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Luca Bertazzi
Guilherme O. Chagas
Leandro C. Coelho
Demetrio Laganà
Francesca Vocaturo

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Bureau de Montréal
Université de Montréal
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Tél : 1-418-656-2073
Télécopie : 1-418-656-2624

Online Algorithms for the Multi-Vehicle Inventory-Routing Problem with Real-Time Demands

Luca Bertazzi[1], Guilherme O. Chagas[2,3], Leandro C. Coelho[2,4,*], Demetrio Laganà[5] Francesca Vocaturo[6]

1. Dipartimento di Economie e Management, Universitá degli Studi di Brescia, Italy
2. Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation (CIRRELT)
3. Faculty of Business Administration, Université Laval
4. Canada Research Chair in Integrated Logistics, Université Laval
5. Dipartimento di Ingegneria Meccanica, Energetica e Gestionale, Universitá della Calabria, Italy
6. Dipartimento di Economia, Statistica e Finanza "Giovanni Anania", Universitá della Calabria, Italy

Abstract. The increasing availability of sophisticated information and communication technology has stimulated new research within the distribution logistics area in the last few decades. Real-time information is crucial to ensure not only the competitiveness of a company but also its survival in the e-commerce era. Companies try to offer delivery to their customers within a few hours of receiving a request. In addition, real-time information can be exploited in systems that operate under emergencies, where response time is critical. We model and solve a multi-vehicle inventory-routing problem in which new service requests are revealed dynamically over time, called real-time or online. For this problem, we propose a class of online algorithms and present a competitive analysis to evaluate its performance from a theoretical perspective. Our approach is based on the solution of an integer programming model through a tailored branch-and-cut method in which several families of valid inequalities are separated and dynamically introduced in the model. The results of an extensive computational experience are also described. Based on these tests, we show that the ratios between the results obtained by our online algorithms and the offline ones are low, demonstrating the effectiveness of our approach.

Keywords: Decision making under uncertainty, real-time information, online optimization, integer programming, branch-and-cut

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* Corresponding author: leandro.coelho@fsa.ulaval.ca

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1 Introduction

Over the last few decades, Information and Communication Technology (ICT) has been widely recognized as the primary enabler for increasing supply chain performance and supporting critical logistics services, especially in transportation contexts [26]. To highlight the value of information, Viet et al. [59] examine papers published from 2006 to 2017. In particular, they point out that the literature is rich in assessing the value of information in inventory decisions. Yet, there are opportunities for research in other areas, such as transportation and supply. Furthermore, they emphasize how access to more information in a supply chain can be challenging and that information sharing requires a high level of trust between the parties. In addition, collecting and transferring information need investments in ICT infrastructures. This investment per se does not ensure improved performance, even if it is significant. To reach its full potential, one must make the most intelligent usage possible of the hardware being deployed and the huge amount of data it provides.

Management Science and Operations Research play a key role in this challenge since developing powerful algorithms to support decision-making processes is fundamental in real-world applications. Within these disciplines, the same decision problems can be classified according to information availability. Specifically, the problems concerning supply chain and logistics can be distinguished based on an important dimension of information, i.e., *evolution* [53], which is related to the availability of information that changes over time. Based on this dimension, two main categories of problems can be identified: *static* vs. *dynamic* problems.

A static problem is characterized by input known beforehand. Knowledge may come in various forms. For instance, static and stochastic problems are characterized by inputs known as random variables with a given probability distribution. In most studies concerning this type of problem, decisions are made a priori, and only minor changes are allowed afterward. Instead, in a dynamic problem, a part or all of the input is unknown and revealed incrementally during the design or execution of the plans. Consequently, the plans are redefined in an ongoing fashion. In distribution problems, for instance, logistics support such as geographic information systems, global positioning systems, traffic sensors, and smartphone technologies can be jointly used to collect and transmit data in real-time to different stakeholders. For these characteristics, these problems are referred to as *online* or *real-time* [32].

Here we study a dynamic problem in the distribution logistics area. Specifically, we deal with a dynamic version of the classical inventory-routing problem (IRP) introduced by Archetti et al. [5] and widely studied in the last decades. The IRP is an integrated combinatorial optimization problem arising when customers transfer the responsibility for inventory replenishment to the supplier, who must decide when to visit each customer, how much to deliver, and how to sequence customers in one or more vehicle routes. The business model associated with IRPs is known as vendor-managed inventory (VMI). In the version studied in this paper, a fleet of vehicles has to serve requests incrementally revealed over a finite time horizon and unpredictable in advance (*real-time* or *online demands*). Therefore, in the following, we also refer to our problem as the Online IRP (O-IRP). With access to real-time information, suppliers can react to changes in demand patterns and make online decisions continuously to adjust and improve routes accordingly.

In designing the O-IRP we have been inspired by current practices in VMI systems and the new features of routing problems in the e-commerce era, which is strongly characterized by uncertainty [45]. While in the industrial sector, one of the main assumptions in IRPs is related to how deliveries can be anticipated concerning the day the product is needed, in the e-commerce era, orders are placed online, and there is an explicit request, or at least an expectation, of quick delivery times. Therefore, all the demand should be satisfied promptly without backlogging [4]. Furthermore, the O-IRP arises in emergency settings, e.g., in the blood supply chain, where a blood collection center monitors the demands at hospitals and determines an optimal distribution scheme [48]. Blood management is particularly complex due to the high uncertainty concerning demand, its perishable nature, strict storage and handling requirements, and the vastness of operations. Satisfying unpredictable demands is vital in this and many other contexts.

To tackle the O-IRP we propose a class of online algorithms that has to make irrevocable decisions without full knowledge of the problem instance. Complying with the nature of the O-IRP, our solution approach has complete knowledge of the past but no or just partial knowledge of the future. This paper makes the following contributions to the literature: (i) we deal with a problem in the area of inventory routing where the literature is scarce with respect to real-time inputs; (ii) we propose a class of online algorithms for the O-IRP; to the best of our knowledge, ours is the first online optimization method associated with theoretical and computational competitive analysis for an IRP; (iii) to solve the integer program on which the online algorithms are based, a tailored branch-and-cut is used; our branch-and-cut uses strong theoretical and methodological results from the literature.

The remainder of this document is organized as follows. Some related studies in the scientific literature are discussed in Section 2. A formal description of the O-IRP is given in Section 3. Section 4 illustrates the class of online algorithms proposed for the O-IRP, while Section 5 presents a theoretical competitive analysis for this class of algorithms. Section 6 illustrates the solution process for the integer programming model on which the online algorithms are based. Section 7 describes the results of an extensive experimental study. Conclusions follow in Section 8.

2 Literature review

This section summarizes the fundamental concepts of online optimization and discusses some pertinent works in this field. In addition, we briefly describe some studies concerning IRPs, with a particular focus on real-time data.

2.1 Approaches to uncertainty and online algorithms

Here we propose a classification of the main modeling frameworks for combinatorial optimization problems, where the position of online optimization can be evaluated in perspective. This classification, inspired by Bianchi et al. [15], considers two aspects: the way uncertain information is formalized (*degree of uncertainty*) and when uncertain information is revealed with respect to when decisions must be taken (*degree of dynamism*). Although it is possible to quantify the degree of dynamism of a problem, we simplify the discussion by only distinguishing between static and dynamic problems as previously discussed (see the horizontal axis of Figure 1, adapted from Bianchi et al. [15]). Concerning the degree of uncertainty (vertical axis of Figure 1), the lowest level is represented by perfect knowledge of input (absence of uncertainty). On the contrary, the highest level is represented by total uncertainty. In between, it is possible to (*i*) consider a probabilistic distribution or process for the problem data, (*ii*) model the data using fuzzy logic, or (*iii*) to introduce sets for uncertain parameters rather than probabilistic distributions. Then, different modeling frameworks are commonly adapted.

Deterministic optimization considers static problem models with perfect knowledge of the input. Stochastic optimization implies that probability distributions are known or can be accurately estimated in static and dynamic settings. Various methods can solve the problems formulated as stochastic programs, as described in Birge and Louveaux [16]. Fuzzy and robust optimization represent alternative frameworks for combinatorial optimization problems when probabilistic distributions are unavailable. In particular, uncertain parameters are represented by fuzzy numbers in fuzzy programming [60] and by sets or intervals in robust optimization [30]. When the degree of uncertainty increases, online optimization comes into play. Especially in real-time applications, the problem data are at most partially available when irrevocable decisions must be taken (e.g., only the demands of some customers are known); in extreme cases, the decision maker has to act with no knowledge of future inputs.

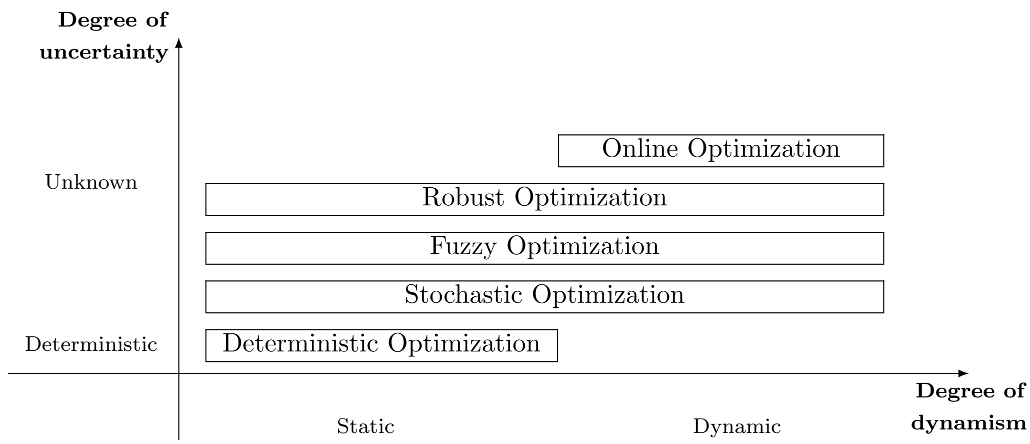


Figure 1: Modeling frameworks for combinatorial optimization problems (conceptual classification, inspired from Bianchi et al. [15])

Here we focus on online optimization, the modeling framework selected to solve the O-IRP described in this paper. Online problems had already been investigated in the 1970s and early 1980s, but an extensive and systematic study was only published in 1985. Specifically, to evaluate the performance of an online algorithm, Sleator and Tarjan [56] suggested comparing it to an exact offline algorithm. After that, Karlin et al. [43] introduced the concept of *competitive analysis*. An online algorithm ALG is c-*competitive* (with $c \geq 1$) if, given any problem instance I, the cost $ALG(I)$ of the solution given by the online algorithm ALG is no more than c times the cost $OPT(I)$ of an exact offline algorithm with full knowledge of the instance in advance:

$$\frac{ALG(I)}{OPT(I)} \leq c, \text{ for any problem instance } I.$$

The smallest c such that $ALG(I) \leq c \times OPT(I)$ is called the *competitive ratio* of ALG. For the sake of simplicity, in the following, we will omit the reference to I. Comprehensive surveys on online optimization are given by Borodin and El-Yaniv [17] and Albers [3], whereas Jaillet and Wagner [40] focus on online vehicle routing problems.

Recently, online algorithms have been successfully applied by Bergé et al. [8] for the online k-Canadian traveler problem in which a traveler has to traverse an undirected graph and can discover blocked edges when arriving at one of its endpoints, and by Chen et al. [19] for a machine minimization problem in which jobs with hard deadlines arrive online over time at their release dates. We refer the interested reader to Höhne et al. [37, 38, 39] for overviews of recent research on online algorithms.

2.2 Inventory-routing

The literature is rich with contributions related to inventory management problems [6] as well as node, arc, and general routing problems [25, 58]. Although integrated problems have been studied less extensively, the IRP has received much attention since it was introduced by Bell et al. [7] for the integrated inventory management and distribution of liquefied industrial gases. Thereafter, the IRP has been associated with a large variety of applications. For instance, Christiansen et al. [20] consider a maritime environment where a heterogeneous fleet of bulk ships transports multiple non-mixable cement products from suppliers to regional silo stations along the coast of Norway. Bertazzi et al. [14] describe the impact of a VMI system applied to nanostores in a South East Asian city. These nanostores belong to the Procter & Gamble supply chain that is resupplied according to a VMI model. McKenna et al. [50] investigate dispatching and routing policies for the resupply of geographically dispersed units operating in a combat environment through unmanned aerial cargo vehicles; in the experimental phase, the authors construct a hypothetical military scenario based on contingency operations in Afghanistan by the United States Army which utilizes a VMI model for its resupply operations. The IRP is also employed by Gaur and Fisher [31] and Laganà et al. [44] to model the distribution of products in supermarket chains, Popović et al. [54] to study the fuel delivery in petrol stations inventory management, Çankaya et al. [61] for the distribution of humanitarian relief supplies in disaster operations management, and Cárdenas-Barrón and Melo [28] for the collection of waste vegetable oil in reverse logistics. We refer to Coelho et al. [23] for a review on the deterministic and stochastic IRPs and their applications and to Bertazzi and Speranza [9, 10] tutorials on the single and multi-vehicle IRPs. Most studies are carried out in static settings using deterministic optimization, which assumes a fully specified input. The problem we study in this paper is a dynamic version of the classical IRP introduced by Archetti et al. [5]. We omit the references to papers that studied this deterministic problem with fully specified input as we focus on studies in which the knowledge of input is not perfect to classify our problem correctly.

Hvattum et al. [36], Niakan and Rahimi [51], and Solyalı et al. [57] investigate uncertain IRPs within stochastic, fuzzy, and robust frameworks, respectively. Bertazzi et al. [11] study an IRP in which a supplier has to serve a set of retailers whose stochastic demands must be satisfied over a given time horizon according to an order-up-to-level policy. An inventory cost is applied to any positive inventory level, while a penalty cost is charged, and the excess demand is not backlogged whenever the inventory level is negative. They propose a hybrid rollout algorithm and evaluate its performance on a large set of randomly generated instances. Coelho et al. [24] solve the dynamic and stochastic version of the IRP under two different policies: the first one uses a reactive framework, in which future delivery decisions are solely based on the current state of the inventory of the customers; the second uses demand forecasts to support future decisions. Amongst other conclusions, the authors show that using stochastic information to generate better solutions is possible, albeit at the expense of more computing time. Bertazzi et al. [12] study an IRP in which the supplier has a limited production capacity and the stochastic demand of the retailers is fulfilled with outsourced transportation services. They show that considering the average values of the customers' demands can produce a total expected cost infinitely worse than the one accounting for the overall probability distribution of the demands. Hence, they propose a stochastic dynamic programming formulation of the problem providing an optimal policy for small-size instances of the problem and allowing to design a matheuristic algorithm that integrates a mixed-integer linear programming model into a lookahead rollout policy, which can solve realistic size problem instances. Bertazzi et al. [13] study an IRP with stochastic demands and an outsourced fleet of vehicles. Since the demand's probability distribution is unknown, they adopt a min-max approach to find robust policies for a dynamic programming formulation of the problem. They propose a min-max matheuristic to solve benchmark instances and show that it is very effective. Brinkmann et al. [18] present a dynamic and stochastic IRP for bike-sharing systems to avoid unsatisfied demand. The authors propose a dynamic lookahead policy that simulates future demand over a predefined horizon to anticipate potential future demands in the current inventory decisions. Achamrah et al. [1] model a dynamic and stochastic IRP that considers two flexible instruments of transshipment and substitution to mitigate shortages at the customer level. An approach based on the hybridization of mathematical programming, genetic algorithms, and deep reinforcement learning is presented, and different demand distributions are examined in the experiments. Other IRPs in uncertain and dynamic settings are studied by Giesen et al. [33, 34, 35].

