
\begin{aligned}
\text{(SCI-A)} \quad \min \; & \sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} C_{ij}^{k} d^{kt} x_{ij}^{kt} + \sum_{(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} F_{ij}^{\ell} y_{ij}^{\ell t} + \\
& \sum_{(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} \overline{f}_{ij}^{\ell} s_{ij}^{\ell t} + \sum_{(i,j)\in\mathcal{A}} \sum_{\ell\in\mathcal{L}} \sum_{t\in\mathcal{T}} \underline{f}_{ij}^{\ell} w_{ij}^{\ell t} && (23)\\
s.t. \; & \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{kt} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{kt} = b_i^k \quad \forall i \in \mathcal{N}, \;\; \forall k \in \mathcal{K}, \;\; \forall t \in \mathcal{T} && (24)\\
& \sum_{k\in\mathcal{K}} d^{kt} x_{ij}^{kt} \leq \sum_{\ell\in\mathcal{L}} U_{ij}^{\ell} y_{ij}^{\ell t} \quad \forall (i,j) \in \mathcal{A}, \;\; \forall t \in \mathcal{T} && (25)\\
& \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} = \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell(t-1)} + \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell t} - \sum_{\ell\in\mathcal{L}} \ell w_{ij}^{\ell t} \quad \forall (i,j) \in \mathcal{A}, \;\; \forall t \in \mathcal{T}^{\setminus\{1\}} && (26)\\
& \sum_{\ell\in\mathcal{L}} \ell y_{ij}^{\ell t} = \sum_{\ell\in\mathcal{L}} \ell s_{ij}^{\ell t} \quad \forall (i,j) \in \mathcal{A}, \;\; t = 1 && (27)\\
& \sum_{\ell\in\mathcal{L}} y_{ij}^{\ell t} = 1 \quad \forall (i,j) \in \mathcal{A}, \;\; \forall t \in \mathcal{T} && (28)\\
& \sum_{\ell\in\mathcal{L}} s_{ij}^{\ell t} = 1 \quad \forall (i,j) \in \mathcal{A}, \;\; \forall t \in \mathcal{T} && (29)\\
& \sum_{\ell\in\mathcal{L}} w_{ij}^{\ell t} = 1 \quad \forall (i,j) \in \mathcal{A}, \;\; \forall t \in \mathcal{T} && (30)\\
& 0 \leq x_{ij}^{kt} \leq 1 \quad \forall (i,j) \in \mathcal{A}, \;\; \forall k \in \mathcal{K}, \;\; \forall t \in \mathcal{T} \\
& y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t} \in \{0,1\} \quad \forall (i,j) \in \mathcal{A}, \;\; \forall \ell \in \mathcal{L}, \;\; \forall t \in \mathcal{T}.
\end{aligned}
$$

The strong linking constraints compatible with the SCI-A formulation are as follows:

$$
x_{ij}^{kt} \leq \sum_{\ell\in\mathcal{L}^{\setminus\{0\}}} y_{ij}^{\ell t} \quad \forall (i,j) \in \mathcal{A}, \;\; \forall k \in \mathcal{K}, \;\; \forall t \in \mathcal{T}. \tag{31}
$$

Note that we explicitly keep out $y$ variables for maintaining an arc at capacity level 0 from the strong linking constraints because a commodity flow can never be routed through this arc.

## C.2 SCI2-A: Alternative SCI2 formulation

The SCI2 formulation differs from the SCI formulation only by equations (10), which does not contain the original continuous variables $x_{ij}^{k\ell t}$. The SCI2-A formulation is therefore given by (23)-(25), (27)-(30) and (10). Note that the strong linking constraints (31) from the SCI-A formulation can also be applied to the SCI2-A formulation.

## C.3 GMC-A: Alternative GMC formulation

Binary variables $y_{ij}^{\ell_1\ell t}$ are defined as in Section 3.4, whereas continuous variables $x_{ij}^{kt}$ are now defined as for the SCI-A formulation in Appendix C.1. The GMC-A formulation therefore writes as follows:

$$\text{(GMC-A)} \quad \min \sum_{(i,j)\in\mathcal{A}}\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} C_{ij}^k d^{kt} x_{ij}^{kt} + \sum_{(i,j)\in\mathcal{A}}\sum_{\ell_1\in\mathcal{L}}\sum_{\ell_2\in\mathcal{L}}\sum_{t\in\mathcal{T}} f_{ij}^{\ell_1\ell_2 t} y_{ij}^{\ell_1\ell_2 t} \tag{32}$$

$$s.t. \quad \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{kt} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{kt} = b_i^k \quad \forall i\in\mathcal{N},\ \forall k\in\mathcal{K},\ \forall t\in\mathcal{T} \tag{33}$$

$$\sum_{k\in\mathcal{K}} d^{kt} x_{ij}^{kt} \le \sum_{\ell\in\mathcal{L}}\sum_{\ell_1\in\mathcal{L}} U_{ij}^{\ell} y_{ij}^{\ell_1\ell t} \quad \forall (i,j)\in\mathcal{A},\ \forall t\in\mathcal{T} \tag{34}$$

$$\sum_{\ell_1\in\mathcal{L}} y_{ij}^{\ell_1\ell(t-1)} = \sum_{\ell_2\in\mathcal{L}} y_{ij}^{\ell\ell_2 t} \quad \forall (i,j)\in\mathcal{A},\ \forall \ell\in\mathcal{L},\ \forall t\in\mathcal{T}^{\setminus\{1\}} \tag{35}$$

$$\sum_{\ell_2\in\mathcal{L}} y_{ij}^{0\ell_2 t} = 1, \quad \forall (i,j)\in\mathcal{A},\ t=1 \tag{36}$$

$$\sum_{\ell_2\in\mathcal{L}}\sum_{\ell_1\in\mathcal{L}} y_{ij}^{\ell_1\ell_2 t} = 1 \quad \forall (i,j)\in\mathcal{A},\ t=1 \tag{37}$$

$$0 \le x_{ij}^{kt} \le 1 \quad \forall (i,j)\in\mathcal{A},\ \forall k\in\mathcal{K},\ \forall t\in\mathcal{T}$$

$$y_{ij}^{\ell_1\ell_2 t} \in \{0,1\} \quad \forall (i,j)\in\mathcal{A},\ \forall \ell_1\in\mathcal{L},\ \forall \ell_2\in\mathcal{L},\ \forall t\in\mathcal{T}.$$

The corresponding strong linking constraints for the GMC-A formulation can be written as:

$$x_{ij}^{kt} \le \sum_{\ell_1\in\mathcal{L}}\sum_{\ell_2\in\mathcal{L}\setminus\{0\}} y_{ij}^{\ell_1\ell_2 t} \quad \forall (i,j)\in\mathcal{A},\ \forall k\in\mathcal{K},\ \forall t\in\mathcal{T}. \tag{38}$$

Note that we explicitly keep out $y$ variables for reducing an arc to capacity level 0 from the strong linking constraints because a commodity flow can never be routed through this arc.

# D Proof of Theorem 3: Dominance over alternative formulations

We here prove Theorem 3, i.e., the dominance of the main formulations over the alternative formulations. To be precise, we will prove the following three statements: (A) the SCI formulation (including strong linking constraints (9)) is stronger (strictly stronger for some instances) than the SCI-A formulation (including strong linking constraints (31)), (B) the SCI2 formulation (including strong linking constraints (9)) is stronger (strictly stronger for some instances) than the SCI2-A formulation (including strong linking constraints (31)), and (C) the GMC formulation (including strong linking constraints (17)) is stronger (strictly stronger for some instances) than the GMC-A formulation (including strong linking constraints (38)). To prove each statement, we will use the same technique as used for the dominance proof in Appendix B.