Jarugumilli and Grasman [41], Nolz et al. [52], Raba et al. [55], Liu and Lin [47], and Cubillos et al. [27] deal with real-time information coming from specific ICT infrastructures. In particular, Jarugumilli and Grasman [41] emphasize the importance of Radio Frequency Identification (RFID) technology in VMI systems and provide a methodology for the dynamic control of RFID-enabled IRPs where a single distribution center serves a set of customers. In this scenario, RFID technology enables efficient control of inventory distribution by exchanging real-time information upon arrival at each location. RFID technology is also considered by Nolz et al. [52] to manage the collection of infectious medical waste. The materials to be collected are produced by patients in self-treatment, stored at pharmacies, and picked up by local authorities for disposal. The reverse logistics problem is formulated as a collector-managed IRP and encompasses stochastic aspects. Real-time information on inventory levels at each pharmacy can be used daily to revise the planning of collection tours. Raba et al. [55] present a reactive approach for the stochastic IRP in the context of a supply chain for the animal-feed industry. Their approach, based on the combination of a biased-randomized algorithm with Monte Carlo simulation, allows using sensors to obtain updated data on customers' demands at the end of each period and reoptimizes the distribution process for the remaining periods of a finite planning horizon. Liu and Lin [47] explicitly consider CO_2 emission cost and propose an online distribution system of an IRP with simultaneous deliveries and returns. In their study, a mobile device, the vehicles' position, and Google Maps are integrated into an online decision support module. Cubillos et al. [27] investigate an IRP with stochastic demand in which the knowledge of the demands can be updated by using sensor information to make delivery decisions.

To our knowledge, online algorithms with a specified competitive analysis for real-time IRPs are missing in the scientific literature. This work contributes to covering this gap in the literature.

3 Problem description

This section formally presents our O-IRP, where demands are revealed over time. At each period, based on the demands revealed so far, a decision is made by an online algorithm that makes irrevocable decisions without full knowledge of the problem instance. Let $G = (V, E)$ be an undirected, weighted, and simple graph, where $V = \{0, 1, \ldots, n\}$ is the vertex set and $E = \{(i, j) \mid i, j \in V, i < j\}$ is the edge set. The set of vertices is divided into the supplier, represented by vertex 0, and customer vertices, defined by $V' = V \setminus \{0\}$. Each vertex $i \in V$ is associated with a unit inventory cost h_i, a maximum inventory level U_i, and an initial inventory level $I_i^0 \leq U_i$. Since using an order-up-to-level policy fulfilling the residual inventory capacity of the retailers can produce high inventory costs due to the unpredictable demands of the customers in the online context, we adopt a maximum-level policy. A routing cost c_{ij} is associated with every edge of $(i, j) \in E$. The fleet is assumed to be homogeneous, and the transportation capacity of the vehicle is defined by Q. We denote by $\mathcal{K} = \{1, \ldots, K\}$ the vehicles set. Let $\mathcal{T} = \{1, 2, \ldots, T\}$ be the set of periods of the time horizon. For each period $t \in \mathcal{T}$, we define the subset $\mathcal{T}_t(\gamma) = \{t, t+1, \ldots, \min[t+\gamma, T]\} \subseteq \mathcal{T}$, where γ is an integer number between 0 and $T-1$.

For each period $\tau \in \mathcal{T}_t(\gamma)$, we know the quantity r_τ made available at the supplier and some information $\hat{d}_i^\tau$ on the demand d_i^τ of each customer $i \in V'$. In particular, at the beginning of period t, we assume to have perfect information on the demand d_i^t of each customer i, i.e., $\hat{d}_i^t = d_i^t$, where $d_i^t \leq U_i$. Instead, for each of the future periods $\tau = t+1$, $t+2$, …, $\min[t+\gamma, T]$, we assume to have perfect information for a given percentage of the customers and no information at all for the remaining customers. Therefore, $\hat{d}_i^\tau$ is either equal to 0 or $d_i^\tau \leq U_i$ for each customer i at these periods. Note that each customer's demand information can differ in different periods. We assume that the quantities r_τ and the number of vehicles K are such that at least a feasible solution exists for any value of γ.

Taking into account that the information on the demand is revealed at each period $t \in \mathcal{T}$ for the set of periods $\tau \in \mathcal{T}_t(\gamma)$ only, the aim is to determine the quantity to deliver to each customer and the routes to travel at each period $t \in \mathcal{T}$ such that the sum of the total inventory cost at the supplier and the customers and the total routing cost is minimized.

4 Online algorithms

In this section, we describe the class of online algorithms we propose, referred to as $\text{ALG}^{(\gamma)}$. Using these algorithms, at the beginning of each period $t \in \mathcal{T}$, the following mixed-integer linear programming model O-IRP$_t(\gamma)$ is solved, and only the values of the variables at time t are stored (see Algorithm 1).

Algorithm 1: Class of online algorithms $\text{ALG}^{(\gamma)}$

```
Input:
for t ∈ 𝒯 do
    Solve model O-IRP_t(γ)
    Store the values of the variables at time t
end
Return the corresponding value of the objective function
```

Let I_i^τ be a continuous variable representing the inventory level at vertex $i \in V$ at the end of period $\tau \in \mathcal{T}_t(\gamma)$ for a given $t \in \mathcal{T}$. Let $y_i^{k\tau}$ be a binary variable equal to one if vertex $i \in V$ is visited by vehicle $k \in \mathcal{K}$ in period $\tau \in \mathcal{T}_t(\gamma)$. Routing variable $x_{ij}^{k\tau}$ is equal to the number of times the edge $(i,j) \in E$ is used by vehicle $k \in \mathcal{K}$ in period $\tau \in \mathcal{T}_t(\gamma)$. Since G is a complete graph, these variables are binary for each edge $(i, j) \in E$ such that $i \neq 0$ while they can assume values in $\{0, 1, 2\}$ for each edge incident to 0. The quantity of product delivered by vehicle k in period τ to customer i is given by $q_i^{k\tau}$. Let $\hat{D}_i^t = \max\left\{\sum_{\tau\in\mathcal{T}_t(\gamma)} \hat{d}_i^\tau - I_i^{t-1}, 0\right\}$ be the total quantity that must be delivered to customer i in periods $\tau \in \mathcal{T}_t(\gamma)$ starting from the beginning of period t. Then, the O-IRP$_t(\gamma)$ formulation is given by (1a)–(1o).

$$\text{O-IRP}_t(\gamma) = \min \sum_{i\in V} \sum_{\tau\in\mathcal{T}_t(\gamma)} h_i I_i^\tau + \sum_{(i,j)\in E} \sum_{k\in\mathcal{K}} \sum_{\tau\in\mathcal{T}_t(\gamma)} c_{ij} x_{ij}^{k\tau} \tag{1a}$$

subject to

$$I_0^\tau = I_0^{\tau-1} + r^\tau - \sum_{i\in V'} \sum_{k\in\mathcal{K}} q_i^{k\tau},\ \forall \tau \in \mathcal{T}_t(\gamma) \tag{1b}$$

$$I_i^\tau = I_i^{\tau-1} + \sum_{k\in\mathcal{K}} q_i^{k\tau} - \hat{d}_i^\tau,\ \forall i \in V', \forall \tau \in \mathcal{T}_t(\gamma) \tag{1c}$$

$$\sum_{k\in\mathcal{K}} q_i^{k\tau} \leq U_i - I_i^{\tau-1},\ \forall i \in V', \forall \tau \in \mathcal{T}_t(\gamma) \tag{1d}$$

$$q_i^{k\tau} \leq U_i y_i^{k\tau},\ \forall i \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1e}$$

$$\sum_{\tau\in\mathcal{T}_t(\gamma)} \sum_{k\in\mathcal{K}} q_i^{k\tau} = \hat{D}_i^t,\ \forall i \in V' \tag{1f}$$

$$\sum_{i\in V'} q_i^{k\tau} \leq Q y_0^{k\tau},\ \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1g}$$

$$\sum_{j\in V, i<j} x_{ij}^{k\tau} + \sum_{j\in V, j<i} x_{ji}^{k\tau} = 2y_i^{k\tau},\ \forall i \in V, \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1h}$$

$$\sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}, i<j} x_{ij}^{k\tau} \leq \sum_{i\in\mathcal{S}} y_i^{k\tau} - y_m^{k\tau},\ \forall \mathcal{S} \subseteq V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma), \forall m \in \mathcal{S} \tag{1i}$$

$$\sum_{k\in\mathcal{K}} y_i^{k\tau} \leq 1,\ \forall i \in V', \forall \tau \in \mathcal{T}_t(\gamma) \tag{1j}$$

$$I_i^\tau \in \mathbb{R}^+,\ \forall i \in V, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1k}$$

$$q_i^{k\tau} \in \mathbb{R}^+,\ \forall i \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1l}$$

$$x_{0j}^{k\tau} \in \{0, 1, 2\},\ \forall j \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1m}$$

$$x_{ij}^{k\tau} \in \{0, 1\},\ \forall i \in V', \forall j \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{1n}$$

$$y_i^{k\tau} \in \{0, 1\},\ \forall i \in V, \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma). \tag{1o}$$

The objective function (1a) minimizes the inventory costs at the supplier and at customers' locations and the routing costs. Constraints (1b) and (1c) impose inventory conservation for the supplier and the customers. Constraints (1d) define the inventory capacity of the customers. Constraints (1e) link the delivery variables q to visiting variables y. Constraints (1f) guarantee that the total quantity sent to each customer satisfies the total demand of the customer exactly. Constraints (1g) impose the vehicle capacity. Constraints (1h) and (1i) are degree and subtour elimination, respectively, and constraints (1j) ensure that at most one vehicle visits a customer at each period. Constraints (1k)–(1o) define the nature and domain of the variables.

To solve O-IRP$_t(\gamma)$ in Algorithm 1, we propose a branch-and-cut algorithm (see Section 6). Note that the *offline problem*, i.e., the deterministic problem in which the demands of all customers in periods $1, 2, \ldots, T$ are known at time 1, corresponds to model (1a)–(1o) in which $t = 1$, $\gamma = T-1$ and perfect information on the demand of each customer for each period of the corresponding set $\mathcal{T}_t(\gamma)$ is assumed. The offline problem is fundamental to evaluating the performance of our online algorithms. Although the above-introduced notation and model do not refer to information level (for readability issues), in the following, we will use O-IRP$_1(T-1)$ to denote the offline problem.

We consider the following online algorithms in the class ALG$^{(\gamma)}$:

- ALG$^{(0)}$: we have perfect information on the demands of all customers in the current period t, i.e., $\hat{d}_i^t = d_i^t$ for $i \in V'$, and no information on the remaining demands, i.e., $\hat{d}_i^\tau = 0$, for $i \in V'$ and $\tau = t+1, t+2, \ldots, \min[t+\gamma, T]$;

- ALG$^{(1)}$: we have perfect information on the demands of all customers in the current period t and the demands of *some* customers in period $t+1$, and no information on the remaining demands;

- ALG$^{(2)}$: we have perfect information on the demands of all customers in the current period t and the demands of *some* customers in periods $t+1$ and $t+2$, and no information on the remaining demands;

- ALG$^{(3)}$: we have perfect information on the demands of all customers in the current period t and the demands of *some* customers in periods $t+1$, $t+2$, and $t+3$, and no information on the remaining demands;

- ALG$^{(4)}$: we have perfect information on the demands of all customers in the current period t and the demands of *some* customers in periods $t+1$, $t+2$, $t+3$, and $t+4$, and no information on the remaining demands.

The percentage of the customers for which the demand is known at each time period $\tau = t+1, t+2, \ldots, \min[t+\gamma, T]$ leads to different versions of the corresponding algorithms. In the experimental phase, we focus on one version of algorithms ALG$^{(\gamma)}$, with $\gamma = 0, 1, 2, 3, 4$, in which the demand of 100% of the customers is known for each period in the corresponding set $\mathcal{T}_t(\gamma)$.

5 Competitive analysis of ALG$^{(\gamma)}$

To derive a theoretical competitive analysis of our class of algorithms ALG$^{(\gamma)}$, the main ideas rely on building a tight competitive ratio by approximating with an upper bound the cost obtained by online algorithm ALG$^{(\gamma)}$, and with a lower bound on the optimal cost of the offline problem O-IRP$_1(T-1)$. Specifically, the offline problem O-IRP$_1(T-1)$ is challenging since it generalizes the Vehicle Routing Problem (VRP), an NP-hard problem [58]. Therefore, a good approximation of the overall routing cost of an optimal offline solution is obtained by solving an instance of the Split Delivery VRP (SDVRP). When all the customer demands are known a priori, the routing cost of the offline problem O-IRP$_1(T-1)$ cannot be less than or equal to the optimal cost of an SDVRP instance. By allowing deliveries to be split, we approximate the multi-period aspect of the IRP, where several deliveries to a customer can take place in different periods.

In this section, we provide a competitive analysis of algorithms ALG$^{(\gamma)}$ when the number of vehicles K is such that a feasible solution exists for any online algorithm. Let $z^{SDVRP(D_i)}$ be the optimal cost of the SDVRP when quantity D_i is delivered to customer i, $h_{\max} = \max_{i \in V} h_i$, $h_{\min} = \min_{i \in V} h_i$, $ALG^{(\gamma)}$ be the cost obtained by the online algorithm ALG$^{(\gamma)}$, and OPT be the optimal cost of the offline IRP.

Theorem 1. $\frac{ALG^{(\gamma)}}{OPT} \leq \max\{\frac{h_{\max}}{h_{\min}}, T\}$.

Proof. Consider first the inventory cost. Note that, for each period $t \in \mathcal{T}$, the total inventory level, obtained by summing up I_0^t in equation (1b) and $\sum_{i \in V'} I_i^t$ in equations (1c), i.e., $B_t = \sum_{i \in V} I_i^t = \sum_{i \in V} I_i^0 + \sum_{\rho=1}^{t} r_\rho - \sum_{i \in V'} \sum_{\rho=1}^{t} d_i^\rho$, is independent of the value of the variables of the problem. Therefore, the total inventory cost is not lower than $h_{\min} B$ and not greater than $h_{\max} B$, where $B = \sum_{t \in \mathcal{T}} B_t$.

Consider now the routing cost. A lower bound is obtained by solving the SDVRP when the quantity delivered to each customer i is $D_i = \max\left\{\sum_{t \in \mathcal{T}} d_i^t - I_i^0, 0\right\}$, i.e., the routing cost is not lower than $z^{SDVRP(D_i)}$. The set of routes in any feasible solution of the offline problem O-IRP$_1(T-1)$ provides a feasible solution to the SDVRP when the quantity delivered to each customer i is $\max\left\{\sum_{t \in \mathcal{T}} d_i^t - I_i^0, 0\right\}$.

Let us now compute an upper bound on the cost generated by the algorithm ALG$^{(\gamma)}$. As shown before, an upper bound on the inventory cost is given by $h_{\max}B$. An upper bound on the routing cost of this solution can be obtained by using, at each period $t \in \mathcal{T}$, the routes found in the lower bound computation. These routes are feasible for any quantity delivered to each customer i at each period t less than or equal to the total quantity $\sum_{t\in\mathcal{T}} d_i^t - I_i^0$ delivered to i over the time horizon $\mathcal{T}$. Therefore, an upper bound on the routing cost can be obtained by delivering a quantity in $[0, \sum_{t\in\mathcal{T}} d_i^t - I_i^0]$ to each customer in each $t \in \mathcal{T}$. Hence, the corresponding routing cost is $z^{SDVRP(D_i)}T$. Therefore,

$$ALG^{(\gamma)} \leq h_{\max}B + z^{SDVRP(D_i)}T,$$

while

$$OPT \geq h_{\min}B + z^{SDVRP(D_i)}.$$

Therefore,

$$\frac{ALG^{(\gamma)}}{OPT} \leq \frac{h_{\max}B + z^{SDVRP(D_i)}T}{h_{\min}B + z^{SDVRP(D_i)}} \leq \max\left\{\frac{h_{\max}}{h_{\min}}, T\right\}.$$

The last inequality holds because, if $\frac{a}{c} = \phi$ and $\frac{b}{d} = \eta$, then $\frac{a+b}{c+d} = \frac{c\phi+d\eta}{c+d} \leq \frac{(c+d)\max\{\phi,\eta\}}{c+d} = \max\{\phi, \eta\} = \max\{\frac{a}{c}, \frac{b}{d}\}$. □

Since the competitive ratio provided in the previous theorem is based on an upper bound of the cost of online algorithms ALG(γ) and a lower bound based on the optimal cost of the offline problem, it is now important to show that the ratio is not overestimated in the worst case for the simplest algorithm in the class, i.e., when $\gamma = 0$.

Consider first the following instance to prove that the bound $\frac{h_{\max}}{h_{\min}}$ is tight: time horizon T, $n = 2$ customers, $c_{0i} = 2$ for each customer i, $c_{12} = 0$, initial inventory level at the supplier $I_0^0 = 2$, initial inventory level at each customer $I_i^0 = 0$, inventory cost at the supplier h_0, inventory cost at each customer i $h_i < h_0$, maximum inventory level at each customer i $U_i = 1$, transportation capacity $Q = 2$, production rate $r^t = 0$ in periods $1, 2, \ldots, T-1$ and $r^T = 2$, demand of customer i in periods $t = 1, 2, \ldots, T-1$ $d_i^t = 0$ and $d_i^T = 1$ for the remaining time periods. Note that in this instance, three different routes can be used: $0 \to 1 \to 0$, $0 \to 2 \to 0$, $0 \to 1 \to 2 \to 0$. The cost of each of these routes is 4. Let us first focus on the solution provided by ALG$^{(0)}$. Since the demand in the period $t = 1$ is 0 units, given constraints (1f), then 0 units have to be delivered to each customer. The same happens for all successive periods less than T. At period T, since the demand $d_i^T = 1$ for both customers, 1 unit is delivered to each of them using the route $0 \to 1 \to 2 \to 0$. Hence, the total cost of the solution provided by ALG$^{(0)}$ is

$$ALG^{(0)} = 2h_0(T-1) + 2h_0 + 4.$$

Consider now the following feasible solution to the offline problem: in period 1, 1 unit is delivered to each customer; no deliveries occur in successive periods. Since the corresponding cost is $2h_i(T-1) + 2h_0 + 4$, then

$$OPT \leq 2h_i(T-1) + 2h_0 + 4$$

Therefore, in this instance

$$\frac{ALG^{(0)}}{OPT} \geq \frac{2h_0(T-1) + 2h_0 + 4}{2h_i(T-1) + 2h_0 + 4} \to \frac{h_0}{h_i} = \frac{h_{\max}}{h_{\min}} \qquad T \to \infty.$$

Consider now the following instance to prove that the bound T is tight: time horizon T, $n = 2$ customers located at unit distance from the depot and at zero distance from each other, initial inventory level at the supplier $I_0^0 = 2T - 2$, initial inventory level at each customer $I_i^0 = 0$, inventory cost at each customer i $h_i < 1$, inventory cost at the supplier $h_0 = h_i/T$, maximum inventory level at each customer i $U_i = T$, transportation capacity $Q = 2T$, production rate $r^t = 2$ at each period t, demand $d_i^t = 1$ at a customer i at each period t. Note that in this instance, three different routes can be used: $0 \to 1 \to 0$, $0 \to 2 \to 0$, $0 \to 1 \to 2 \to 0$. The cost of each of these routes is 2.

Let us focus on the solution provided by ALG$^{(0)}$. Since for each customer i the initial inventory level I_i^0 is 0, and the demand in period 1 is 1 unit, and given constraints (1f), 1 unit has to be delivered to each customer i. The same happens for all successive periods.

Hence, the total cost of the solution provided by ALG$^{(0)}$ is

$$ALG^{(0)} = h_i 2(T-1) + 2T.$$

Consider now the following feasible solution: T units are delivered to each customer in period 1, and zero units in the successive periods. Since the corresponding cost is:

$$h_0[2+4+\ldots+2(T-1)] + 2h_i[(T-1)+(T-2)+\ldots+0] + 2 = \frac{h_i}{T} 2 \frac{T(T-1)}{2} + 2h_i \frac{(T-1)T}{2} + 2 =$$

$$= h_i(T^2 - 1) + 2,$$

then $OPT \leq h_i(T^2 - 1) + 2$.

Therefore, in this instance

$$\frac{ALG^{(0)}}{OPT} \geq \frac{h_i 2(T-1) + 2T}{h_i(T^2 - 1) + 2} \to T \qquad h_i \to 0.$$

We remark that competitive analysis provides a worst-case measure since we ask that the outcome expressed by Theorem 1 be worth every instance. Alternatively, the performance of the online algorithms must be within a given threshold on inputs generated by a worst-case adversary. From this perspective, any competitive analysis has the disadvantage of being pessimistic. However, in some complex real-world applications such as inventory routing, there may be situations in which (worst-case) guarantees on performance are necessary. In this case, competitive analysis or other similar investigations are fundamental.

As Karlin [42] suggested, theoretical measures should be complemented by empirical studies to evaluate the performance of online algorithms. Sometimes, competitive algorithms associated with high ratios may perform better in practice than the theory would suggest. A thorough analysis combining competitive ratios and average results provides a better assessment of the performance of an algorithm than just using average results. While a competitive ratio provides an understanding of the worst possible performance of an algorithm, computational results are limited to the set of instances used in the computational experiments. To this end, in Section 7, we present an extensive computational study of the online algorithms proposed in this article where we show that the practical results lead to actual ratios are much lower than $\max\left\{\frac{h_{\max}}{h_{\min}}, T\right\}$. In fact, we observe empirical ratios seldom larger than 1.5.

6 Solution algorithm for the O-IRP$_t(\gamma)$

We solve the O-IRP$_t(\gamma)$ through a branch-and-cut algorithm. We point out that the branch-and-cut algorithm does not represent the core of this work, but it is used as a tool to solve the O-IRP$_t(\gamma)$ model. Hence, the following section provides a quick overview of some known valid inequalities that are used to strengthen the LP of the mathematical formulation (1a)–(1o) at the nodes of the branch-and-bound tree. We remark that all the valid inequalities reviewed in Section 6.1 are facets for the polytope of the linear relaxation of (1a)–(1o). Section 6.2 summarizes how the algorithm works.

6.1 Valid inequalities

Valid inequalities (2)–(4) were introduced by Archetti et al. [5] and adapted to the IRP by Coelho et al. [22]. Symmetry breaking constraints (5) and (6) were used by Coelho and Laporte [21].

$$x_{0i}^{k\tau} \leq 2y_i^{k\tau}, \ \forall i \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma) \tag{2}$$

$$x_{ij}^{k\tau} \leq y_i^{k\tau}, \ \forall i,j \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma). \tag{3}$$

Constraints (2) and (3) are referred to as logical inequalities. They enforce the condition that if the supplier is the predecessor of customer $i \in V'$ in the route of vehicle k at time τ, i.e., $x_{0i}^{k\tau} = 1$ or 2, then i must be visited by the same vehicle at time τ, i.e., $y_i^{k\tau} = 1$. Similary, if the edge (i,j) is traveled by vehicle k at time τ, i.e., $x_{ij}^{k\tau} = 1$, then i must be visited by the same vehicle at time τ, i.e., $y_i^{k\tau} = 1$.

$$y_i^{k\tau} \leq y_0^{k\tau}, \ \forall i \in V', \forall k \in \mathcal{K}, \tau \in \mathcal{T}_t(\gamma). \tag{4}$$

Constraints (4) include the supplier in the route of vehicle k if at least one customer is visited by this vehicle at time τ.

At each period τ, there are two main symmetry issues due to the presence of identical vehicles. First, if $K_\tau \leq K$ denotes the number of vehicles to dispatch in period τ, there are $\binom{K}{K_\tau}$ ways to select K_τ vehicles from K. To break this symmetry, constraints (5) allow vehicle k to be dispatched only if vehicle $k-1$ is also dispatched. Second, among the K_τ vehicles, there are $K_\tau!$ ways to exchange the vehicles among the routes they can be assigned to. Hence, the total number of symmetric solutions in period τ is $\left[\binom{K}{K_\tau} \cdot K_\tau!\right]$. To address the second symmetry issue, constraints (6) come into play. They state that if the customer i is assigned to the vehicle k in period τ, vehicle $k-1$ must serve a customer with an index smaller than i in the same period. These constraints are inspired by those proposed by Fischetti et al. [29] and by Albareda-Sambola et al. [2], and already used in on IRP setting by Coelho and Laporte [21].

$$y_0^{k\tau} \leq y_0^{k-1,\tau}, \ \forall k \in \mathcal{K} \setminus 1, \forall \tau \in \mathcal{T}_t(\gamma) \tag{5}$$

$$y_i^{k\tau} \leq \sum_{j<i} y_j^{k-1,\tau}, \ \forall i \in V', \forall k \in \mathcal{K} \setminus 1, \forall \tau \in \mathcal{T}_t(\gamma). \tag{6}$$

Inequalities (7) and (8) were presented in Lefever [46] and they improve the bounds on the continuous variables I_i^τ and $q_i^{k\tau}$.

$$I_i^{\tau} \geq I_i^{0,\tau}, \ \forall i \in V', \forall \tau \in \mathcal{T}_t(\gamma) \tag{7}$$

$$q_i^{k\tau} \leq U_i - I_i^{0,\tau}, \ \forall i \in V', \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}_t(\gamma), \tag{8}$$

where

$$I_i^{0,\tau} = \max\left\{0, I_i^0 - \sum_{t'=1}^{\tau} d_i^{t'}\right\}, \ \forall i \in V', \forall \tau \in \mathcal{T}_t(\gamma).$$

6.2 A Branch-and-cut algorithm

The branch-and-cut algorithm solves the formulation presented in Section 4 by separating the subtour elimination constraints (1i) at each node of the branch-and-bound tree. The initial LP is defined by (1a)–(1h), (1j)–(1l), and the linear relaxation of the non-negativity and integrality conditions (1m)–(1o). Moreover, valid inequalities (2)–(8) are added to the initial LP to improve the root node lower bound quality. Subtour elimination constraints (1i) are separated using the CVRPSEP package of Lysgaard et al. [49]. All the violated inequalities found are added to the LP.

7 Computational results

This section presents the computational experiments designed to evaluate our proposed approach. We first provide details about the instances and hardware used in the experimental phase. In Section 7.1, we present the numerical results concerning the five classes of online algorithms described in Section 4 ($\gamma = 0, 1, 2, 3, 4$). In Section 7.2, we present the numerical results concerning online algorithms associated with particular demand scenarios and belonging to a further class.

All implementations were in C++ under Gurobi's API, version 9.5.2. Computational tests were executed on a computer with an AMD Rome 7532 CPU 2.40GHz $\times$ 24 processor with 256MiB *cache* memory and 48GiB of RAM. A time limit of 7200 seconds was considered for the branch-and-cut algorithm. All detailed solutions appear in appendices described later and online at `https://www.leandro-coelho.com/online-irp-algorithms/`.

The tests were carried out on the instances introduced in Archetti et al. [5], where the number of customers ranges between 5 and 50 and the number of periods is $T = 3$ or $T = 6$. In particular, for $T = 6$ we only selected the instances with 30 customers at most. This is because the branch-and-cut algorithm described in Section 6.2 deteriorates quickly when the number of customers is greater than 30, which impacts the evaluation of the empirical competitive ratio for $\text{ALG}^{(\gamma)}$. The instances are divided into two classes according to their inventory holding costs. Instances of the class "Low Cost" have $h_i \in [0.01, 0.05]$ and $h_0 = 0.03$, while "High Cost" refers to the instances with $h_i \in [0.1, 0.5]$ and $h_0 = 0.3$.

For each instance under consideration, we have used $K = 1, 2, 3, 4, 5$. The multi-vehicle instances are obtained as in Coelho and Laporte [21], dividing the original vehicle capacity Q by K. This sometimes leads to infeasibility as customer demand exceeds the vehicle capacity. So we have not considered these instances. To distinguish the results where these cases occur, we have used a superscript value l, with $l = 1, \ldots, 4$, to denote the number of feasible instances in which the results are averaged. If no superscript value is shown, then all five instances are feasible. In particular, we used this superscript in Tables 5 and 6, and the tables of Appendices A–C.

7.1 ALG$^{(\gamma)}$ results

Table 1 summarizes the computational results of these first experiments on all instances with $T = 6$. Each row of the table represents one version of our algorithm with average results over all numbers of customers (ranging from 5 to 30) and vehicles (ranging from 1 to 5). Each version of the algorithm differs by the amount of information known beforehand, i.e., the uncertainty level. The columns indicate the solution value (UB), lower bound (LB), gap (Gap (%)), the runtime in seconds (Time (s)), and the empirical competitive ratio Z obtained by dividing the solution of the considered algorithm by the best lower bound of the deterministic problem:

$$Z = \frac{UB(\text{ALG}^{(\gamma)})}{LB(\text{O-IRP}_1(T-1))}. \tag{9}$$

The results of Table 1 show that even a sophisticated state-of-the-art branch-and-cut algorithm cannot solve the deterministic problem O-IRP$_1(T-1)$ to optimality for all instances. Indeed, for the low cost instances, the average optimality gap remains at 7.53% with a runtime of almost 1h. As the number of customers and vehicles increases, the problem becomes much harder, and several instances are not solved to optimality within the 2h time limit. For high cost instances, the average gap is lower, but the conclusions remain the same. When we apply algorithm ALG$^{(0)}$, i.e., the version of the algorithm in which only the demands of the customers of the current period are known, most subproblems are small enough to be solved to optimality in about 5 min on average, as we solve the problem with little information one period at a time. This lack of information leads to solutions where the decision maker must route the vehicles and make delivery plans that are not ideal, as indicated by the ratio Z of 1.58 (low cost) and 1.32 (high cost). This means that these solutions are 58% (32%) more costly than the deterministic solution with full information. As the information about the demand of the customers is considered for period $t+1$ in algorithm $\text{ALG}^{(1)}$, we observe runtimes that start to increase as the problems solved contain a bit more information, but the ratios decrease from 1.58 to 1.36 (low cost) and 1.32 to 1.19 (high cost). As most information becomes available in version $\text{ALG}^{(4)}$, the solutions become similar to those of the deterministic version, the runtimes also increase to similar levels, and the ratios become only 1.11 and 1.05.

Table 1: Average computational results of each algorithm for all instances with $T = 6$

	Low cost					High cost				
Algorithm	UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
$\text{O-IRP}_1(T-1)$	8081.13	7186.91	7.53	3390.56	–	14152.68	13305.72	4.19	3473.81	–
$\text{ALG}^{(0)}$	11377.62	11368.32	0.06	323.08	1.58	17509.02	17501.05	0.03	328.68	1.32
$\text{ALG}^{(1)}$	9822.76	9760.79	0.42	957.01	1.36	15854.20	15803.58	0.21	920.24	1.19
$\text{ALG}^{(2)}$	8756.35	8556.12	1.45	1700.46	1.20	14853.08	14653.21	0.88	1620.86	1.11
$\text{ALG}^{(3)}$	8159.83	7684.83	3.77	2448.26	1.12	14275.60	13812.29	2.16	2315.48	1.06
$\text{ALG}^{(4)}$	8127.77	7211.25	7.54	3338.27	1.11	14206.36	13375.75	4.08	3208.88	1.05

These results strongly indicate two meaningful conclusions. First, one can achieve results very similar to those of the deterministic version of the problem without full knowledge of the instance data. In fact, version $\text{ALG}^{(4)}$ sees the demand of the customers in the current period and four periods ahead (five out of the six periods of these instances) but finds solutions in a rolling horizon fashion that are sometimes just as good. Secondly, the ratios observed are very tight and, indeed, much lighter in practice than anticipated in theory; recall that from the theoretical analysis of Section 5, we would expect competitive ratios for the online algorithms of $\max\left\{\frac{h_{\max}}{h_{\min}}, T\right\}$, which in these instances would mean a ratio of $\max\left\{\frac{0.05}{0.01}, 6\right\} = 6$ (low cost) and $\max\left\{\frac{0.5}{0.1}, 6\right\} = 6$ (high cost). However, we see ratios close to 1.0 and often less than 1.5. Detailed results over the different number of customers and vehicles, presented in the tables of Appendix A, show that looking only at the current period and three periods ahead ($\text{ALG}^{(3)}$) is sometimes sufficient for the algorithm to find the same optimal solution as the algorithm with full information. When looking four periods ahead, in most instances with one and two vehicles, the solution is optimal as in the version with full knowledge of the demand.