## D.1 Proof of statement (A)

In the first step, we prove that the SCI formulation is at least as strong as the SCI-A formulation with inequalities (31) by showing that from any solution $\{x_{ij}^{k\ell t}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ feasible in $\overline{\text{SCI}}$ (i.e., the LP relaxation of SCI), we can construct a solution $\{x_{ij}^{kt}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ that is feasible in $\overline{\text{SCI-A}}$ (i.e., the LP relaxation of SCI-A) and has the same objective function value. In the second step, we show that for a specific problem instance, the optimal solution provided by $\overline{\text{SCI}}$ is strictly superior to the one provided by $\overline{\text{SCI-A}}$.

### (a) Construction of feasible $\overline{\text{SCI-A}}$ solution from a $\overline{\text{SCI}}$ solution

Let $\{x_{ij}^{k\ell t}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ be any solution feasible in $\overline{\text{SCI}}$. We now construct an equivalent solution $\{x_{ij}^{kt}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ that is feasible in $\overline{\text{SCI-A}}$ (i.e., it satisfies all of its constraints) and has the same objective function value.

We first set the $y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}$ variables in the $\overline{\text{SCI-A}}$ solution to take, respectively, the same values as the $y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}$ variables in the $\overline{\text{SCI}}$ formulation. Given that constraints (4)-(8) are satisfied in the $\overline{\text{SCI}}$

formulation, the equivalent constraints (26)-(30) hold in the $\overline{\text{SCI-A}}$ formulation. The domain constraints for the $y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}$ variables also hold in the $\overline{\text{SCI-A}}$ formulation.

We then set the values of the $x_{ij}^{kt}$ variables based on the values of the $\overline{\text{SCI}}$ variables as follows:

$$x_{ij}^{kt} = \sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t} \qquad \forall (i,j) \in \mathcal{A}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T}. \tag{39}$$

Summing the strong linking constraints (9) over the capacity level set $\mathcal{L}$ and taking into consideration constraints (6) and equations (39) gives the following inequality:

$$0 \leq x_{ij}^{kt} = \sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t} \leq \sum_{\ell \in \mathcal{L}} y_{ij}^{\ell t} = 1 \;\; \forall (i,j) \in \mathcal{A}, \;\; \forall k \in \mathcal{K}, \;\; \forall t \in \mathcal{T}. \tag{40}$$

Inequality (40) guarantees that variables $x_{ij}^{kt}$ respect the lower bound of 0 and the upper bound of 1 in the domain constraints of the $\overline{\text{SCI-A}}$ formulation. Since constraints (2) of the $\overline{\text{SCI}}$ formulation hold for the $\overline{\text{SCI}}$ solution, it follows that constraints (24) of the $\overline{\text{SCI-A}}$ formulation also hold after replacing $\sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t}$ by $x_{ij}^{kt}$ according to equations (39).

Summing over constraints (3) over the capacity level set $\mathcal{L}$ and taking into consideration equations (39) gives inequality (41), demonstrating that constraints (25) of the $\overline{\text{SCI-A}}$ formulation are satisfied:

$$\sum_{k \in \mathcal{K}} d^{kt} x_{ij}^{kt} = \sum_{k \in \mathcal{K}} d^{kt} \sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t} \leq \sum_{\ell \in \mathcal{L}} U_{ij}^{\ell} y_{ij}^{\ell t} \;\; \forall (i,j) \in \mathcal{A}, \;\; \forall t \in \mathcal{T}. \tag{41}$$

Inequality (41) is also key to show that the strong linking constraints (31) are satisfied. Note that, for all feasible values of $y_{ij}^{\ell=0,t}$, inequality (41) forces $x_{ij}^{kt}$ to be zero, as the capacity at capacity level $\ell = 0$ $U_{ij}^{\ell}$ is always zero. Thus, $y_{ij}^{\ell=0,t}$ can be omitted from the rightmost side of inequality (40) to match the strong linking constraints (31), and it follows that the strong linking constraints (31) are satisfied.

Finally, it is straightforward to see that the value of objective function (23) of the $\overline{\text{SCI-A}}$ formulation is equal to the value of objective function (1) of the $\overline{\text{SCI}}$ formulation after replacing $x_{ij}^{kt}$ according to equations (39). Recall that the routing cost $C_{ij}^{k\ell}$ is assumed to not vary among the different capacity levels $\ell$ (i.e., $C_{ij}^{k\ell} = C_{ij}^{k} \; \forall \ell \in \mathcal{L}$). This concludes the proof that any solution feasible in $\overline{\text{SCI}}$ may be written as an equivalent solution feasible in $\overline{\text{SCI-A}}$ with the same objective function value.

### (b) Example of problem instance where $\overline{\text{SCI}}$ is strictly stronger than $\overline{\text{SCI-A}}$

As a consequence of the the previous step, the solution space of $\overline{\text{SCI-A}}$ is at least as large as the solution space of $\overline{\text{SCI}}$. $\overline{\text{SCI}}$ is therefore at least as strong as $\overline{\text{SCI-A}}$. However, for certain problem instances, $\overline{\text{SCI-A}}$ may therefore find solutions that are not part of the $\overline{\text{SCI}}$ solution space, and have a lower objective function value than any of the feasible $\overline{\text{SCI}}$ solutions, therefore providing bounds that are less tight in relation to the optimal integer solution of the problem. We now provide such a problem instance, which proves that the bound provided by $\overline{\text{SCI}}$ is strictly stronger than the bound provided by $\overline{\text{SCI-A}}$.

Consider a problem instance similar to the one presented in Appendix B. Specifically, the problem instance has a single time period $t_1$, a single commodity $k_1$, two nodes $n_1$ and $n_2$ and a single arc $(n_1, n_2)$, on which 15 units of commodity $k_1$ must be routed from $n_1$ to $n_2$. The arc has two capacity levels, i.e., therefore $\mathcal{L} = \{0, 1, 2\}$. Capacity levels 1 and 2 have capacities of $U_{ij}^{1} = 10$ and $U_{ij}^{2} = 20$, respectively. The expansion costs to construct such capacities are $\overline{f}_{n_1 n_2}^{\ell=1}$ =10\$ and $\overline{f}_{n_1 n_2}^{\ell=2}$ =15\$ for capacity levels 1 and 2, respectively. The maintenance costs to operate the arc throughout the time period are $F_{n_1 n_2}^{1}$ =10\$ and $F_{n_1 n_2}^{2}$ =20\$, respectively, for the two capacity levels. Routing costs and reduction costs are all set to 0.

For this problem instance, the optimal solution of $\overline{\text{SCI-A}}$ partially constructs the arc on capacity level 2 ($s_{n_1 n_2}^{\ell=0,t_1} = 0.25$ and $s_{n_1 n_2}^{\ell=2t_1} = 0.75$), which is the most economical decision for construction, and maintains capacity open half on capacity level 1 and half on capacity level 2 ($y_{n_1 n_2}^{\ell=1,t_1} = 0.5$ and $y_{n_1 n_2}^{\ell=2,t_1} = 0.5$), which is the most economical decision for maintenance while having enough capacity to route the flow and respecting the strong linking constraints (31) of the $\overline{\text{SCI-A}}$ formulation. The flow is routed via $x_{n_1 n_2}^{k_1,t_1} = 1.0$. Constructing partially on one capacity level and maintaining partially on two capacity levels allows the formulation to find a solution of low objective function value 26.25\$.