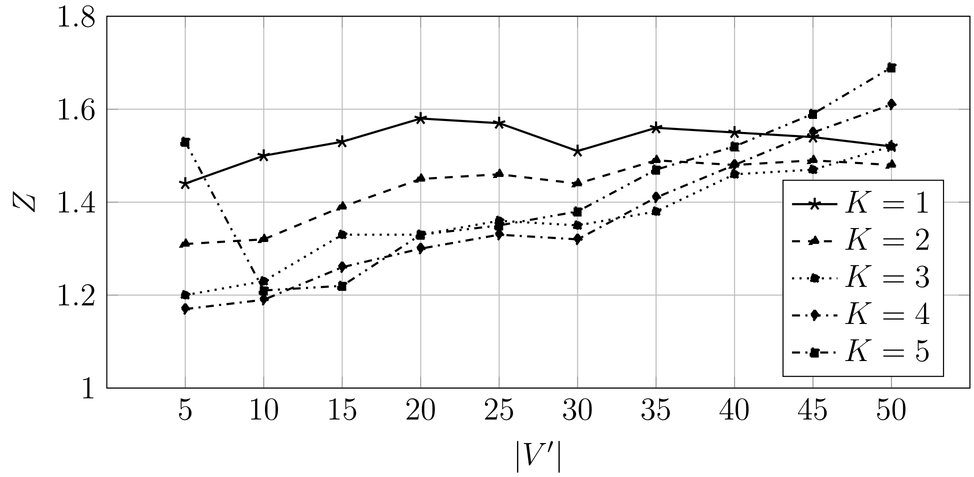
The results for the instances with a shorter planning horizon of $T = 3$ are shown in Table 2. Here, the information described is the same as that of Table 1, but it only makes sense to use $\text{ALG}^{(\gamma)}$ with $\gamma = \{0, 1\}$. We start again with the solution of the deterministic algorithm with full knowledge of the demand $\text{O-IRP}_1(T-1)$, where we see that the branch-and-cut algorithm has a smaller optimality gap of 4.67% (low cost instances) and 2.23% (high cost instances). Solving the problem one period at a time and without any information on the future demands, we see from $\text{ALG}^{(0)}$ that even though the runtimes are shorter (smaller problems solved at every period), the solution quality deteriorates with a ratio of 1.42 (low cost) and 1.19 (high cost). As these instances are smaller, the deviation with respect to the instances of six periods is also smaller, and a ratio of less than 50% worse solutions is attained. By increasing the amount of information available ($\text{ALG}^{(1)}$), one decreases the uncertainty and improves the quality of the solutions, with ratios decreasing to 1.26 (low cost) and 1.10 (high cost).

Table 2: Average computational results of each algorithm for all instances with $T = 3$

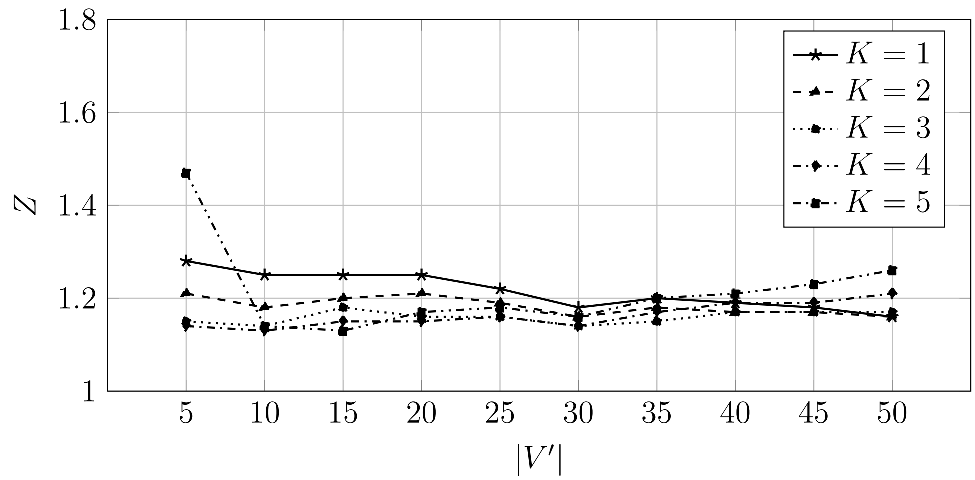
	Low cost					High cost				
Algorithm	UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
$\text{O-IRP}_1(T-1)$	3740.28	3449.05	4.67	2154.23	–	8453.68	8150.83	2.23	2129.40	–
$\text{ALG}^{(0)}$	4929.67	4883.92	0.61	823.65	1.42	9640.09	9594.99	0.30	835.66	1.19
$\text{ALG}^{(1)}$	4407.98	4231.31	2.07	1440.36	1.26	9042.75	8919.42	0.82	1404.76	1.10

Detailed results for each algorithm, shown in the tables of Appendix B, indicate that ratios as tight as 1.03 are obtained for some instances. Again, this confirms that this kind of online algorithm performs very well in practice, solving difficult problems in small steps, yielding very competitive solutions, and presenting empirical competitive ratios much better than the theoretical guarantees of $\max\left\{\frac{0.05}{0.01}, 3\right\} = 5$ (low cost) and $\max\left\{\frac{0.5}{0.1}, 3\right\} = 5$ (high cost).

A fundamental outcome of our computational experiments is that the numbers of customers and vehicles do not significantly affect the value of Z. First, we focus on the change in the number of customers. The variation of the ratio Z does not have a net increasing or decreasing trend. For example, we focus on the online algorithm with the lowest degree of information, i.e., $\text{ALG}^{(0)}$. In addition, we consider the case $T = 3$ to have a higher variation in the number of customers (recall that we solved instances with up to 50 customers for $T = 3$, and up to 30 customers for $T = 6$). Figure 2, which refers to Table 15 of Appendix B, plots the average values of ratio Z as the number of customers varies by distinguishing these values based on the number of vehicles K. The case "High Cost" (Figure 2b) emerges a rather constant trend represented by flat curves. The performance is less clear for the case "Low Cost" (Figure 2a). A slightly increasing trend would arise when the number of vehicles increases. However, higher Z values as the number of customers and vehicles increases are mainly due to the difficulty in solving the IRP instances at each time t, as already highlighted.



(a) Low Cost



(b) High Cost

Figure 2: Average value of ratio Z as the number of customers varies for $\mathrm{ALG}^{(0)}$ and $T = 3$

Second, to better analyze the performance of the online approach for the number of vehicles, we grouped all the instances without distinguishing them based on the number of customers. Tables 3 and 4 show the average values obtained for each value of K. We can state that the variation produced by the number of vehicles is not significant.

By analyzing the results reported in Tables 3 and 4, we confirm previous observations that as the value of γ increases, i.e., more information is available to the algorithm, the value of Z consistently decreases. From this perspective, these tables also offer the reader a clear idea of the value of additional information in this context, particularly for larger instances with $T = 6$ (Table 3). Finally, Tables 1–4 clearly show that the class "High Cost" is associated with lower Z values than the class "Low Cost". This can be explained by considering that the IRP model tends not to anticipate deliveries when the inventory costs of the customers are high. Hence, the routes activated by the proposed class of online algorithms are quite similar to the ones in the offline problem solution.

Table 3: Average computational results obtained by combining all instances with the same number of vehicles (instances with up to 30 customers for $T = 6$)

Algorithm	K	Low cost					High cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
ALG$^{(0)}$	1	9260.58	9260.58	0.00	1.92	1.63	15392.75	15392.75	0.00	1.67	1.32
	2	10207.48	10207.48	0.00	13.36	1.57	16339.65	16339.65	0.00	15.61	1.31
	3	11164.55	11164.55	0.00	78.71	1.53	17296.72	17296.72	0.00	81.86	1.29
	4	12354.68	12349.11	0.04	631.76	1.55	18486.85	18481.35	0.02	645.54	1.31
	5	13900.83	13859.90	0.25	889.65	1.59	20029.12	19994.78	0.13	898.70	1.34
ALG$^{(1)}$	1	7991.09	7991.07	0.00	6.91	1.41	14050.88	14050.88	0.00	5.40	1.21
	2	8617.46	8617.46	0.00	74.54	1.33	14622.98	14622.98	0.00	84.18	1.17
	3	9442.69	9433.83	0.07	1023.79	1.29	15439.79	15431.61	0.03	625.81	1.15
	4	10952.03	10871.27	0.60	1465.49	1.38	16820.94	16754.73	0.30	1489.61	1.19
	5	12110.53	11890.33	1.42	2214.31	1.39	18336.40	18157.70	0.72	2396.21	1.22
ALG$^{(2)}$	1	6585.30	6585.28	0.00	26.59	1.16	12638.34	12638.28	0.00	27.99	1.07
	2	7443.67	7438.27	0.06	740.07	1.14	13451.95	13451.87	0.00	594.59	1.07
	3	8553.77	8479.71	0.70	1685.20	1.18	14661.72	14570.51	0.42	1685.00	1.09
	4	9886.80	9564.08	2.47	2735.20	1.25	15999.56	15702.83	1.39	2683.36	1.13
	5	11312.19	10713.27	4.02	3315.24	1.30	17513.85	16902.55	2.58	3113.37	1.17
ALG$^{(3)}$	1	5798.52	5798.52	0.00	19.22	1.02	11881.63	11881.58	0.00	14.48	1.01
	2	6731.58	6721.32	0.11	1132.72	1.04	12816.94	12803.17	0.07	1036.92	1.02
	3	8033.06	7708.80	3.16	2597.53	1.10	14068.87	13763.71	1.59	2483.04	1.04
	4	9271.25	8562.69	5.93	3446.69	1.17	15511.76	14787.91	3.43	3284.31	1.09
	5	10964.72	9632.83	9.64	5045.14	1.26	17098.77	15825.09	5.70	4758.66	1.14
ALG$^{(4)}$	1	5684.36	5684.36	0.00	40.26	1.00	11777.99	11777.99	0.00	27.33	1.00
	2	6641.85	6498.18	1.70	1672.58	1.02	12749.29	12568.11	0.96	1807.04	1.01
	3	7954.18	7270.55	6.67	3517.74	1.09	14017.21	13375.33	3.38	3180.95	1.04
	4	9370.06	7885.07	12.66	5501.63	1.18	15456.83	14142.07	6.52	5301.03	1.09
	5	10988.39	8718.09	16.67	5959.12	1.26	17030.46	15015.26	9.51	5728.08	1.13

Table 4: Average computational results obtained by combining all instances with the same number of vehicles (instances with up to 50 customers for $T = 3$)

Algorithm	K	Low cost					High cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
ALG$^{(0)}$	1	4295.07	4295.07	0.00	42.95	1.53	8996.30	8996.28	0.00	30.79	1.21
	2	4522.51	4522.51	0.00	63.25	1.43	9223.74	9223.74	0.00	78.10	1.18
	3	4790.43	4784.41	0.09	341.76	1.36	9492.81	9485.66	0.05	312.54	1.16
	4	5246.91	5177.71	0.97	1593.39	1.36	9949.24	9887.62	0.41	1566.69	1.16
	5	5793.44	5639.92	2.01	2076.90	1.43	10538.33	10381.65	1.01	2190.17	1.22
ALG$^{(1)}$	1	3624.08	3624.08	0.00	14.82	1.29	8304.17	8304.14	0.00	10.51	1.11
	2	3924.46	3924.45	0.00	277.80	1.24	8535.44	8535.37	0.00	143.72	1.09
	3	4233.63	4160.88	1.29	1255.14	1.20	8917.80	8818.35	0.69	1195.89	1.08
	4	4759.18	4560.20	2.94	2500.22	1.24	9432.90	9279.48	1.09	2488.66	1.10
	5	5498.55	4886.92	6.13	3153.84	1.33	10023.44	9659.74	2.50	3185.04	1.13

7.2 Alternative uncertainty scenarios

The scenarios studied so far solve the problem on a rolling horizon for each period, where all the demands of the considered period t are known, besides all the demands of the periods $t+\gamma$. Our solution approach is general and can be applied to demand scenarios where the realization of the demand occurs differently, for example, where only some demands of some periods become gradually known. To this end, we consider the following further cases in which the perfect information on future periods relates only to some customers:

- *Scenario 1*: 100% of the customers have their demands known at t, 50% of the customers have their demands known at $t+1$, 25% of the customers have their demands known at $t+2$, and 10% of the customers have their demands known for all $t' \geq t+3$;
- *Scenario 2*: 100% of the customers have their demands known at t; in addition, the demand of 100% of the customers having demands greater than the average demand is known for each period $t' > t$.

These scenarios correspond to two versions in the class ALG$^{(T-t)}$ which are denoted as ALG$^{(T-t),s}$ where $s \in \{1,2\}$ represents the scenario. We used a two-phase procedure to select the customers for which we have demand information in Scenario 1. In the first phase, a customer is randomly chosen and put in a set L of selected customers. The remaining customers are chosen in the second phase in an iterative way, where at each iteration, a pair of customers $\{i,j\}$ with $i = \text{argmin}_{h\in V'\setminus L}\sum_{\ell\in L\cup\{0\}} c_{h\ell}$ and $j = \text{argmax}_{h\in V'\setminus L}\sum_{\ell\in L\cup\{0\}} c_{h\ell}$ is chosen and put in L until the percentage of elements to be selected is reached.

Tables 5 and 6 report the average numerical results obtained by our online algorithms in the class $\text{ALG}^{(T-t),s}$ with $s \in \{1,2\}$ for the instances with $T = 6$. For completeness, the equivalent tables with $T = 3$ are presented in Appendix C. These tables show the ratio Z between the average upper bounds obtained by $\text{ALG}^{(T-t),s}$ and the average lower bounds associated with the offline problem reported earlier in algorithm O-IRP$_1$($T-1$), i.e., $\text{ALG}^{(\gamma)}$ with $\gamma \in \{0,1,2,3,4\}$ is replaced by $\text{ALG}^{(T-t),s}$ with $s \in \{1,2\}$ in equation (9). Note that each instance was executed five times due to the randomness involved in scenario 1. Since there are up to five instances at each group of instances (recall that we have removed the infeasible ones), then the values shown in Table 5 are averaged over up to 25 runs. Note that for Table 6 there is no randomness associated with the realization of the demands.

Table 5: Average computational results for $\text{ALG}^{(T-t),1}$ in instances with $T = 6$

K	$\lvert V'\rvert$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	3283.88	3283.88	0.00	0.22	1.06	5173.45	5173.45	0.00	1.17	1.03
	10	5044.17	5044.17	0.00	0.63	1.11	8395.40	8395.40	0.00	0.83	1.06
	15	5773.72	5773.72	0.00	4.97	1.09	10976.46	10976.46	0.00	4.19	1.04
	20	7348.07	7348.07	0.00	108.21	1.13	14183.90	14183.90	0.00	63.50	1.06
	25	8130.49	8130.49	0.00	797.15	1.14	16460.37	16460.37	0.00	863.29	1.07
	30	8424.71	8424.71	0.00	1746.82	1.12	19403.00	19403.00	0.00	1275.34	1.05
2	5	4201.52	4201.52	0.00	1.73	1.08	6026.65	6026.65	0.00	1.44	1.04
	10	6362.49	6362.49	0.00	5.05	1.12	9588.14	9588.14	0.00	6.43	1.05
	15	6563.63	6563.63	0.00	18.85	1.05	11970.61	11970.61	0.00	20.27	1.04
	20	8168.52	8168.52	0.00	391.08	1.10	15185.56	15185.51	0.00	348.90	1.07
	25	9023.11	9023.11	0.00	1696.92	1.17	17325.36	17325.36	0.00	1481.84	1.08
	30	9413.58	9413.58	0.00	2504.94	1.19	20275.13	20275.13	0.00	2441.84	1.08
3	5	5320.94	5320.94	0.00	2.81	1.07	7172.01	7172.01	0.00	3.19	1.05
	10	7578.11	7578.11	0.00	14.95	1.10	10981.17	10981.17	0.00	12.09	1.07
	15	7642.98	7642.98	0.00	73.80	1.05	13172.42	13172.42	0.00	76.92	1.05
	20	9577.91	9577.91	0.00	1631.35	1.22	16401.10	16401.10	0.00	1805.28	1.11
	25	10352.00	10352.00	0.00	3036.73	1.26	18652.66	18652.66	0.00	2924.43	1.12
	30	10452.27	10452.27	0.00	3126.70	1.29	21302.21	21302.21	0.00	3251.37	1.12
4	5	6391.71	6391.71	0.00	3.84	1.07	8125.48	8125.48	0.00	5.66	1.04
	10	8883.64	8883.64	0.00	36.22	1.14	12408.10	12408.00	0.00	34.71	1.11
	15	8847.47	8847.47	0.00	242.24	1.10	14005.34	14005.34	0.00	234.81	1.05
	20	11237.13	11237.13	0.00	3901.35	1.35	17950.34	17950.34	0.00	3418.16	1.17
	25	11910.47	11910.47	0.00	4022.07	1.35	20295.79	20295.59	0.00	4075.44	1.18
	30	11789.34	11789.34	0.00	5069.89	1.43	22506.31	22506.31	0.00	4863.27	1.17
5	5	7846.28[4]	7846.28[4]	0.00[4]	4.35[4]	1.05	9773.88[4]	9773.88[4]	0.00[4]	6.15[4]	1.04
	10	10312.45	10312.45	0.00	90.59	1.16	13762.06	13762.06	0.00	95.99	1.12
	15	10218.29	10218.29	0.00	1023.34	1.17	15353.93	15353.93	0.00	918.53	1.10
	20	12553.48	12553.48	0.00	4281.53	1.40	19437.66	19437.66	0.00	4270.36	1.22
	25	13718.70	13718.65	0.00	4329.20	1.45	21757.96	21757.96	0.00	5437.54	1.22
	30	13334.92	13334.92	0.00	6506.69	1.55	23902.49	23902.49	0.00	5547.09	1.22
Avg		8656.87	8656.86	0.00	1489.14	1.19	14730.83	14730.82	0.00	1543.00	1.09

The values of ratio Z are very low, especially for Scenario 1. For this scenario, the increase in ratio Z as the number of vehicles increases does not seem too signifi-

Table 6: Average computational results for $\text{ALG}^{(T-t),2}$ in instances with $T = 6$

K	$\lvert V'\rvert$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	4966.20	4966.20	0.00	0.03	1.60	6873.44	6873.44	0.00	0.03	1.37
	10	7433.10	7433.10	0.00	0.16	1.63	10828.13	10828.13	0.00	0.15	1.36
	15	8644.95	8644.95	0.00	0.42	1.62	13924.85	13924.85	0.00	0.40	1.32
	20	10421.15	10421.15	0.00	1.43	1.60	17318.53	17318.53	0.00	1.42	1.30
	25	11552.66	11552.66	0.00	7.76	1.62	19893.44	19893.44	0.00	8.07	1.29
	30	12266.50	12266.30	0.00	10.29	1.63	23236.03	23235.83	0.00	11.74	1.26
2	5	5687.40	5687.40	0.00	0.14	1.46	7594.64	7594.64	0.00	0.55	1.31
	10	8615.70	8615.70	0.00	1.67	1.51	12010.73	12010.73	0.00	1.88	1.32
	15	9340.75	9340.75	0.00	3.42	1.50	14620.65	14620.65	0.00	3.24	1.27
	20	11467.75	11467.55	0.00	12.39	1.54	18365.13	18364.93	0.00	12.62	1.29
	25	12625.46	12625.46	0.00	32.71	1.64	20966.24	20966.24	0.00	34.59	1.30
	30	12978.30	12978.30	0.00	49.97	1.64	23947.83	23947.83	0.00	63.38	1.27
3	5	7004.17	7004.17	0.00	0.15	1.41	8909.66	8909.66	0.00	0.45	1.30
	10	9696.50	9696.50	0.00	2.45	1.40	13091.53	13091.53	0.00	2.20	1.27
	15	10371.95	10371.95	0.00	6.45	1.42	15651.85	15651.85	0.00	7.53	1.25
	20	12494.95	12494.75	0.00	83.68	1.59	19392.33	19392.13	0.00	96.52	1.31
	25	13371.66	13371.66	0.00	176.33	1.62	21712.44	21712.44	0.00	130.60	1.31
	30	13748.30	13748.30	0.00	1101.02	1.70	24717.83	24717.83	0.00	963.33	1.30
4	5	7709.06	7709.06	0.00	0.11	1.29	9615.09	9615.09	0.00	0.12	1.23
	10	11198.90	11198.90	0.00	3.90	1.43	14595.29	14595.29	0.00	4.26	1.31
	15	11527.75	11527.75	0.00	11.26	1.43	16807.65	16807.65	0.00	10.71	1.26
	20	13907.35	13907.35	0.00	136.98	1.66	20804.73	20804.73	0.00	139.99	1.36
	25	14893.06	14893.06	0.00	1633.91	1.69	23238.04	23238.04	0.00	1858.82	1.35
	30	14785.10	14764.30	0.13	2851.30	1.80	25707.23	25663.03	0.15	2749.52	1.34
5	5	10147.30[3]	10147.30[3]	0.00[3]	0.10[3]	1.36	12057.52[3]	12057.52[3]	0.00[3]	0.10[3]	1.29
	10	12499.39	12499.39	0.00	4.01	1.41	15894.68	15894.68	0.00	4.27	1.30
	15	12909.15	12909.15	0.00	17.42	1.48	18189.05	18189.05	0.00	17.39	1.30
	20	15591.75	15591.75	0.00	553.65	1.74	22489.13	22489.13	0.00	592.73	1.41
	25	16323.26	16202.86	0.69	3141.48	1.73	24707.44	24549.44	0.60	3152.26	1.39
	30	15915.10	15880.50	0.23	2700.72	1.85	26932.83	26908.23	0.10	2097.66	1.38
Avg		11336.49	11330.61	0.03	418.18	1.57	17469.80	17462.22	0.03	398.88	1.31

cant. Moreover, the results seem relatively stable with an increase in the number of customers. As before, instances with high inventory costs find better solutions, as routing and inventory holding become more balanced in the objective function.

Table 7 is similar to Tables 3 and 4 and summarizes the results obtained by grouping instances with the same number of vehicles K and different values for the number of customers $|V'|$. Some results seem to improve with a higher number of vehicles. This might be explained by the smaller capacity of each vehicle, which again limits the flexibility offered to the algorithm to satisfy the demands. In all cases, we observe that the empirical competitive ratios are very tight and significantly better than the theoretical guarantees.

Table 7: Average computational results obtained for the additional scenarios by combining all instances with the same number of vehicles (instances with up to 50 customers for $T = 3$ and 30 customers for $T = 6$)

Cost	Algorithm	K	$T=3$ UB	LB	Gap (%)	Time (s)	Z	$T=6$ UB	LB	Gap (%)	Time (s)	Z
Low	$\text{ALG}^{(T-t),1}$	1	3506.77	3506.63	0.00	292.79	1.24	6334.17	6334.17	0.00	443.00	1.11
		2	3830.69	3828.94	0.03	507.98	1.21	7288.81	7288.81	0.00	769.76	1.12
		3	4241.19	4239.52	0.03	1307.37	1.21	8487.37	8487.37	0.00	1314.39	1.16
		4	4669.97	4663.95	0.09	1998.98	1.21	9843.29	9843.29	0.00	2212.60	1.24
		5	5184.12	5178.27	0.08	2476.88	1.26	11330.69	11330.68	0.00	2705.95	1.30
	$\text{ALG}^{(T-t),2}$	1	4285.91	4285.91	0.00	14.01	1.52	9214.09	9214.06	0.00	3.35	1.62
		2	4483.13	4483.13	0.00	124.13	1.41	10119.23	10119.19	0.00	16.72	1.55
		3	4796.32	4791.17	0.07	459.47	1.36	11114.59	11114.55	0.00	228.35	1.53
		4	5217.27	5143.06	1.01	1486.48	1.35	12336.87	12333.40	0.02	772.91	1.55
		5	5709.18	5549.40	2.04	2275.26	1.39	13897.66	13871.83	0.15	1069.56	1.59
High	$\text{ALG}^{(T-t),1}$	1	8204.47	8204.35	0.00	243.53	1.09	12432.10	12432.10	0.00	368.05	1.05
		2	8497.64	8496.42	0.01	474.35	1.08	13395.24	13395.23	0.00	716.79	1.06
		3	8930.07	8925.90	0.03	1437.76	1.08	14613.60	14613.60	0.00	1345.54	1.09
		4	9465.65	9460.10	0.04	2017.16	1.10	15881.89	15881.84	0.00	2105.34	1.12
		5	9883.23	9875.39	0.05	2477.43	1.12	17331.33	17331.33	0.00	3179.28	1.15
	$\text{ALG}^{(T-t),2}$	1	8986.96	8986.94	0.00	12.93	1.21	15345.74	15345.70	0.00	3.64	1.32
		2	9184.34	9184.34	0.00	104.49	1.17	16250.87	16250.84	0.00	19.37	1.30
		3	9495.10	9489.50	0.04	545.14	1.16	17245.94	17245.91	0.00	200.11	1.29
		4	9939.11	9865.36	0.49	1474.77	1.16	18461.34	18453.97	0.03	793.90	1.31
		5	10377.66	10234.24	0.90	2154.71	1.17	20045.11	20014.67	0.12	977.40	1.34

8 Conclusions

Combinatorial optimization methods are largely applied to complex problems from a broad spectrum of real-world applications. Generally, most techniques assume that all input data are known beforehand. However, there are many practical cases where the problem instance is revealed incrementally, and decisions must be made before complete knowledge is available. In this setting, the problems are referred to as *dynamic* or *online*. Online algorithms represent practical approaches to real-time problems. In particular, these algorithms produce a partial solution as soon as a new piece of information becomes known.

In this paper, we have presented a class of online algorithms for a fundamental and well-studied problem in the distribution logistics area, called the inventory-routing problem (IRP), in which a supplier manages the inventory replenishment of its customers under uncertain conditions. In the version of the problem studied in this paper, the customers' demands are gradually revealed over time. In particular, at each period of a planning horizon, a decision has to be made without full knowledge of the problem instance but based on the demands revealed until that moment. Our class of online algorithms works by iteratively solving an integer programming model through a tailored branch-and-cut method.

Our class of online algorithms represents the first online optimization approach associated with a theoretical competitive analysis of an IRP. In particular, we have defined a bound on the ratio between the optimal cost obtained by *any* online algorithm of the class and the optimal cost associated with the deterministic instance in which the demands of all customers are known a priori. We have proved that the bound is $\max\left\{\frac{h_{max}}{h_{min}}, T\right\}$, which in the classical instances used in our paper would lead to competitive ratios of 5 or 6. We have theoretically proved that this bound is tight for the simplest algorithm in the class. In extensive computational experiments, we have empirically demonstrated the effective ratio is (much) lower than in these cases. Our results have shown that the average ratio is much smaller, even for the simplest algorithm in the class. In particular, for the first group of algorithms (Section 7.1), it is just 1.26 and 1.10 for the best online algorithm when $T = 3$ and the inventory cost is low and high, respectively, while it is just 1.11 and 1.05 for the best online algorithm when $T = 6$ and the inventory cost is low and high, respectively. Considering the online setting and that these ratios are computed using a lower bound on the optimal cost of the offline problem, these results are very satisfactory. Overall, we have observed ratios very close to 1.0, and rarely larger than 1.5.

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A Results for the offline and online problem for $T = 6$

Table 8 presents the results of our algorithm for the offline problem (O-IRP$_1(T-1)$). Tables 9–13 show the results of our online algorithms ALG$^{(\gamma)}$ with $\gamma = \{0, 1, 2, 3, 4\}$.

Table 8: Average computational results for O-IRP$_1(T-1)$ in instances with $T = 6$

K	$\|V'\|$	Low cost				High cost			
		UB	LB	Gap (%)	Time (s)	UB	LB	Gap (%)	Time (s)
1	5	3101.51	3101.51	0.00	0.36	5001.04	5000.98	0.00	1.00
	10	4547.22	4547.22	0.00	0.88	7953.63	7953.63	0.00	0.99
	15	5320.82	5320.82	0.00	2.48	10567.84	10567.84	0.00	2.46
	20	6499.09	6499.09	0.00	28.83	13350.25	13350.25	0.00	15.94
	25	7109.54	7109.54	0.00	24.49	15399.79	15399.79	0.00	17.00
	30	7506.84	7506.84	0.00	475.26	18395.37	18395.37	0.00	310.49
2	5	3888.74	3888.74	0.00	1.87	5783.64	5783.64	0.00	1.90
	10	5689.86	5689.86	0.00	26.02	9103.53	9103.53	0.00	23.96
	15	6231.31	6231.31	0.00	74.45	11480.70	11480.70	0.00	64.19
	20	7481.32	7450.90	0.36	2439.95	14331.54	14254.58	0.49	2280.09
	25	8113.39	7714.76	4.66	4350.78	16354.56	16080.64	1.64	4349.96
	30	8317.45	7903.34	4.82	4672.05	19230.09	18802.08	2.17	4546.99
3	5	4955.60	4955.60	0.00	13.72	6849.62	6849.62	0.00	15.48
	10	6912.25	6912.25	0.00	1672.76	10326.66	10286.80	0.36	2086.83
	15	7280.59	7280.59	0.00	1870.34	12527.79	12527.79	0.00	1046.40
	20	9053.23	7841.64	12.03	6008.79	15793.01	14788.01	6.01	5642.47
	25	9673.84	8239.99	13.73	4997.85	17948.43	16594.19	7.07	5007.85
	30	9721.83	8103.88	16.16	7209.66	20529.82	19070.03	6.99	7215.12
4	5	5954.86	5954.86	0.00	23.94	7839.69	7839.69	0.00	21.41
	10	8095.26	7804.59	3.22	4073.60	11513.92	11160.78	2.89	5164.50
	15	8415.93	8044.60	4.40	7203.72	13647.24	13333.34	2.24	7203.00
	20	10544.91	8353.34	19.22	7211.15	17357.76	15280.75	11.36	7209.90
	25	11321.39	8830.46	20.02	7208.20	19489.42	17173.25	11.07	7208.75
	30	11490.03	8229.61	27.71	7215.04	22666.34	19193.87	15.15	7217.61
5	5	7470.29[4]	7470.29[4]	0.00[4]	65.21[4]	9373.94[4]	9373.94[4]	0.00[4]	34.80[4]
	10	9356.59	8875.02	4.73	5995.53	12752.49	12254.91	3.63	5964.85
	15	9551.06	8742.36	8.44	7204.40	14820.42	13986.49	5.56	7206.23
	20	12157.73	8960.58	24.64	7210.59	19021.57	15948.43	15.48	7220.99
	25	13520.37	9461.40	27.27	7223.04	21512.01	17780.18	16.19	7116.70
	30	13150.97	8582.21	34.48	7211.69	23658.15	19556.47	17.32	7216.50
Avg		8081.13	7186.91	7.53	3390.56	14152.68	13305.72	4.19	3473.81

Table 9: Average computational results for $\text{ALG}^{(0)}$ in instances with $T = 6$