In contrast, the optimal solution of $\overline{\text{SCI}}$ constructs and maintains at capacity level 2 ($s_{n_1n_2}^{\ell=2,t_1} = 1.0$ and $y_{n_1n_2}^{\ell=2,t_1} = 1.0$), routing the flow completely through capacity level 2 with $x_{n_1n_2}^{k_1,\ell=2,t_1} = 1.0$. This solution has an objective function value of 35.00\$, which is also equivalent to the optimal integer solution.

Note that the dominance of the SCI formulation over the SCI-A formulation does neither exclusively rely on the capacity constraints separated by level (3) nor exclusively on the strong linking constraints (9), but rather the combination of both. In other words, the capacity constraints separated by level (3) and the strong linking constraints (9) impose a tighter relationship between the $x_{ij}^{k\ell t}$ and $y_{ij}^{\ell t}$ variables. In the following, we will give an intuition why this is the case. The $\overline{\text{SCI}}$ formulation for this problem instance has the following form after (I) removing variables that must be equal to zero or have trivial values, as well as redundant constraints, and (II) omitting the $i$, $j$, $k$, and $t$ indexes of the $x_{ij}^{k\ell t}$, $y_{ij}^{\ell t}$, and $s_{ij}^{\ell t}$ variables as there is only one arc $(n_1, n_2)$, one commodity $k_1$, and one time period $t_1$:

$$
\begin{aligned}
\min \quad & 10s^1 + 15s^2 + 10y^1 + 20y^2 \\
s.t. \quad & x^0 + x^1 + x^2 = 1 \\
& 15x^0 \le 0y^0 \\
& 15x^1 \le 10y^1 \\
& 15x^2 \le 20y^2 \\
& y^0 + y^1 + y^2 = 1 \\
& s^0 + s^1 + s^2 = 1 \\
& y^1 + 2y^2 = s^1 + 2s^2 \\
& x^0 \le y^0 \\
& x^1 \le y^1 \\
& x^2 \le y^2 \\
& 0 \le x^\ell \le 1 \quad \forall \ell \in \mathcal{L} \\
& 0 \le y^\ell \le 1 \quad \forall \ell \in \mathcal{L} \\
& 0 \le s^\ell \le 1 \quad \forall \ell \in \mathcal{L}.
\end{aligned}
$$

Given that variables $s^1$, $s^2$, $y^1$ and $y^2$ have a positive coefficient in the objective function and this is a minimization problem, the optimal solution takes the lowest values for variables $s^1$, $s^2$, $y^1$ and $y^2$ in the feasible region. The lower bound of variable $y^1$ is $\frac{3}{2}x^1$, which comes from the capacity constraint, whereas the lower bound of variable $y^2$ is $1x^2$, which comes from the strong linking constraint. On the other hand, the values of variables $s^1$ and $s^2$ are regulated by the equality constraints. Note that variable $x^0$ must be equal to zero due to constraint $15x^0 \le 0y^0$.

Since the pair of variables $(x^0, y^0)$, $(x^1, y^1)$ $(x^2, y^2)$ only interact with each other in the feasible region with the exception of four equality constraints, it is possible to partially rewrite the feasible region of the $\overline{\text{SCI}}$ formulation as a system of equations, taking into consideration that variables $y^1$ and $y^2$ must be at their lower bound as previously explained:

$$\begin{cases} x^0 + x^1 + x^2 = 1 \\ y^0 + y^1 + y^2 = 1 \\ s^0 + s^1 + s^2 = 1 \\ y^1 + 2y^2 - s^1 - 2s^2 = 0 \end{cases} \rightarrow \begin{cases} x^1 + x^2 = 1 \\ y^0 + \frac{3}{2}x^1 + x^2 = 1 \\ s^0 + s^1 + s^2 = 1 \\ \frac{3}{2}x^1 + 2x^2 - s^1 - 2s^2 = 0 \end{cases} \rightarrow \begin{cases} y^0 = -2 + \alpha + 2\beta \\ x^1 = 4 - 2\alpha - 4\beta \\ x^2 = -3 + 2\alpha + 4\beta \\ s^0 = 1 - \alpha - \beta \\ s^1 = \alpha \\ s^2 = \beta \end{cases} \rightarrow$$

$$\begin{cases} y^0 = -2 + \alpha + 2\beta \\ y^1 = \frac{3}{2}(4 - 2\alpha - 4\beta) \\ y^2 = -3 + 2\alpha + 4\beta \\ s^0 = 1 - \alpha - \beta \\ s^1 = \alpha \\ s^2 = \beta \end{cases}$$

Only $\alpha = 0$ and $\beta = 1$ provides feasible values for variables $s^0 = 0$, $s^1 = 0$, $s^2 = 1$, $y^0 = 0$, $y^1 = 0$ and $y^2 = 1$ in terms of domain constraints, which results in the solution presented earlier.

The two steps (a) and (b) above prove statement (A) of Theorem 3, validating that the SCI formulation dominates the SCI-A formulation in terms of LP relaxation strength.

## D.2 Proof of statement (B)

In the first step, we prove that the SCI2 formulation is at least as strong as the SCI2-A formulation by showing that from any solution $\{x_{ij}^{k\ell t}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ feasible in $\overline{\text{SCI2}}$ (i.e., the LP relaxation of SCI2), we can construct a solution $\{x_{ij}^{kt}, y_{ij}^{\ell t}, s_{ij}^{\ell t}, w_{ij}^{\ell t}\}$ that is feasible in $\overline{\text{SCI2-A}}$ (i.e., the LP relaxation of SCI2-A) and has the same objective function value. In the second step, we show that for a specific problem instance, the optimal solution provided by $\overline{\text{SCI2}}$ is strictly superior to the one provided by $\overline{\text{SCI2-A}}$.

The SCI2 formulation differs from the SCI formulation only by one constraint that does not contain flow variables. Therefore, the first step to prove statement (B) follows trivially from step (a) to prove statement (A). For the same problem instance presented for the proof of statement (A), the optimal solution of the $\overline{\text{SCI2}}$ formulation is the same as the solution of the $\overline{\text{SCI}}$ formulation, and the optimal solution of the $\overline{\text{SCI2-A}}$ formulation is the same as the solution of the $\overline{\text{SCI-A}}$ formulation. Therefore, the second step to prove statement (B) also follows trivially from step (b) to prove statement (A). Thus, statement (B) of Theorem 3 holds, validating that the SCI2 formulation dominates the SCI2-A formulation in terms of LP relaxation strength.

## D.3 Proof of statement (C)

In the first step, we prove that the GMC formulation is at least as strong as the GMC-A formulation by showing that from any solution $\{x_{ij}^{k\ell t}, y_{ij}^{\ell_1\ell_2 t}\}$ feasible in $\overline{\text{GMC}}$ (i.e., the LP relaxation of GMC), we can construct a solution $\{x_{ij}^{kt}, y_{ij}^{\ell_1\ell_2 t}\}$ that is feasible in $\overline{\text{GMC-A}}$ (i.e., the LP relaxation of GMC-A) and has the same objective function value. In the second step, we show that for a specific problem instance, the optimal solution provided by $\overline{\text{GMC}}$ is strictly superior to the one provided by $\overline{\text{GMC-A}}$.

### (a) Construction of feasible $\overline{\text{GMC-A}}$ solution from a $\overline{\text{GMC}}$ solution

Let $\{x_{ij}^{k\ell t}, y_{ij}^{\ell_1\ell_2 t}\}$ be any solution feasible in $\overline{\text{GMC}}$. We now construct an equivalent solution $\{x_{ij}^{kt}, y_{ij}^{\ell_1\ell_2 t}\}$ that is feasible in $\overline{\text{GMC-A}}$ (i.e., it satisfies all of its constraints) and has the same objective function value.