K	$\|V'\|$	Low Cost					High cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	5101.79	5101.79	0.00	0.03	1.64	7008.59	7008.59	0.00	0.08	1.40
	10	7441.35	7441.35	0.00	0.07	1.64	10836.90	10836.90	0.00	0.07	1.36
	15	8697.05	8697.05	0.00	0.15	1.63	13969.38	13969.38	0.00	0.16	1.32
	20	10423.55	10423.55	0.00	0.66	1.60	17321.72	17321.72	0.00	0.64	1.30
	25	11595.24	11595.24	0.00	3.60	1.63	19939.50	19939.50	0.00	3.04	1.29
	30	12304.50	12304.50	0.00	6.98	1.64	23280.41	23280.41	0.00	6.03	1.27
2	5	5885.59	5885.59	0.00	0.07	1.51	7792.39	7792.39	0.00	0.07	1.35
	10	8763.75	8763.75	0.00	1.00	1.54	12159.30	12159.30	0.00	1.00	1.34
	15	9434.65	9434.65	0.00	1.85	1.51	14706.98	14706.98	0.00	1.62	1.28
	20	11464.35	11464.35	0.00	7.77	1.54	18362.52	18362.52	0.00	8.61	1.29
	25	12695.24	12695.24	0.00	20.59	1.65	21039.50	21039.50	0.00	22.56	1.31
	30	13001.30	13001.30	0.00	48.87	1.65	23977.21	23977.21	0.00	59.81	1.28
3	5	7068.39	7068.39	0.00	0.08	1.43	8975.19	8975.19	0.00	0.07	1.31
	10	9723.35	9723.35	0.00	1.49	1.41	13118.90	13118.90	0.00	1.42	1.28
	15	10428.05	10428.05	0.00	4.24	1.43	15700.38	15700.38	0.00	3.62	1.25
	20	12547.35	12547.35	0.00	69.36	1.60	19445.52	19445.52	0.00	57.35	1.31
	25	13444.04	13444.04	0.00	126.48	1.63	21788.30	21788.30	0.00	129.46	1.31
	30	13776.10	13776.10	0.00	270.60	1.70	24752.01	24752.01	0.00	299.25	1.30
4	5	7771.59[4]	7771.59[4]	0.00[4]	0.06[4]	1.31	9678.39	9678.39	0.00	3.24	1.23
	10	11272.35	11272.35	0.00	3.18	1.44	14667.90	14667.90	0.00	2.77	1.31
	15	11571.25	11571.25	0.00	7.08	1.44	16843.58	16843.58	0.00	6.72	1.26
	20	13899.15	13899.15	0.00	131.09	1.66	20797.32	20797.32	0.00	129.34	1.36
	25	14887.24	14853.84	0.22	1772.81	1.69	23231.50	23198.90	0.14	1740.26	1.35
	30	14726.50	14726.50	0.00	1876.34	1.79	25702.41	25702.01	0.00	1990.90	1.34
5	5	10147.31[3]	10147.31[3]	0.00[3]	0.05[3]	1.36	12057.58[3]	12057.58[3]	0.00[3]	0.05[3]	1.29
	10	12499.35	12499.35	0.00	2.75	1.41	15894.90	15894.90	0.00	2.51	1.30
	15	12899.85	12899.85	0.00	13.07	1.48	18172.18	18172.18	0.00	13.80	1.30
	20	15621.35	15621.35	0.00	331.49	1.74	22519.52	22519.52	0.00	344.68	1.41
	25	16343.64	16178.24	0.96	3000.45	1.73	24680.90	24530.51	0.59	3026.32	1.39
	30	15893.50	15813.30	0.52	1990.10	1.85	26849.61	26794.01	0.22	2004.82	1.37
Avg		11377.62	11368.32	0.06	323.08	1.58	17509.02	17501.05	0.03	328.68	1.32

Table 10: Average computational results for $\text{ALG}^{(1)}$ in instances with $T = 6$

K	$\|V'\|$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	4568.27	4568.27	0.00	0.12	1.47	6461.28	6461.28	0.00	0.04	1.29
	10	6230.16	6230.07	0.00	0.14	1.37	9697.47	9697.47	0.00	0.13	1.22
	15	7718.72	7718.72	0.00	0.53	1.45	12739.99	12739.99	0.00	0.55	1.21
	20	8867.31	8867.31	0.00	2.98	1.36	15823.61	15823.61	0.00	2.60	1.19
	25	9200.10	9200.10	0.00	15.43	1.29	17674.87	17674.87	0.00	12.59	1.15
	30	11361.98	11361.98	0.00	22.29	1.51	21908.07	21908.07	0.00	16.46	1.19
2	5	4961.63	4961.63	0.00	0.71	1.28	7054.93	7054.93	0.00	0.59	1.22
	10	7630.32	7630.32	0.00	3.35	1.34	10554.36	10554.36	0.00	2.66	1.16
	15	8428.39	8428.39	0.00	5.69	1.35	13472.03	13472.03	0.00	5.10	1.17
	20	9385.46	9385.46	0.00	44.82	1.26	16304.62	16304.62	0.00	58.22	1.14
	25	10182.53	10182.53	0.00	224.33	1.32	18603.50	18603.50	0.00	200.70	1.16
	30	11116.46	11116.46	0.00	168.35	1.41	21748.42	21748.42	0.00	237.82	1.16
3	5	5503.58	5503.58	0.00	0.81	1.11	7395.54	7395.54	0.00	0.90	1.08
	10	8660.98	8660.98	0.00	6.98	1.25	11651.65	11651.65	0.00	6.12	1.13
	15	8817.45	8817.45	0.00	14.74	1.21	14210.01	14210.01	0.00	15.88	1.13
	20	11017.00	11016.87	0.00	193.77	1.40	17592.99	17592.99	0.00	329.21	1.19
	25	10981.00	10970.38	0.10	3298.91	1.33	19627.44	19627.44	0.00	609.12	1.18
	30	11676.16	11633.74	0.34	2627.55	1.44	22161.09	22112.03	0.20	2793.64	1.16
4	5	7244.57	7244.57	0.00	1.26	1.22	8883.97	8883.97	0.00	1.27	1.13
	10	10021.98	10021.98	0.00	18.03	1.28	12779.01	12779.01	0.00	18.68	1.14
	15	10243.23	10243.23	0.00	35.77	1.27	15644.23	15644.23	0.00	37.27	1.17
	20	11810.03	11810.03	0.00	1329.84	1.41	18652.95	18652.82	0.00	1848.10	1.22
	25	13357.44	13055.91	2.16	4346.14	1.51	20955.52	20759.82	0.93	3894.03	1.22
	30	13034.93	12851.91	1.44	3061.91	1.58	24009.94	23808.53	0.84	3138.28	1.25
5	5	8320.56[4]	8320.45[4]	0.00[4]	0.90[4]	1.11[4]	10172.85[4]	10172.82[4]	0.00[4]	0.98[4]	1.09
	10	10390.18	10390.18	0.00	37.86	1.17	14117.80	14117.80	0.00	37.96	1.15
	15	10929.82	10929.82	0.00	83.55	1.25	16284.84	16284.84	0.00	79.27	1.16
	20	13415.20	13355.10	0.40	3798.96	1.50	20996.19	20857.68	0.59	5090.46	1.32
	25	15003.74	14381.18	3.99	4418.98	1.59	23283.40	22749.67	2.19	4877.82	1.31
	30	14603.71	13965.25	4.14	4945.62	1.70	25163.33	24763.37	1.52	4290.78	1.29
Avg		9822.76	9760.79	0.42	957.01	1.36	15854.20	15803.58	0.21	920.24	1.19

Table 11: Average computational results for ALG$^{(2)}$

		Low Cost					High Cost				
K	$\|V'\|$	UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	3492.02[4]	3492.02[4]	0.00[4]	0.09[4]	1.13	5359.51	5359.45	0.00	0.11	1.07
	10	5332.29	5332.29	0.00	0.45	1.17	8672.29	8672.05	0.00	0.33	1.09
	15	6174.37	6174.33	0.00	1.70	1.16	11238.60	11238.60	0.00	1.43	1.06
	20	7627.39	7627.31	0.00	5.53	1.17	14479.70	14479.67	0.00	5.23	1.08
	25	8308.51	8308.51	0.00	16.41	1.17	16505.28	16505.28	0.00	10.31	1.07
	30	8577.24	8577.24	0.00	135.34	1.14	19574.65	19574.65	0.00	150.55	1.06
2	5	4303.63	4303.63	0.00	1.29	1.11	6259.14	6259.14	0.00	1.18	1.08
	10	6491.32	6491.32	0.00	5.85	1.14	9581.03	9581.03	0.00	4.32	1.05
	15	7077.86	7077.86	0.00	12.87	1.14	12256.87	12256.87	0.00	10.45	1.07
	20	8366.31	8366.31	0.00	333.90	1.12	15161.71	15161.71	0.00	288.69	1.06
	25	9196.03	9196.03	0.00	1115.48	1.19	17464.63	17464.45	0.00	1341.07	1.09
	30	9226.85	9194.47	0.36	2971.00	1.17	19988.34	19988.03	0.00	1921.86	1.06
3	5	5493.45	5493.45	0.00	1.69	1.11	7411.68	7411.68	0.00	1.58	1.08
	10	7703.79	7703.79	0.00	20.48	1.11	10785.85	10785.85	0.00	18.11	1.05
	15	7960.78	7960.78	0.00	45.88	1.09	13355.06	13355.06	0.00	38.99	1.07
	20	9550.30	9550.30	0.00	2103.02	1.22	16316.15	16316.15	0.00	2250.67	1.10
	25	10256.85	10048.27	2.05	3826.62	1.24	18736.73	18675.39	0.33	3870.97	1.13
	30	10357.46	10121.69	2.17	4113.53	1.28	21364.86	20878.95	2.21	3929.66	1.12
4	5	6589.16	6589.16	0.00	578.76	1.11	8179.54	8179.54	0.00	2.27	1.04
	10	9318.81	9318.81	0.00	82.85	1.19	12410.30	12410.30	0.00	80.92	1.11
	15	9200.71	9200.71	0.00	255.45	1.14	14547.81	14547.81	0.00	219.18	1.09
	20	10139.64	9957.11	1.38	3645.52	1.21	17971.52	17600.94	1.92	4488.94	1.18
	25	12154.66	11327.28	6.34	5209.60	1.38	20009.08	19254.80	3.59	4439.08	1.17
	30	11917.84	10991.42	7.11	6639.00	1.45	22879.10	22223.60	2.83	6869.80	1.19
5	5	8205.47[4]	8205.47[4]	0.00[4]	3.00[4]	1.10	10073.15[4]	10073.15[4]	0.00[4]	2.47[4]	1.07
	10	10051.44	10051.44	0.00	714.78	1.13	13427.81	13427.81	0.00	121.30	1.10
	15	10342.48	10342.48	0.00	718.72	1.18	15536.89	15536.89	0.00	693.40	1.11
	20	11904.23	11292.22	4.60	5118.67	1.33	19438.98	18527.68	4.31	5846.64	1.22
	25	13444.85	12109.65	8.97	6135.84	1.42	21717.52	20551.67	4.93	4815.51	1.22
	30	13924.68	12278.39	10.54	7200.43	1.62	24888.76	23298.11	6.25	7200.90	1.27
Avg		8756.35	8556.12	1.45	1700.46	1.20	14853.08	14653.21	0.88	1620.86	1.11

Table 12: Average computational results for ALG$^{(3)}$

		Low Cost					High Cost				
K	$\|V'\|$	UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	3149.92[4]	3149.92[4]	0.00[4]	0.24[4]	1.02	5063.20	5063.20	0.00	0.23	1.01
	10	4633.08	4633.08	0.00	1.20	1.02	8050.52	8050.40	0.00	0.79	1.01
	15	5455.75	5455.75	0.00	2.37	1.03	10588.50	10588.50	0.00	2.70	1.00
	20	6607.38	6607.38	0.00	7.62	1.02	13450.17	13450.17	0.00	7.71	1.01
	25	7277.91	7277.91	0.00	12.77	1.02	15589.79	15589.58	0.00	18.59	1.01
	30	7667.09	7667.09	0.00	91.12	1.02	18547.62	18547.62	0.00	56.87	1.01
2	5	3982.33	3982.33	0.00	1.60	1.02	5884.79	5884.79	0.00	1.47	1.02
	10	5900.46	5900.46	0.00	13.48	1.04	9178.93	9178.93	0.00	10.32	1.01
	15	6337.70	6337.70	0.00	32.46	1.02	11622.56	11622.56	0.00	22.50	1.01
	20	7608.54	7608.54	0.00	1164.84	1.02	14474.13	14474.13	0.00	893.29	1.02
	25	8161.14	8161.14	0.00	2655.03	1.06	16451.83	16451.83	0.00	2288.61	1.02
	30	8399.32	8337.76	0.67	2928.93	1.06	19289.41	19206.78	0.42	3005.32	1.03
3	5	5034.58	5034.58	0.00	99.79	1.02	6911.80	6911.80	0.00	3.22	1.01
	10	7295.44	7295.44	0.00	99.65	1.06	10420.42	10420.42	0.00	76.31	1.01
	15	7389.49	7389.49	0.00	211.09	1.01	12655.61	12655.37	0.00	168.48	1.01
	20	9009.36	8589.01	4.17	4437.12	1.15	15899.35	15407.80	2.91	4432.80	1.08
	25	9779.79	8977.66	7.58	4435.72	1.19	17924.26	17285.68	3.35	4452.92	1.08
	30	9689.69	8966.61	7.22	6301.81	1.20	20601.80	19901.21	3.27	5764.51	1.08
4	5	6097.46	6097.46	0.00	5.02	1.02	7923.37	7923.37	0.00	3.88	1.01
	10	8205.07	8205.07	0.00	677.08	1.05	11623.02	11623.02	0.00	688.29	1.04
	15	8500.91	8500.91	0.00	828.11	1.06	13749.86	13749.86	0.00	748.75	1.03
	20	10052.58	9366.99	5.79	5917.82	1.20	17444.08	16509.15	4.99	5949.39	1.14
	25	11418.80	9884.13	12.15	6049.49	1.29	19935.25	18311.51	7.66	5111.20	1.16
	30	11352.71	9321.58	17.65	7202.59	1.38	22394.99	20610.54	7.92	7204.32	1.17
5	5	7720.50[4]	7720.50[4]	0.00[4]	14.31[4]	1.03	9620.63[4]	9620.63[4]	0.00[4]	9.41[4]	1.03
	10	9532.27	9502.10	0.30	3911.69	1.07	13077.60	13035.88	0.30	2444.12	1.07
	15	9672.05	9394.80	2.86	5210.17	1.11	14887.14	14601.14	1.89	5396.83	1.06
	20	11726.39	10129.76	11.89	6739.11	1.31	19116.17	17252.08	9.23	6632.13	1.20
	25	13399.41	10753.73	18.51	7194.71	1.42	21867.35	19401.12	10.40	6867.94	1.23
	30	13737.70	10296.08	24.27	7200.82	1.60	24023.75	21039.67	12.35	7201.51	1.23
Avg		8159.83	7684.83	3.77	2448.26	1.12	14275.60	13812.29	2.16	2315.48	1.06

Table 13: Average computational results for ALG$^{(4)}$

		Low Cost					High Cost				
K	$\|V'\|$	UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	3101.51[4]	3101.51[4]	0.00[4]	0.37[4]	1.00	5001.04	5001.04	0.00	0.33	1.00
	10	4568.33	4568.33	0.00	0.99	1.00	7953.63	7953.63	0.00	0.95	1.00
	15	5320.82	5320.82	0.00	2.59	1.00	10567.84	10567.84	0.00	2.99	1.00
	20	6499.09	6499.09	0.00	16.09	1.00	13350.25	13350.25	0.00	10.46	1.00
	25	7109.54	7109.54	0.00	25.26	1.00	15399.79	15399.79	0.00	16.86	1.00
	30	7506.84	7506.84	0.00	196.29	1.00	18395.37	18395.37	0.00	132.39	1.00
2	5	3888.74	3888.74	0.00	1.85	1.00	5784.02	5784.02	0.00	1.69	1.00
	10	5848.54	5848.54	0.00	33.21	1.03	9103.53	9103.53	0.00	22.29	1.00
	15	6231.31	6231.31	0.00	74.35	1.00	11480.70	11480.70	0.00	63.14	1.00
	20	7482.69	7391.19	1.09	2147.81	1.00	14337.97	14250.56	0.56	2647.98	1.01
	25	8100.69	7702.93	4.69	4343.93	1.05	16360.09	16057.21	1.81	4339.30	1.02
	30	8299.13	7926.37	4.39	3434.35	1.05	19429.42	18732.63	3.40	3767.82	1.03
3	5	4955.76	4955.76	0.00	8.48	1.00	6850.67	6850.67	0.00	9.22	1.00
	10	7204.88	7204.88	0.00	951.34	1.04	10326.66	10326.66	0.00	834.59	1.00
	15	7280.59	7280.59	0.00	1364.98	1.00	12527.79	12527.72	0.00	775.16	1.00
	20	8620.42	7771.89	8.80	5521.04	1.10	15906.02	14794.01	6.58	5431.94	1.08
	25	9799.03	8209.36	14.89	6047.84	1.19	18050.60	16538.63	7.79	4826.69	1.09
	30	9864.41	8200.78	16.32	7212.73	1.22	20441.53	19214.28	5.94	7208.08	1.07
4	5	6085.26	6085.26	0.00	1114.99	1.02	8005.66	8005.66	0.00	7.83	1.02
	10	8142.45	7970.35	1.90	3531.79	1.04	11521.52	11316.49	1.65	3435.09	1.03
	15	8381.90	8165.53	2.54	6737.50	1.04	13647.02	13412.46	1.63	6745.43	1.02
	20	10175.03	8160.63	18.24	7210.44	1.22	17205.96	15292.63	10.47	7205.22	1.13
	25	11721.68	8645.52	24.56	7207.15	1.33	19916.35	17246.60	12.58	7207.86	1.16
	30	11714.04	8283.13	28.72	7207.93	1.42	22444.47	19578.61	12.81	7204.72	1.17
5	5	7588.72[4]	7588.67[4]	0.00[4]	25.88[4]	1.02	9510.18[4]	9510.18[4]	0.00[4]	16.13[4]	1.01
	10	9341.43	8912.32	4.12	6894.59	1.05	12918.66	12492.17	3.11	5529.82	1.05
	15	9580.38	8742.12	8.72	7204.98	1.10	14826.54	13897.75	6.20	7204.36	1.06
	20	11507.08	8956.72	20.84	7205.80	1.28	18819.14	16174.56	13.62	7206.46	1.18
	25	13938.09	9448.11	30.15	7222.64	1.47	21743.61	18233.90	15.31	7210.32	1.22
	30	13974.67	8660.60	36.19	7200.81	1.63	24364.63	19782.99	18.83	7201.37	1.25
Avg		8127.77	7211.25	7.54	3338.27	1.11	14206.36	13375.75	4.08	3208.88	1.05

B Results for the offline and online problem for $T = 3$

The results presented in this section in Tables 14–16 are similar to those of the previous section and refer to the instances with $T = 3$.

Table 14: Average computational results for O-IRP$_1(T-1)$ in instances with $T = 3$

K	$\|V'\|$	Low Cost UB	Low Cost LB	Low Cost Gap (%)	Low Cost Time (s)	High Cost UB	High Cost LB	High Cost Gap (%)	High Cost Time (s)
1	5	1250.02	1250.02	0.00	0.08	1940.95	1940.95	0.00	0.05
	10	1842.37	1842.37	0.00	0.13	3655.35	3655.35	0.00	0.22
	15	2118.49	2118.49	0.00	0.60	4541.56	4541.56	0.00	0.45
	20	2538.33	2538.33	0.00	1.37	5959.57	5959.57	0.00	0.94
	25	2822.90	2822.90	0.00	1.05	7331.46	7331.46	0.00	3.15
	30	3082.55	3082.55	0.00	7.63	8821.11	8821.04	0.00	3.16
	35	3232.74	3232.74	0.00	1.74	9239.59	9239.59	0.00	2.66
	40	3459.68	3459.68	0.00	41.22	10099.29	10099.29	0.00	8.01
	45	3592.98	3592.98	0.00	30.79	11120.06	11120.06	0.00	6.67
	50	4018.07	4018.07	0.00	96.85	12333.50	12333.50	0.00	45.90
2	5	1563.61	1563.61	0.00	0.27	2250.09	2250.09	0.00	0.79
	10	2318.40	2318.40	0.00	1.39	4128.03	4128.03	0.00	1.18
	15	2465.87	2465.87	0.00	2.91	4889.46	4889.46	0.00	2.59
	20	2948.48	2948.48	0.00	28.20	6369.14	6369.14	0.00	18.34
	25	3206.80	3206.80	0.00	362.77	7716.86	7716.86	0.00	169.14
	30	3365.41	3365.41	0.00	50.02	9103.91	9103.91	0.00	41.64
	35	3518.20	3518.20	0.00	88.19	9526.88	9526.88	0.00	73.41
	40	3737.88	3737.88	0.00	1239.80	10377.49	10377.49	0.00	1234.76
	45	3874.45	3852.29	0.53	1969.74	11413.81	11357.20	0.49	1946.97
	50	4462.20	4312.20	3.28	2805.56	12656.94	12591.89	0.53	2800.07
3	5	1936.72	1936.72	0.00	0.39	2616.05	2616.05	0.00	0.37
	10	2830.37	2830.37	0.00	3.53	4636.72	4636.72	0.00	3.06
	15	2862.67	2862.67	0.00	5.85	5285.02	5285.02	0.00	6.75
	20	3431.56	3431.56	0.00	198.49	6851.48	6851.48	0.00	195.66
	25	3714.76	3655.46	1.52	1562.28	8206.77	8191.08	0.19	1510.73
	30	3734.55	3734.55	0.00	1192.78	9472.24	9472.24	0.00	2173.14
	35	4007.53	3929.38	1.95	5048.07	10023.60	9927.16	0.96	4571.59
	40	4173.76	3934.07	5.51	4397.12	10795.40	10573.40	2.00	4369.54
	45	4326.61	4025.99	6.50	5783.38	11907.90	11537.40	3.07	5780.78
	50	5640.15	4378.16	20.62	7223.04	13780.05	12718.45	7.64	7214.17
4	5	2238.76	2238.76	0.00	0.71	2919.53	2919.51	0.00	0.54
	10	3342.54	3342.54	0.00	15.44	5148.41	5148.41	0.00	11.86
	15	3348.51	3348.51	0.00	26.35	5766.28	5766.28	0.00	35.84
	20	3971.17	3915.46	1.37	2651.09	7381.91	7381.91	0.00	1321.11
	25	4239.68	4120.53	2.64	2919.06	8751.11	8629.74	1.38	2655.29
	30	4166.19	4039.87	3.08	4632.89	9945.40	9754.00	1.86	4390.93
	35	4509.20	4187.98	7.01	5658.21	10494.45	10179.73	3.03	6573.66
	40	4687.69	4139.87	10.94	4986.60	11349.54	10715.24	5.46	5121.41
	45	5254.76	4095.92	19.02	5804.27	13381.43	11621.63	11.24	5843.65
	50	6390.16	4582.42	27.38	7211.39	14641.23	12882.25	11.97	7228.53
5	5	2578.35	2578.35	0.00	0.80	3259.60	3259.60	0.00	0.74
	10	3723.26	3723.26	0.00	24.02	5530.40	5530.40	0.00	30.35
	15	3796.73	3796.73	0.00	177.66	6215.89	6215.89	0.00	163.44
	20	4369.98	4220.94	3.08	4370.20	7781.54	7727.08	0.60	2806.22
	25	4680.89	4391.62	5.71	4086.55	9259.95	8833.77	4.33	4346.88
	30	4662.19	4180.93	10.10	5855.60	10401.67	9928.52	4.57	6088.28
	35	5103.00	4347.16	14.64	7223.50	11260.88	10278.00	8.65	7268.71
	40	5449.98	4249.43	19.95	6676.41	12129.95	10869.98	9.94	7211.40
	45	6185.36	4247.89	28.25	6010.02	13836.84	11731.94	14.18	5957.77
	50	8237.47	4738.06	40.31	7235.35	16177.70	13005.19	19.35	7227.30
Avg		3740.28	3449.05	4.67	2154.23	8453.68	8150.83	2.23	2129.40

Table 15: Average computational results for $\text{ALG}^{(0)}$ in instances with $T = 3$

K	$\|V'\|$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	1797.55	1797.55	0.00	0.01	1.44	2483.01	2483.01	0.00	0.01	1.28
	10	2766.79	2766.79	0.00	0.04	1.50	4574.84	4574.84	0.00	0.04	1.25
	15	3236.44	3236.44	0.00	0.09	1.53	5657.45	5657.45	0.00	0.11	1.25
	20	4003.39	4003.39	0.00	0.53	1.58	7421.50	7421.50	0.00	0.48	1.25
	25	4424.17	4424.17	0.00	2.42	1.57	8935.44	8935.44	0.00	3.04	1.22
	30	4655.32	4655.32	0.00	6.98	1.51	10405.79	10405.79	0.00	6.13	1.18
	35	5041.80	5041.80	0.00	6.24	1.56	11053.01	11053.01	0.00	7.27	1.20
	40	5371.97	5371.97	0.00	394.98	1.55	12003.85	12003.65	0.00	273.58	1.19
	45	5538.61	5538.61	0.00	10.52	1.54	13066.90	13066.90	0.00	8.62	1.18
	50	6114.61	6114.61	0.00	7.72	1.52	14361.22	14361.22	0.00	8.62	1.16
2	5	2040.75	2040.75	0.00	0.03	1.31	2726.21	2726.21	0.00	0.04	1.21
	10	3058.39	3058.39	0.00	0.27	1.32	4866.44	4866.44	0.00	0.23	1.18
	15	3433.64	3433.64	0.00	0.53	1.39	5854.65	5854.65	0.00	0.58	1.20
	20	4261.19	4261.19	0.00	2.94	1.45	7679.30	7679.30	0.00	2.94	1.21
	25	4678.37	4678.37	0.00	9.26	1.46	9189.64	9189.64	0.00	7.26	1.19
	30	4830.52	4830.52	0.00	48.97	1.44	10580.99	10580.99	0.00	24.37	1.16
	35	5249.20	5249.20	0.00	30.17	1.49	11260.41	11260.41	0.00	26.01	1.18
	40	5545.37	5545.37	0.00	45.58	1.48	12177.25	12177.25	0.00	42.45	1.17
	45	5739.81	5739.81	0.00	57.60	1.49	13268.10	13268.10	0.00	55.28	1.17
	50	6387.86	6387.86	0.00	437.16	1.48	14634.47	14634.47	0.00	621.81	1.16
3	5	2321.15	2321.15	0.00	0.03	1.20	3006.61	3006.61	0.00	0.03	1.15
	10	3480.39	3480.39	0.00	0.55	1.23	5288.44	5288.44	0.00	0.68	1.14
	15	3809.84	3809.84	0.00	1.16	1.33	6230.85	6230.85	0.00	1.34	1.18
	20	4562.79	4562.79	0.00	13.72	1.33	7980.90	7980.90	0.00	12.40	1.16
	25	4953.17	4953.17	0.00	54.16	1.36	9464.44	9464.44	0.00	51.50	1.16
	30	5045.92	5045.92	0.00	55.20	1.35	10796.39	10796.39	0.00	64.08	1.14
	35	5426.60	5426.60	0.00	232.34	1.38	11437.81	11437.81	0.00	170.96	1.15
	40	5732.77	5732.77	0.00	608.83	1.46	12364.65	12364.65	0.00	427.34	1.17
	45	5921.81	5921.81	0.00	386.70	1.47	13450.10	13450.10	0.00	332.07	1.17
	50	6649.86	6589.61	0.91	2064.93	1.52	14907.97	14836.47	0.49	2065.03	1.17
4	5	2628.87	2628.87	0.00	0.02	1.17	3332.40[4]	3332.40[4]	0.00[4]	0.02[4]	1.14
	10	3986.59	3986.59	0.00	1.29	1.19	5794.64	5794.64	0.00	0.99	1.13
	15	4215.44	4215.44	0.00	2.18	1.26	6636.45	6636.45	0.00	2.35	1.15
	20	5070.79	5070.79	0.00	25.44	1.30	8488.90	8488.90	0.00	22.69	1.15
	25	5482.97	5482.97	0.00	317.66	1.33	9994.24	9994.24	0.00	294.92	1.16
	30	5336.12	5336.12	0.00	557.74	1.32	11086.59	11086.39	0.00	654.77	1.14
	35	5925.20	5900.00	0.39	2231.87	1.41	11944.01	11906.81	0.27	2168.60	1.17
	40	6122.37	6032.57	1.43	2204.34	1.48	12761.85	12664.45	0.72	2091.82	1.19
	45	6340.41	6211.41	1.92	3384.74	1.55	13853.10	13749.70	0.74	3216.08	1.19
	50	7360.30	6912.30	6.01	7208.66	1.61	15600.22	15222.22	2.41	7214.61	1.21
5	5	3945.02[2]	3945.02[2]	0.00[2]	14.04[2]	1.53	4800.06[1]	4800.06[1]	0.00[1]	7.34[1]	1.47
	10	4499.59	4499.59	0.00	1.05	1.21	6307.64	6307.64	0.00	0.98	1.14
	15	4621.44	4621.44	0.00	2.70	1.22	7042.45	7042.45	0.00	23.09	1.13
	20	5595.59	5595.59	0.00	118.20	1.33	9013.70	9013.70	0.00	119.31	1.17
	25	5931.37	5931.37	0.00	459.80	1.35	10442.64	10442.64	0.00	685.01	1.18
	30	5761.32	5761.32	0.00	1331.81	1.38	11511.79	11511.79	0.00	1217.31	1.16
	35	6378.60	6254.80	1.84	4594.13	1.47	12380.41	12272.61	0.81	4532.16	1.20
	40	6479.57	6315.17	2.44	3212.39	1.52	13113.05	12943.85	1.21	3156.40	1.21
	45	6733.13	6424.38	4.15	3827.34	1.59	14393.90	13992.10	2.73	4940.11	1.23
	50	7988.79	7050.54	11.63	7207.57	1.69	16377.66	15489.66	5.36	7219.99	1.26
Avg		4929.67	4883.92	0.61	823.65	1.42	9640.09	9594.99	0.30	835.66	1.19

Table 16: Average computational results for $\text{ALG}^{(1)}$ in instances with $T = 3$