We first set the $y_{ij}^{\ell_1\ell_2 t}$ variables in the $\overline{\text{GMC-A}}$ solution to take the same values as the $y_{ij}^{\ell_1\ell_2 t}$ variables in the $\overline{\text{GMC}}$ formulation. Given that constraints (14)-(16) are satisfied in the $\overline{\text{GMC}}$ formulation, the equivalent constraints (35)-(37) hold in the $\overline{\text{GMC-A}}$ formulation. By construction, the domain constraints for the $y_{ij}^{\ell_1\ell_2 t}$ variables also hold in the $\overline{\text{GMC-A}}$ formulation.

We then set the values of the $x_{ij}^{kt}$ variables based on the values of the $\overline{\text{GMC}}$ variables as follows:

$$x_{ij}^{kt} = \sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t} \qquad \forall (i,j) \in \mathcal{A}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T}. \tag{42}$$

Summing over the strong linking constraints (17) over the capacity level set $\mathcal{L}$ and taking into consideration constraints (14)-(16) and equations (42) gives the following inequality:

$$0 \leq x_{ij}^{kt} = \sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t} \leq \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} y_{ij}^{\ell_1 \ell t} = 1 \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall k \in \mathcal{K}, \ \ \forall t \in \mathcal{T}. \tag{43}$$

Inequality (43) guarantees that variables $x_{ij}^{kt}$ respect the lower bound of 0 and the upper bound of 1 in the domain constraints of the $\overline{\text{GMC-A}}$ formulation. Since constraints (12) of the $\overline{\text{GMC}}$ formulation hold for the $\overline{\text{GMC}}$ solution, it follows that constraints (33) of the $\overline{\text{GMC-A}}$ formulation also hold after replacing $\sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t}$ by $x_{ij}^{kt}$ according to equations (42).

Summing over constraints (13) over the capacity level set $\mathcal{L}$, which can be done because $x_{ij}^{k\ell t} \geq 0$, $y_{ij}^{\ell_1 \ell t} \geq 0$, $U_{ij}^{\ell} \geq 0$ and $d^{kt} \geq 0$ in the $\overline{\text{GMC}}$ formulation, and taking into consideration equations (42) gives inequality (44), proving that constraints (34) of the $\overline{\text{GMC-A}}$ are satisfied:

$$\sum_{k \in \mathcal{K}} d^{kt} x_{ij}^{kt} = \sum_{k \in \mathcal{K}} d^{kt} \sum_{\ell \in \mathcal{L}} x_{ij}^{k\ell t} \leq \sum_{\ell \in \mathcal{L}} \sum_{\ell_1 \in \mathcal{L}} U_{ij}^{\ell} y_{ij}^{\ell_1 \ell t} \ \ \forall (i,j) \in \mathcal{A}, \ \ \forall t \in \mathcal{T}. \tag{44}$$

Again, inequality (44) is also key to show that the strong linking constraints (38) are satisfied. Note that, for all feasible values of $y_{ij}^{\ell_1, \ell=0, t} \, \forall \ell_1 \in \mathcal{L}$, inequality (44) forces $x_{ij}^{kt}$ to be zero, as the capacity at capacity level $\ell = 0$ $U_{ij}^{\ell}$ is always zero. Thus, $y_{ij}^{\ell_1, \ell=0, t} \, \forall \ell_1 \in \mathcal{L}$ can be omitted from the rightmost side of inequality (43) to match the strong linking constraints (38), and it follows that the strong linking constraints (38) are satisfied.

Finally, it is straightforward to see that the objective function (32) of the $\overline{\text{GMC-A}}$ formulation is equal to the objective function (11) of the $\overline{\text{GMC}}$ formulation after replacing $x_{ij}^{kt}$ according to equations (42). Recall that the routing cost $C_{ij}^{k\ell}$ is assumed to not vary among the different capacity levels $\ell$ (i.e., $C_{ij}^{k\ell} = C_{ij}^{k} \, \forall \ell \in \mathcal{L}$). This concludes the proof that any solution feasible in $\overline{\text{GMC}}$ may be written as an equivalent solution feasible in $\overline{\text{GMC-A}}$ with the same objective function value.

### (b) Example of problem instance where $\overline{\text{GMC}}$ is strictly stronger than $\overline{\text{GMC-A}}$

As a consequence of the the previous step, the solution space of $\overline{\text{GMC-A}}$ is at least as large as the solution space of $\overline{\text{GMC}}$. $\overline{\text{GMC}}$ is therefore at least as strong as $\overline{\text{GMC-A}}$. However, for certain problem instances, $\overline{\text{GMC-A}}$ may therefore find solutions that are not part of the $\overline{\text{GMC}}$ solution space, and have a lower objective function value than any of the feasible $\overline{\text{GMC}}$ solutions, therefore providing bounds that are less tight in relation to the optimal integer solution of the problem. We now provide such a problem instance, which proves that the bound provided by $\overline{\text{GMC}}$ is stronger than the bound provided by $\overline{\text{GMC-A}}$.

Consider the same problem instance presented for the proof of statement (A). For this problem instance, the optimal solution of $\overline{\text{GMC-A}}$ constructs the arc half on capacity level 1 and half on capacity level 2 ($y_{n_1 n_2}^{\ell_1=0, \ell_2=1, t_1} = 0.5$ and $y_{n_1 n_2}^{\ell_1=0, \ell_2=2, t_1} = 0.5$), which is the most economical decision for construction and maintenance while providing enough capacity to route the flow and respecting the strong linking constraints (38) of the $\overline{\text{GMC-A}}$ formulation. The flow is routed via $x_{n_1 n_2}^{k_1, t_1} = 1.0$. Constructing and maintaining partially on two capacity levels allow the formulation to find a solution of low objective function value 27.50$.

In contrast, the optimal solution of $\overline{\text{GMC}}$ constructs and maintains the arc at capacity level 2 ($y_{n_1 n_2}^{\ell_1=0, \ell_2=2, t_1} = 1.0$), routing the flow completely through capacity level 2 with $x_{n_1 n_2}^{k_1, \ell=2, t_1} = 1.0$. This solution has an objective function value of 35.00$, which is also equivalent to the optimal integer solution. As it was the case for the SCI and SCI-A formulations, the dominance of the GMC formulation over the GMC-A formulation does neither exclusively rely on the capacity constraints separated by level (13) nor exclusively on the strong linking constraints (17), but the combination of both. In other words, the combination of capacity constraints separated by level (13) and the strong linking constraints (17) impose a tighter relationship between the $x_{ij}^{k\ell t}$

and $y_{ij}^{\ell_1\ell_2t}$ variables. In the following, we provide an intuition why this is the case. The $\overline{\text{GMC}}$ formulation for this problem instance has the following form after (I) removing variables that must be equal to zero and redundant constraints, and (II) omitting the $i$, $j$, $k$, and $t$ indexes of the $x_{ij}^{k\ell t}$ and $y_{ij}^{\ell_1\ell_2t}$ variables as there is only one arc $(n_1, n_2)$, one commodity $k_1$, and one time period $t_1$:

$$\begin{aligned}
\min \quad & 20y^{01} + 35y^{02} \\
s.t. \quad & x^0 + x^1 + x^2 = 1 \\
& 15x^0 \leq 0y^{00} \\
& 15x^1 \leq 10y^{01} \\
& 15x^2 \leq 20y^{02} \\
& y^{00} + y^{01} + y^{02} = 1 \\
& x^0 \leq y^{00} \\
& x^1 \leq y^{01} \\
& x^2 \leq y^{02} \\
& 0 \leq x^\ell \leq 1 \quad \forall \ell \in \mathcal{L} \\
& 0 \leq y^{0\ell} \leq 1 \quad \forall \ell \in \mathcal{L}
\end{aligned}$$

Given that variables $y^{01}$ and $y^{02}$ have a positive coefficient in the objective function and this is a minimization problem, the optimal solution takes the lowest values for variables $y^{01}$ and $y^{02}$ in the feasible region. The lower bound of variable $y^{01}$ is $\frac{3}{2}x^1$, which comes from the capacity constraint, whereas the lower bound of variable $y^{02}$ is $1x^2$, which comes from the strong linking constraint. Note that variable $x^0$ must be equal to zero due to constraint $15x^0 \leq 0y^{00}$.