K	$\|V'\|$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	1588.05	1588.05	0.00	0.02	1.27	2154.90	2154.90	0.00	0.02	1.11
	10	2282.77	2282.77	0.00	0.04	1.24	4235.69	4235.69	0.00	0.08	1.16
	15	2616.61	2616.61	0.00	0.29	1.24	5025.75	5025.75	0.00	0.34	1.11
	20	3519.37	3519.37	0.00	2.28	1.39	6617.22	6617.22	0.00	2.49	1.11
	25	3165.20	3165.20	0.00	13.73	1.12	7979.80	7979.80	0.00	6.42	1.09
	30	4649.62	4649.62	0.00	24.05	1.51	10021.44	10021.44	0.00	12.22	1.14
	35	3607.41	3607.41	0.00	31.43	1.12	9616.24	9616.04	0.00	21.87	1.04
	40	4473.15	4473.15	0.00	21.30	1.29	10940.79	10940.79	0.00	29.24	1.08
	45	4735.61	4735.61	0.00	15.14	1.32	12618.14	12618.14	0.00	15.68	1.13
	50	5602.96	5602.96	0.00	39.94	1.39	13831.73	13831.61	0.00	16.75	1.12
2	5	1885.76	1885.76	0.00	0.16	1.21	2477.05	2477.05	0.00	0.28	1.10
	10	2660.32	2660.32	0.00	1.06	1.15	4587.21	4587.21	0.00	0.74	1.11
	15	2910.38	2910.38	0.00	1.51	1.18	5283.36	5283.34	0.00	1.25	1.08
	20	3806.33	3806.33	0.00	13.51	1.29	6985.71	6985.71	0.00	13.38	1.10
	25	3540.00	3540.00	0.00	146.96	1.10	8317.84	8317.84	0.00	28.43	1.08
	30	4824.49	4824.49	0.00	53.56	1.43	10215.93	10215.71	0.00	55.95	1.12
	35	3885.67	3885.67	0.00	116.67	1.10	9891.47	9891.47	0.00	127.62	1.04
	40	4875.84	4875.84	0.00	989.98	1.30	11168.30	11168.30	0.00	92.70	1.08
	45	4967.18	4967.18	0.00	248.65	1.29	12834.50	12834.50	0.00	170.65	1.13
	50	5888.58	5888.52	0.00	1205.95	1.37	13592.97	13592.57	0.00	946.24	1.08
3	5	2076.43	2076.43	0.00	0.37	1.07	2707.44	2707.44	0.00	0.32	1.03
	10	3092.34	3092.34	0.00	1.97	1.09	4956.22	4956.22	0.00	2.00	1.07
	15	3110.29	3110.29	0.00	3.79	1.09	5694.09	5694.09	0.00	3.69	1.08
	20	4120.06	4120.06	0.00	59.76	1.20	7365.80	7365.80	0.00	164.71	1.08
	25	4175.21	4175.21	0.00	256.19	1.14	8679.61	8679.61	0.00	178.34	1.06
	30	5025.18	5025.15	0.00	561.84	1.35	10255.76	10255.76	0.00	719.80	1.08
	35	4311.99	4266.43	1.10	3113.45	1.10	10322.23	10275.34	0.44	3198.54	1.04
	40	4940.32	4940.32	0.00	1565.69	1.26	11683.49	11683.49	0.00	314.22	1.10
	45	5433.51	5433.51	0.00	1817.52	1.35	12906.78	12906.48	0.00	1682.95	1.12
	50	6050.94	5369.03	11.76	5170.86	1.38	14606.55	13659.23	6.45	5694.30	1.15
4	5	2496.26	2496.26	0.00	0.45	1.12	3095.31	3095.31	0.00	0.40	1.06
	10	3548.75	3548.75	0.00	4.58	1.06	5550.88	5550.88	0.00	4.54	1.08
	15	3707.70	3707.70	0.00	11.53	1.11	6102.14	6102.14	0.00	9.89	1.06
	20	4556.67	4556.67	0.00	197.45	1.16	7963.90	7963.74	0.00	138.64	1.08
	25	4794.32	4794.32	0.00	1115.60	1.16	9256.64	9256.60	0.00	1278.21	1.07
	30	5011.93	4927.87	1.78	2940.68	1.24	10516.04	10437.15	0.75	2921.00	1.08
	35	5161.18	4988.47	3.07	5025.32	1.23	11051.42	10896.94	1.29	4916.36	1.09
	40	5380.39	5253.55	2.29	3865.56	1.30	11978.09	11866.16	0.95	3885.67	1.12
	45	5870.84	5558.53	4.64	4620.09	1.43	13236.59	12808.02	3.13	4504.44	1.14
	50	7063.78	5769.92	17.62	7220.95	1.54	15577.98	14817.87	4.77	7227.50	1.21
5	5	2875.23[4]	2875.23[4]	0.00[4]	0.34[4]	1.12	3683.91[4]	3683.91[4]	0.00[4]	0.63[4]	1.13
	10	4019.55	4019.55	0.00	7.95	1.08	5917.63	5917.63	0.00	6.65	1.07
	15	4091.10	4091.10	0.00	22.73	1.08	6513.55	6513.48	0.00	31.27	1.05
	20	5005.70	5005.70	0.00	441.20	1.19	8414.29	8414.29	0.00	542.83	1.09
	25	5241.82	5176.67	1.20	3623.90	1.19	9754.73	9651.41	1.02	4112.64	1.10
	30	5418.13	5299.94	2.11	3230.82	1.30	11149.71	11012.86	1.20	3148.35	1.12
	35	6009.19	5607.52	6.46	5884.24	1.38	11682.70	11215.01	4.02	5873.17	1.14
	40	6246.82	5815.58	6.48	5807.86	1.47	12622.07	12302.17	2.57	5419.94	1.16
	45	5964.11	5121.02	12.44	5312.55	1.40	14073.25	13210.04	5.77	5503.47	1.20
	50	10113.83	5856.94	32.64	7206.76	2.13	16422.59	14676.59	10.47	7211.43	1.26
Avg		4407.98	4231.31	2.07	1440.36	1.26	9042.75	8919.42	0.82	1404.76	1.10

C Results for the alternative scenarios, $T = 3$

Tables 17 and 18 below present the results for the two alternative scenarios for instances with three periods.

Table 17: Average computational results for $\text{ALG}^{(T-t),1}$ in instances with $T = 3$

K	$\lvert V'\rvert$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	1427.70	1427.70	0.00	0.03	1.14	2185.06	2185.06	0.00	8.01	1.13
	10	1879.66	1879.66	0.00	0.09	1.02	3776.58	3776.58	0.00	0.26	1.03
	15	2467.43	2467.43	0.00	0.31	1.16	4865.20	4865.20	0.00	0.39	1.07
	20	3357.01	3357.01	0.00	6.14	1.32	6783.42	6783.42	0.00	5.69	1.14
	25	3735.67	3735.67	0.00	16.57	1.32	8352.27	8352.27	0.00	58.11	1.14
	30	3888.52	3888.52	0.00	521.18	1.26	9580.87	9580.67	0.00	374.87	1.09
	35	4203.53	4203.53	0.00	513.06	1.30	10170.32	10170.27	0.00	424.50	1.10
	40	4430.67	4429.20	0.04	1133.11	1.28	11069.09	11068.87	0.00	499.46	1.10
	45	4613.54	4613.54	0.00	425.73	1.28	12136.62	12136.22	0.00	63.62	1.09
	50	5064.00	5064.00	0.00	311.68	1.26	13125.23	13124.93	0.00	1000.42	1.06
2	5	1724.12	1724.12	0.00	0.15	1.10	2402.68	2402.68	0.00	0.24	1.07
	10	2332.75	2332.75	0.00	1.16	1.01	4178.86	4178.86	0.00	0.80	1.01
	15	2814.49	2814.49	0.00	2.20	1.14	5220.41	5220.41	0.00	2.00	1.07
	20	3719.05	3719.05	0.00	16.39	1.26	7177.09	7177.09	0.00	16.41	1.13
	25	4105.15	4105.15	0.00	46.37	1.28	8576.39	8576.39	0.00	48.15	1.11
	30	4168.01	4168.01	0.00	346.16	1.24	9845.95	9845.90	0.00	572.69	1.08
	35	4540.66	4540.66	0.00	797.96	1.29	10393.42	10393.31	0.00	777.41	1.09
	40	4716.96	4716.96	0.00	846.93	1.26	11224.55	11224.44	0.00	495.53	1.08
	45	4816.83	4816.83	0.00	571.09	1.25	12389.11	12388.95	0.00	907.80	1.09
	50	5368.83	5351.37	0.33	2451.36	1.25	13567.91	13556.18	0.08	1922.47	1.08
3	5	2107.45	2107.45	0.00	0.36	1.09	2737.07	2737.07	0.00	0.37	1.05
	10	2901.69	2901.69	0.00	2.30	1.03	4709.75	4709.75	0.00	2.65	1.02
	15	3251.48	3251.48	0.00	4.93	1.14	5678.61	5678.61	0.00	5.30	1.07
	20	4131.53	4131.53	0.00	45.99	1.20	7529.72	7529.72	0.00	54.54	1.10
	25	4503.74	4503.74	0.00	262.34	1.23	9060.06	9060.01	0.00	276.75	1.11
	30	4561.92	4561.92	0.00	1186.35	1.22	10192.28	10192.23	0.00	1617.88	1.08
	35	4886.35	4886.35	0.00	1755.70	1.24	10948.81	10948.56	0.00	2070.61	1.10
	40	4908.79	4908.79	0.00	1603.05	1.25	11520.65	11520.48	0.00	1597.46	1.09
	45	5171.95	5171.95	0.00	2376.36	1.28	12784.03	12782.33	0.01	3131.40	1.11
	50	5987.01	5970.26	0.27	5836.34	1.37	14139.69	14100.19	0.27	5620.61	1.11
4	5	2319.48	2319.48	0.00	0.97	1.04	3009.05	3009.05	0.00	0.55	1.03
	10	3469.62	3469.62	0.00	4.44	1.04	5340.74	5340.74	0.00	3.79	1.04
	15	3671.21	3671.21	0.00	10.84	1.10	6080.85	6080.85	0.00	9.92	1.05
	20	4597.57	4597.57	0.00	170.38	1.17	8101.19	8101.19	0.00	227.90	1.10
	25	4827.60	4827.60	0.00	836.88	1.17	9397.98	9397.98	0.00	1195.73	1.09
	30	4820.48	4820.48	0.00	1674.33	1.19	10661.76	10661.71	0.00	1619.28	1.09
	35	5250.78	5250.78	0.00	3865.03	1.25	11493.19	11493.04	0.00	3534.44	1.13
	40	5380.68	5380.68	0.00	3495.02	1.30	11970.01	11969.85	0.00	3536.62	1.12
	45	5703.16	5703.16	0.00	3528.14	1.39	13285.80	13280.45	0.04	3538.40	1.14
	50	6659.10	6598.94	0.88	6403.78	1.45	15315.92	15266.09	0.31	6504.98	1.19
5	5	2766.06[4]	2766.06[4]	0.00[4]	0.47[4]	1.07	3472.47[4]	3472.47[4]	0.00[4]	0.57[4]	1.07
	10	3919.90	3919.90	0.00	6.90	1.05	5803.46	5803.46	0.00	6.34	1.05
	15	4104.04	4104.04	0.00	24.67	1.08	6552.89	6552.89	0.00	20.63	1.05
	20	5071.41	5071.41	0.00	446.34	1.20	8443.55	8443.55	0.00	300.91	1.09
	25	5406.25	5406.25	0.00	1915.85	1.23	9869.36	9869.36	0.00	2455.05	1.12
	30	5196.76	5196.76	0.00	2912.08	1.24	10922.34	10922.29	0.00	2771.31	1.10
	35	5864.49	5864.49	0.00	4465.69	1.35	11815.68	11815.58	0.00	4379.71	1.15
	40	5822.84	5820.16	0.04	4316.86	1.37	12392.71	12386.26	0.05	4350.27	1.14
	45	6132.07	6124.12	0.11	4355.40	1.44	13604.48	13604.27	0.00	3822.40	1.16
	50	7557.44	7509.56	0.63	6324.52	1.60	15955.38	15883.79	0.44	6667.13	1.23
Avg		4286.55	4283.46	0.05	1316.80	1.22	8996.21	8992.43	0.02	1330.05	1.09

Table 18: Average computational results for $\mathrm{ALG}^{(T-t),2}$ in instance with $T = 3$

K	$\|V'\|$	Low Cost					High Cost				
		UB	LB	Gap (%)	Time (s)	Z	UB	LB	Gap (%)	Time (s)	Z
1	5	1723.13	1723.13	0.00	0.19	1.38	2408.96	2408.96	0.00	0.15	1.24
	10	2784.19	2784.19	0.00	0.05	1.51	4590.66	4590.66	0.00	0.14	1.26
	15	3218.58	3218.58	0.00	0.09	1.52	5643.90	5643.90	0.00	0.11	1.24
	20	4033.99	4033.99	0.00	0.78	1.59	7453.06	7453.06	0.00	0.61	1.25
	25	4414.18	4414.18	0.00	3.07	1.56	8923.29	8923.29	0.00	3.07	1.22
	30	4645.53	4645.53	0.00	4.32	1.51	10396.41	10396.41	0.00	3.80	1.18
	35	4997.75	4997.75	0.00	3.57	1.55	11015.82	11015.82	0.00	2.86	1.19
	40	5391.38	5391.38	0.00	105.02	1.56	12022.16	12021.96	0.00	92.76	1.19
	45	5549.03	5549.03	0.00	6.64	1.54	13067.08	13067.08	0.00	6.47	1.18
	50	6101.36	6101.36	0.00	16.41	1.52	14348.22	14348.22	0.00	19.32	1.16
2	5	1832.04	1832.04	0.00	0.04	1.17	2515.72	2515.72	0.00	0.03	1.12
	10	3030.79	3030.79	0.00	0.27	1.31	4841.06	4841.06	0.00	0.27	1.17
	15	3419.38	3419.38	0.00	0.68	1.39	5844.70	5844.70	0.00	0.66	1.20
	20	4249.79	4249.79	0.00	3.07	1.44	7668.86	7668.86	0.00	3.52	1.20
	25	4668.38	4668.38	0.00	11.64	1.46	9177.49	9177.49	0.00	9.16	1.19
	30	4783.93	4783.93	0.00	23.57	1.42	10534.81	10534.81	0.00	18.21	1.16
	35	5226.35	5226.35	0.00	31.91	1.49	11244.42	11244.42	0.00	24.23	1.18
	40	5531.18	5531.18	0.00	533.31	1.48	12161.96	12161.96	0.00	573.67	1.17
	45	5718.83	5718.83	0.00	38.56	1.48	13236.88	13236.88	0.00	46.46	1.17
	50	6370.61	6370.61	0.00	598.29	1.48	14617.47	14617.47	0.00	368.68	1.16
3	5	2277.34	2277.34	0.00	0.06	1.18	2961.04	2961.04	0.00	0.04	1.13
	10	3515.79	3515.79	0.00	0.63	1.24	5322.26	5322.26	0.00	0.58	1.15
	15	3754.38	3754.38	0.00	1.71	1.31	6179.70	6179.70	0.00	1.51	1.17
	20	4584.79	4584.79	0.00	17.16	1.34	8003.86	8003.86	0.00	14.02	1.17
	25	4929.78	4929.78	0.00	90.26	1.35	9438.89	9438.89	0.00	102.10	1.15
	30	5067.13	5067.13	0.00	146.44	1.36	10818.01	10818.01	0.00	168.37	1.14
	35	5450.95	5450.95	0.00	863.93	1.39	11469.02	11469.02	0.00	589.99	1.16
	40	5770.78	5770.78	0.00	787.95	1.47	12401.56	12401.56	0.00	751.05	1.17
	45	5936.63	5936.63	0.00	461.77	1.47	13454.68	13454.68	0.00	1396.80	1.17
	50	6675.61	6624.11	0.75	2224.82	1.52	14901.97	14845.97	0.38	2426.90	1.17
4	5	2547.71	2547.71	0.00	9.11	1.14	3228.81	3228.81	0.00	7.76	1.11
	10	3975.39	3975.39	0.00	1.04	1.19	5784.55	5784.55	0.00	1.04	1.12
	15	4144.58	4144.58	0.00	3.18	1.24	6569.90	6569.90	0.00	3.19	1.14
	20	5102.19	5102.19	0.00	37.73	1.30	8521.26	8521.26	0.00	35.68	1.15
	25	5487.98	5487.98	0.00	491.87	1.33	9997.09	9996.89	0.00	535.75	1.16
	30	5321.73	5321.73	0.00	501.40	1.32	11072.61	11072.61	0.00	430.99	1.14
	35	5884.55	5856.95	0.42	1818.03	1.41	11902.62	11875.62	0.19	1824.29	1.17
	40	6126.98	6038.38	1.40	2301.38	1.48	12756.56	12674.16	0.61	2499.23	1.19
	45	6203.75	6159.42	0.67	2485.99	1.51	13845.54	13741.88	0.74	2193.24	1.19
	50	7377.79	6796.29	7.64	7215.02	1.61	15712.19	15187.94	3.33	7216.54	1.22
5	5	2818.74[4]	2818.74[4]	0.00[4]	0.69[4]	1.09	3523.95[3]	3523.95[3]	0.00[3]	0.05[3]	1.08
	10	4470.19	4470.19	0.00	1.03	1.20	6266.52	6266.52	0.00	0.97	1.13
	15	4578.98	4578.98	0.00	3.80	1.21	7004.30	7004.30	0.00	3.63	1.13
	20	5530.39	5530.39	0.00	55.62	1.31	8949.46	8949.46	0.00	53.27	1.16
	25	5936.58	5936.58	0.00	1033.03	1.35	10445.69	10445.69	0.00	925.03	1.18
	30	5766.33	5766.33	0.00	1056.66	1.38	11517.21	11517.01	0.00	975.06	1.16
	35	6285.75	6205.75	1.16	4551.14	1.45	12307.02	12223.02	0.60	4353.86	1.20
	40	6519.78	6302.18	3.18	4405.37	1.53	12871.99	12833.24	0.29	3669.98	1.18
	45	6820.23	6483.23	4.64	4444.54	1.61	14364.08	14008.28	2.43	4370.46	1.22
	50	8364.86	7401.61	11.41	7200.71	1.77	16526.34	15570.94	5.71	7194.78	1.27
Avg		4898.36	4850.53	0.63	871.87	1.41	9596.63	9552.07	0.29	858.41	1.17