Since the pair of variables $(x^0, y^{00})$, $(x^1, y^{01})$ $(x^2, y^{02})$ only interact with each other in the feasible region with the exception of two equality constraints, it is possible to partially rewrite the feasible region of the $\overline{\text{GMC}}$ formulation as a system of equations, taking into consideration that variables $y^{01}$ and $y^{02}$ must be at their lower bound as previously explained:

$$\begin{cases} x^0 + x^1 + x^2 = 1 \\ y^{00} + y^{01} + y^{02} = 1 \end{cases} \rightarrow \begin{cases} x^1 + x^2 = 1 \\ y^{00} + \frac{3}{2}x^1 + x^2 = 1 \end{cases} \rightarrow \begin{cases} y^{00} = -\frac{1}{2} + \frac{1}{2}\lambda \\ x^1 = 1 - \lambda \\ x^2 = \lambda \end{cases} \rightarrow \begin{cases} y^{00} = -\frac{1}{2} + \frac{1}{2}\lambda \\ y^{01} = \frac{3}{2}(1 - \lambda) \\ y^{02} = \lambda \end{cases}$$

Trivially, only $\lambda = 1$ provides feasible values for variables $y^{00} = 0$, $y^{01} = 0$ and $y^{02} = 1$ in terms of domain constraints, which results in the solution presented above.

The two steps (a) and (b) above prove statement (C) of Theorem 3, validating that the GMC formulation dominates the GMC-A formulation in terms of LP relaxation strength.

# E Generation of Problem Instances

This appendix presents the instance generation procedure employed to adapt real-world networks from Pazour et al. (2010) to a multi-period setting with problem-specific features. This instance generation procedure is similar to the one used in Fragkos et al. (2021), which has been inspired by the one presented in Crainic et al. (2001).

Pazour et al. (2010) generate problem instances representing continental high-speed networks for freight transportation. They generate four networks. The JBH50 and JBH98 networks are based on data provided by the largest truckload carrier in the USA, J.B. Hunt Transport Services Inc., whereas the USC30 and USC53 networks are based on data from the 2002 Commodity Flow Survey. The resulting networks are reported in the appendix of Pazour (2008). Details on how the original data is transformed into the final networks can be found in Pazour (2008) and Pazour et al. (2010).

Each network is composed of a set of arcs and a set of commodities. Each arc has an origin node, a destination node, and an euclidean distance. Each commodity has an origin node, a destination node, and the commodity demand value (in number of units to be sent). Following the instance generation procedure proposed by Fragkos et al. (2021), we only consider networks JBH50, USC30 and USC53. In the same spirit, we also decrease the size of the three networks by considering only a subset of commodities per network. The subset of commodities per network has been chosen by selecting commodities with largest demand values until reaching a certain percentage of the original total demand. The reduced networks therefore have a smaller number of commodities, but they still account for most of the demand in the original networks. Also note that the USC30 network has five redundant arcs, which have been manually removed from the network. The resulting reduced networks are referred to with a "red"-suffix added to their name. The JBH50red network has a subset of commodities representing 79% of the original total demand, the USC30red network has a subset of commodities representing 85% of the original total demand, and the USC53red networks has a subset of commodities representing 80% of the original total demand.

We use the resulting JBH50red, USC30red, and USC53red networks to generate the problem instances for our experiments. The instance generation procedure has three main steps: the adaptation of real-world networks to generate single-period instances (Appendix E.1), the expansion of single-period instances to a multi-period setting (Appendix E.2) and, finally, the adjustments to generate features that are specific to the MPND-CA (Appendix E.3).

## E.1 Single-period adaptation

This section presents the adaptation of the real-world networks to generate single-period instances. In its original format, the JBH50red, USC30red, and USC53red networks have the following input data. Each arc $(i, j) \in \mathcal{A}$ has an euclidean distance $w_{ij}$, and each commodity $k \in \mathcal{K}$ has an origin node $o(k)$, a destination node $d(k)$ and a demand value $d^k$ (in number of demand units). The instance generation procedure is given by the following steps for each of the three networks:

Step 1 Store origin $i$, destination $j$ and distance $w_{ij}$ of each arc $(i, j) \in \mathcal{A}$.

Step 2 Calculate a routing cost $c_{ij}$ for each arc $(i, j) \in \mathcal{A}$ by multiplying distance $w_{ij}$ by a cost per mile constant (0.3665\$/mile [3]) that can change according to the application.

Step 3 Store origin $o(k)$, destination $d(k)$, and demand value $d^k$ of each commodity $k \in \mathcal{K}$.

Step 4 Calculate a total demand indicator $D = \sum_{k \in \mathcal{K}} d_k$ based on the demand values.

Step 5 Set loose or tight capacity values for each arc $(i, j) \in \mathcal{A}$ by drawing a capacity value $u_{ij}$ from a random distribution $\mathcal{U}[0.5 \cdot D/C, 1.5 \cdot D/C]$, where $C \in \{1, 8\}$ represents respectively loose or tight capacities and $D$ is the total demand indicator.

Step 6 Set low or high fixed costs for each arc $(i, j) \in \mathcal{A}$ by drawing a fixed cost $f_{ij}$ from a random distribution $\mathcal{U}[0.5 \cdot F \cdot D \cdot c_{ij}, 1.5 \cdot F \cdot D \cdot c_{ij}]$, where $F \in \{0.01, 0.1\}$ represents, respectively, low or high fixed costs and $c_{ij}$ and $D$ are as defined above.

Step 7 Update the previously calculated routing cost $c_{ij}$ and the previously drawn fixed cost $f_{ij}$ for each arc $(i, j) \in \mathcal{A}$ following two different approaches: in the first approach (Euclidean), the previously calculated routing cost $c_{ij}$ and the previously drawn fixed cost $f_{ij}$ for each arc $(i, j) \in \mathcal{A}$ are maintained and therefore consider the Euclidean distance between origin and destination; in the second approach (random), previously calculated routing costs and previously drawn fixed costs are randomly shuffled among arcs to decouple distances from transportation costs.

Step 8 Update the previously drawn fixed costs to force a negative correlation between fixed costs and routing costs. If the correlation between fixed costs and routing costs is greater than $-0.5$, the arc $(i, j)$ with the $n$-th highest fixed cost $f_{ij}$ receives the fixed cost of the arc with the $n$-th lowest routing cost

[3]The cost per mile has been obtained from http://www.rtsfinancial.com/guides/trucking-calculations-formulas.

$c_{ij}$. Then, the fixed costs are shuffled among a subset of arcs to avoid having a too strong negative correlation. This subset of arcs is randomly chosen among all arcs, and has size $0.10 \cdot |\mathcal{A}|$ for instances with Euclidean routing costs and $0.33 \cdot |\mathcal{A}|$ for instances with random routing costs.

The methodology to build loose or tight capacity values, as well as low or high fixed costs, comes from Crainic et al. (2001). Note that the extensions described in this section are identical to those proposed by Fragkos et al. (2021). The latter, however, additionally consider medium capacity values, medium fixed costs, mixed routing costs and random correlation (i.e., no forced negative correlation between fixed and routing costs) for the generation of single-period instances. For the generation of problem instances specific to the MPND-CA, we did not consider those four additional attribute values in order to have a set of problem instances of reasonable size. Given the two types of capacity values, two types of fixed costs, and two types of routing costs, each real-world network generates 8 different single-period instances, resulting in a total of $3 \cdot 8 = 24$ single-period instances.

## E.2 Multi-period extension

This section presents the extension of single-period instances to a multi-period setting. The extension to multiple time periods involves defining demand values $d^{kt}$ of a commodity $k$ for the different time periods $t$, based on the demand value $d^k$ of commodity $k$ in the single-period instance.

We consider two different demand patterns. The first demand pattern is an increasing demand over the planning horizon with small perturbations and is identical to the demand pattern proposed by Fragkos et al. (2021). Here, the generalized logistic function (Richards, 1959) is used to obtain demand values for all time periods. The demand values $d^{kt}$ are calculated as follows:

$$d^{kt} = (1 + r^{kt}) \left( \lambda + \frac{\mu - \lambda}{(1 + e^{-\beta(t - \frac{T}{2})})^{\frac{1}{v}}} \right) d^k, \forall k \in \mathcal{K}, t \in \mathcal{T},$$

where $r^{kt}$ is a perturbation coefficient, $\lambda$ is the minimum asymptotic demand, $\mu$ is the maximum asymptotic demand, $\beta$ is the growth rate, and $v$ is a location parameter. Fragkos et al. (2021) randomly draw $r^{kt}$ from the uniform distribution $\mathcal{U}[-0.1, 0.1]$, fix $\mu = 0.3$, $\beta = 0.4$, and $v = 1$, and choose different values of $\lambda$ for each size $T$ of the planning horizon such that the average demand value over all time periods is close to the demand value $d^k$ used in the single-period instance. Specifically, Fragkos et al. (2021) select $\lambda = 1.583$ for $T = 5$, $\lambda = 1.601$ for $T = 10$, $\lambda = 1.621$ for $T = 15$, and $\lambda = 1.636$ for $T = 20$.

The second demand pattern is a random demand evolution over time. To this end, we draw, for each commodity $k \in \mathcal{K}$ and time period $t \in \mathcal{T}$, a demand value $d^{kt}$ from the normal distribution $N(d^k, 0.5 \cdot d^k)$, where $d^k$ is the demand value in the single-period instance.

Given the two types of demand expansion, each single-period instance generates 2 multi-period instances with different demand patterns. Therefore, there are a total of $2 \cdot 24 = 48$ multi-period instances for each length $T$ of the planning horizon. Note that, with the exception of the demand value, all other data from the single-period instances are used without further modifications in the different time periods of the multi-period instances.

## E.3 Problem-specific adjustments

This section presents the adjustments of the previously defined multi-period instances to obtain all features required for the MPND-CA. Specifically, we extend the multi-period instances by generating (a) capacity values for each capacity level based on the previously stored capacity value $u_{ij}$ for each arc $(i, j) \in \mathcal{A}$, and (b) costs for routing, maintenance, capacity expansion, and capacity reduction at different capacity levels based on the previously stored routing cost $c_{ij}$ and the previously stored fixed cost $f_{ij}$ for each arc $(i, j) \in \mathcal{A}$.

We first define the capacity values for the different capacity levels. The capacity value of level 0 is zero:

$$U^0_{ij} = 0 \ \ \forall (i, j) \in \mathcal{A}.$$

Let $q$ be the number of capacity levels. Each next capacity level adds $g_{ij}$ units of additional capacity, defined as:

$$g_{ij} = \frac{u_{ij}}{q} \ \ \forall (i, j) \in \mathcal{A}.$$

The capacity value $U_{ij}^{\ell} \in \{1, \ldots, q\}$ of an arc $(i, j) \in \mathcal{A}$ open at capacity level $\ell$ is therefore:

$$U_{ij}^{\ell} = \ell \cdot g_{ij}.$$

We next generate the costs for routing, maintenance, capacity expansion and capacity reduction at the different capacity levels based on routing costs $c_{ij}$, and fixed costs $f_{ij}$.

We define the routing cost $C_{ij}^{k\ell}$ for commodity $k$ on arc $(i, j)$ at level $\ell$ as the same as routing cost $c_{ij}$ of arc $(i, j)$, i.e., the routing cost $C_{ij}^{k\ell}$ does not vary among commodities or capacity levels. Note that the presented formulations can easily accommodate routing costs that depend on time period $t$. This may be the case for seasonal variations of the costs.

We set the costs to maintain an arc at capacity level 0, as well as the costs to expand or reduce capacity by 0 levels, to zero:

$$F_{ij}^{0} = \overline{f}_{ij}^{0} = \underline{f}_{ij}^{0} = 0 \qquad \forall (i, j) \in \mathcal{A}.$$

We now define the costs to maintain an arc capacity at level 1 and the costs to expand and to reduce capacity by one capacity level. Let $A_{ij} = 1.9 \cdot f_{ij}$ be an auxiliary cost defined for each arc $(i.j) \in \mathcal{A}$, representing the cost of opening arc $(i, j)$ in the first time period and maintaining it open until the end of the planning horizon (as defined by Fragkos et al., 2021). We split the auxiliary cost $A_{ij}$ into two different cost parameters (maintenance and expansion costs) and distribute them over the planning horizon. Let $P_C^E$ be a scale factor that represents the costs of expanding one capacity level in respect to the costs of maintaining one capacity level. The costs to maintain an arc $(i, j)$ open at capacity level $\ell = 1$ for one time period is then computed as:

$$F_{ij}^{1} = \frac{A_{ij}}{|T| + P_C^E} \cdot \frac{1}{q} \ \ \forall (i, j) \in \mathcal{A}.$$

As an example, consider the setting with a single capacity level greater than zero (i.e., $q = 1$), such as it is the case in Fragkos et al. (2021). Isolating $A_{ij}$ on the left-hand side gives $A_{ij} = P_C^E \cdot F_{ij}^1 + |T| \cdot F_{ij}^1$, which highlights how the auxiliary cost comprises both the costs to construct arc $(i, j)$ (i.e., $P_C^E \cdot F_{ij}^1$) and the maintenance cost throughout the entire planning horizon (i.e., $|T| \cdot F_{ij}^1$).

The cost to expand the capacity of an arc $(i, j)$ by one capacity level is given as:

$$\overline{f}_{ij}^{1} = P_C^E \cdot F_{ij}^{1} \ \ \forall (i, j) \in \mathcal{A}.$$

Next, we define the capacity reduction costs as a fraction of the capacity expansion costs. Thus, let $P_C^R$ be the scale factor that represents the costs of reducing one capacity level in respect to the costs of expanding one capacity level. The costs to reduce the capacity of an arc by one capacity level is:

$$\underline{f}_{ij}^{1} = P_C^R \cdot \overline{f}_{ij}^{1} \ \ \forall (i, j) \in \mathcal{A}.$$

Finally, we compute the costs to maintain arc capacity at a level greater than 1, and the costs to expand or reduce capacity by more than one capacity level based on the corresponding costs for capacity level $\ell = 1$. In order to enable the representation of economies of scale, we define $P_{ES}^M$, $P_{ES}^E$ and $P_{ES}^R$ as the scale factors that represent the economies of scale when operating on higher levels for capacity maintenance, capacity expansion and capacity reduction. The costs to maintain an arc open at a capacity level $\ell \in \{2, \cdots, q\}$ and to increase or reduce the capacity at an arc by $\ell$ capacity levels, where $\ell \in \{2, \cdots, q\}$, are given by:

$$F_{ij}^{\ell} = F_{ij}^{(\ell-1)} + P_{ES}^M \cdot (F_{ij}^{(\ell-1)} - F_{ij}^{(\ell-2)}) \ \ \forall \ell \in \{2, \cdots, q\} \ \ \forall (i, j) \in \mathcal{A},$$

$$\overline{f}_{ij}^{\ell} = \overline{f}_{ij}^{(\ell-1)} + P_{ES}^E \cdot (\overline{f}_{ij}^{(\ell-1)} - \overline{f}_{ij}^{(\ell-2)}) \ \ \forall \ell \in \{2, \cdots, q\} \ \ \forall (i, j) \in \mathcal{A},$$

and

$$\underline{f}_{ij}^{\ell} = \underline{f}_{ij}^{(\ell-1)} + P_{ES}^R \cdot (\underline{f}_{ij}^{(\ell-1)} - \underline{f}_{ij}^{(\ell-2)}) \ \ \forall \ell \in \{2, \cdots, q\} \ \ \forall (i, j) \in \mathcal{A}.$$

The problem-specific parameters here used have the following values. Parameter $P_C^R = 0.1$ (i.e., reducing capacity is much cheaper than expanding capacity) has been fixed for all problem instances. We consider two different numbers of capacity levels $q \in \{3, 5\}$ and two different correlations between maintenance and expansion costs $P_C^E \in \{1.0, 5.0\}$, i.e., the cost to expand capacity is, respectively, the same or five times as expensive as maintaining a capacity open during one time period. We also consider two settings for economies of scale within maintenance, expansion and reduction costs: positive economies of scale ($P_{ES}^M = 0.85$, $P_{ES}^E = 0.85$, $P_{ES}^R = 0.85$) and inverse economies of scale ($P_{ES}^M = 1.15$, $P_{ES}^E = 0.85$ and $P_{ES}^R = 0.85$).

We consider all possible combinations of the values for the parameters described above. Therefore, each multi-period instance as defined in the previous section can be equipped with 8 different sets of problem-specific parameter values. For the MPND-CA, we therefore have a total of $8 \cdot 48 = 384$ problem instances for each size $T$ of the planning horizon, which have been used in the computational experiments.

# F Computational Results for the Problem Variant with Unsplittable Flows

This appendix presents the detailed results for the problem variant with unsplittable commodity flows, i.e., the flow to route a commodity from its origin to its destination has to be routed on the same path. This path may be different in each time period.

Tables 8-11 present the results for the same experiments previously explored for the problem variant with splittable flows. Specifically, Table 8 indicates that the integrality gaps for this problem variant tend to be significantly higher, which suggests that the corresponding mixed-integer programming formulations are more difficult to solve. Table 9, however, shows that the computing times to solve the models remain relatively unchanged, and are actually slightly faster than before for the two SCI formulations. These conclusions are supported by the results in Table 10, where all three formulations have a similar number of instances for which no feasible solution has been found, and similar optimality gaps. It therefore appears that, under CPLEX default settings, the solver has effective resources (i.e., heuristics and cut generation) to compensate for the difficulty of the problem during the optimization process. Finally, this suspicion seems to be confirmed by the results based on an emulated basic branch-and-cut environment in Table 11. Without pre-processing, heuristics and cut-generation, the SCI formulation cannot find feasible solutions for more than 200 of the 267 instances, and require much more computing time to solve the problem. This suggests, once again, that the strength of the GMC formulation may be an important consideration when using less advanced general purpose MIP solvers.

| Description | | | GMC | | SCI | | SCI2 | |
|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | Avg. LPR time (min.) | Avg. int. gap | Avg. LPR time (min.) | Avg. int. gap | Avg. LPR time (min.) | Avg. int. gap |
| Network | JBH50red | 71 | 13.0 | 0.00% | 10.2 | 0.42% | 8.0 | 0.42% |
| | USC30red | 116 | 0.4 | 0.97% | 0.2 | 1.61% | 0.2 | 1.61% |
| | USC53red | 80 | 9.3 | 0.09% | 9.0 | 0.63% | 4.8 | 0.63% |
| $T$ | 10 | 267 | 6.4 | 0.45% | 5.5 | 1.00% | 3.7 | 1.00% |
| $q$ | 3 | 133 | 7.8 | 0.44% | 7.1 | 0.87% | 4.7 | 0.87% |
| | 5 | 134 | 5.1 | 0.47% | 4.0 | 1.13% | 2.7 | 1.13% |
| Fixed costs | Low | 167 | 1.1 | 0.27% | 0.8 | 0.48% | 0.8 | 0.48% |
| | High | 100 | 15.3 | 0.75% | 13.4 | 1.86% | 8.5 | 1.86% |
| Capacities | Loose | 176 | 9.0 | 0.02% | 8.0 | 0.60% | 5.2 | 0.60% |
| | Tight | 91 | 1.5 | 1.28% | 0.8 | 1.78% | 0.8 | 1.78% |
| Routing costs | Euclidean | 135 | 8.2 | 0.58% | 5.7 | 1.14% | 4.2 | 1.14% |
| | Random | 132 | 4.6 | 0.32% | 5.3 | 0.86% | 3.2 | 0.86% |
| Demand behavior | Increasing | 128 | 8.8 | 0.35% | 7.5 | 0.88% | 4.5 | 0.88% |
| | Random | 139 | 4.3 | 0.54% | 3.7 | 1.11% | 2.9 | 1.11% |
| $P_E^C$ | 1 | 128 | 4.5 | 0.41% | 3.5 | 0.64% | 3.2 | 0.64% |
| | 5 | 139 | 8.2 | 0.49% | 7.4 | 1.33% | 4.1 | 1.33% |
| Economies of scale | Inverse | 132 | 5.9 | 0.49% | 5.3 | 1.05% | 3.9 | 1.05% |
| | Positive | 135 | 6.9 | 0.41% | 5.8 | 0.95% | 3.4 | 0.95% |
| All | | 267 | 6.4 | 0.45% | 5.5 | 1.00% | 3.7 | 1.00% |

Table 8: Average LPR solution time and integrality gaps of the formulations for instances for which the optimal integer solution is known (problem variant with unsplittable flows).

| Description | | | GMC | SCI | SCI2 |
|---|---|---|---|---|---|
| Attribute | Value | # | Avg. MIP time (min.) | Avg. MIP time (min.) | Avg. MIP time (min.) |
| Network | JBH50red | 66 | 23.4 | 30.6 | 24.2 |
| | USC30red | 115 | 20.9 | 28.5 | 28.4 |
| | USC53red | 73 | 27.6 | 44.6 | 41.9 |
| $T$ | 10 | 254 | 23.5 | 33.7 | 31.2 |
| $q$ | 3 | 127 | 18.4 | 24.3 | 20.6 |
| | 5 | 127 | 28.6 | 43.0 | 41.7 |
| Fixed costs | Low | 164 | 16.9 | 19.2 | 20.5 |
| | High | 90 | 35.5 | 60.0 | 50.5 |
| Capacities | Loose | 167 | 8.8 | 17.6 | 12.8 |
| | Tight | 87 | 51.6 | 64.5 | 66.5 |
| Routing costs | Euclidean | 131 | 22.4 | 29.9 | 25.2 |
| | Random | 123 | 24.7 | 37.7 | 37.5 |
| Demand behavior | Increasing | 121 | 23.6 | 33.5 | 26.6 |
| | Random | 133 | 23.4 | 33.8 | 35.3 |
| $P_E^C$ | 1 | 125 | 22.5 | 24.7 | 29.7 |
| | 5 | 129 | 24.5 | 42.4 | 32.5 |
| Economies of scale | Inverse | 125 | 29.1 | 39.1 | 33.1 |
| | Positive | 129 | 18.1 | 28.4 | 29.3 |
| All | | 254 | 23.5 | 33.7 | 31.2 |

Table 9: Average MIP solution time (CPLEX default settings) for instances that have been solved by all three formulations (problem variant with unsplittable flows).

| Description | | | GMC | | | | SCI | | | | SCI2 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | # nfs | opt gap | # os | # nos | # nfs | opt gap | # os | # nos | # nfs | opt gap | # os | # nos |
| Network | JBH50red | 62 | 58 | 0.02% | 1 | 3 | 54 | 0.07% | 4 | 4 | 55 | 0.01% | 4 | 3 |
| | USC30red | 13 | 0 | 0.09% | 1 | 12 | 0 | 0.14% | 0 | 13 | 0 | 0.13% | 0 | 13 |
| | USC53red | 55 | 27 | 0.47% | 7 | 21 | 30 | 0.68% | 2 | 23 | 30 | 0.67% | 1 | 24 |
| $T$ | 10 | 130 | 85 | 0.32% | 9 | 36 | 84 | 0.42% | 6 | 40 | 85 | 0.41% | 5 | 40 |
| $q$ | 3 | 65 | 34 | 0.43% | 5 | 26 | 38 | 0.64% | 1 | 26 | 37 | 0.61% | 1 | 27 |
| | 5 | 65 | 51 | 0.06% | 4 | 10 | 46 | 0.10% | 5 | 14 | 48 | 0.08% | 4 | 13 |
| Fixed costs | Low | 28 | 23 | 0.02% | 2 | 3 | 22 | 0.09% | 1 | 5 | 23 | 0.01% | 2 | 3 |
| | High | 102 | 62 | 0.36% | 7 | 33 | 62 | 0.47% | 5 | 35 | 62 | 0.46% | 3 | 37 |
| Capacities | Loose | 25 | 3 | 0.25% | 6 | 16 | 1 | 0.42% | 5 | 19 | 1 | 0.37% | 3 | 21 |
| | Tight | 105 | 82 | 0.38% | 3 | 20 | 83 | 0.42% | 1 | 21 | 84 | 0.45% | 2 | 19 |
| Routing costs | Euclidean | 61 | 42 | 0.29% | 4 | 15 | 45 | 0.46% | 2 | 14 | 43 | 0.28% | 0 | 18 |
| | Random | 69 | 43 | 0.34% | 5 | 21 | 39 | 0.40% | 4 | 26 | 42 | 0.49% | 5 | 22 |
| Demand behavior | Increasing | 71 | 48 | 0.17% | 4 | 19 | 45 | 0.23% | 3 | 23 | 47 | 0.22% | 3 | 21 |
| | Random | 59 | 37 | 0.47% | 5 | 17 | 39 | 0.66% | 3 | 17 | 38 | 0.62% | 2 | 19 |
| $P_E^C$ | 1 | 67 | 43 | 0.38% | 1 | 23 | 46 | 0.47% | 2 | 19 | 43 | 0.53% | 2 | 22 |
| | 5 | 63 | 42 | 0.25% | 8 | 13 | 38 | 0.37% | 4 | 21 | 42 | 0.26% | 3 | 18 |
| Economies of scale | Inverse | 67 | 44 | 0.27% | 5 | 18 | 42 | 0.50% | 3 | 22 | 44 | 0.34% | 3 | 20 |
| | Positive | 63 | 41 | 0.37% | 4 | 18 | 42 | 0.31% | 3 | 18 | 41 | 0.48% | 2 | 20 |
| All | | 130 | 85 | 0.32% | 9 | 36 | 84 | 0.42% | 6 | 40 | 85 | 0.41% | 5 | 40 |

Table 10: Number of instances without feasible solution, with optimal solution and non-optimal solution, as well as average optimality gap (CPLEX default settings) for instances which have not been solved to optimality by at least one of the formulations (problem variant with unsplittable flows).

| Description | | | GMC | | | | SCI | | | | SCI2 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Attribute | Value | # | # nfs | # os | opt gap | time (min.) | # nfs | # os | opt gap | time (min.) | # nfs | # os | opt gap | time (min.) |
| Network | JBH50red | 71 | 7 | 64 | 0.00% | 10.4 | 66 | 2 | 0.03% | 594.9 | 70 | 1 | 0.01% | 600.0 |
| | USC30red | 116 | 42 | 74 | 0.00% | 20.5 | 59 | 57 | 0.01% | 23.3 | 59 | 56 | 0.01% | 35.7 |
| | USC53red | 80 | 33 | 46 | 0.00% | 61.1 | 76 | 1 | 0.06% | 636.3 | 75 | 0 | 0.10% | 720.0 |
| $T$ | 10 | 267 | 82 | 184 | 0.00% | 27.3 | 201 | 60 | 0.01% | 103.7 | 204 | 57 | 0.02% | 99.0 |
| $q$ | 3 | 133 | 38 | 95 | 0.00% | 17.9 | 92 | 35 | 0.02% | 138.9 | 96 | 33 | 0.02% | 101.2 |
| | 5 | 134 | 44 | 89 | 0.00% | 37.2 | 109 | 25 | 0.01% | 46.0 | 108 | 24 | 0.02% | 95.8 |
| Fixed costs | Low | 167 | 53 | 114 | 0.00% | 18.5 | 126 | 35 | 0.02% | 136.6 | 129 | 33 | 0.02% | 112.8 |
| | High | 100 | 29 | 70 | 0.00% | 41.5 | 75 | 25 | 0.01% | 49.9 | 75 | 24 | 0.01% | 78.1 |
| Capacities | Loose | 176 | 9 | 166 | 0.00% | 18.9 | 110 | 60 | 0.01% | 103.7 | 113 | 57 | 0.02% | 99.0 |
| | Tight | 91 | 73 | 18 | 0.01% | 105.5 | 91 | 0 | - | - | 91 | 0 | - | - |
| Routing costs | Euclidean | 135 | 35 | 99 | 0.00% | 33.0 | 103 | 30 | 0.01% | 92.3 | 105 | 29 | 0.01% | 65.3 |
| | Random | 132 | 47 | 85 | 0.00% | 20.7 | 98 | 30 | 0.02% | 114.5 | 99 | 28 | 0.02% | 129.7 |
| Demand behavior | Increasing | 128 | 33 | 94 | 0.00% | 38.3 | 100 | 28 | 0.01% | 20.6 | 100 | 28 | 0.01% | 36.3 |
| | Random | 139 | 49 | 90 | 0.00% | 15.7 | 101 | 32 | 0.02% | 165.0 | 104 | 29 | 0.03% | 149.1 |
| $P_E^C$ | 1 | 128 | 37 | 91 | 0.00% | 22.9 | 89 | 35 | 0.01% | 120.0 | 92 | 33 | 0.01% | 106.0 |
| | 5 | 139 | 45 | 93 | 0.00% | 31.6 | 112 | 25 | 0.02% | 80.2 | 112 | 24 | 0.03% | 89.7 |
| Economies of scale | Inverse | 132 | 40 | 91 | 0.00% | 37.3 | 99 | 32 | 0.01% | 83.5 | 100 | 29 | 0.02% | 105.6 |
| | Positive | 135 | 42 | 93 | 0.00% | 17.5 | 102 | 28 | 0.02% | 124.0 | 104 | 28 | 0.02% | 92.2 |
| All | | 267 | 82 | 184 | 0.00% | 27.3 | 201 | 60 | 0.01% | 103.7 | 204 | 57 | 0.02% | 99.0 |

Table 11: Number of instances without feasible solution and with optimal solution, as well as average optimality gap and solution time (basic branch-and-cut environment for problem variant with unsplittable flows).