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A Unifying Framework for Selective Routing Problems

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Abstract. We present a unifying framework for Selective Routing Problems (SRPs) through a systematic analysis. The common goal in SRPs is to determine an optimal route to serve a subset of vertices while covering another subset. To present a unifying framework for different but related problems, we associate the notion of service with coverage and argue that routing is a tool of service. We classify SRPs according to their selectiveness degree and emphasize the breadth and depth of this problem in terms of its characteristics. This SRP framework helps us identify research gaps as well as potential future research areas. We present a generic mathematical model, use it to describe the connections among these problems and identify some identical problems presented under different names.

Keywords: Combinatorial optimization, Framework, Selective routing problems, Selectiveness degree.

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1. Introduction

Routing problems are widely studied in operations research due to their extensive real-life applications. The common goal in these problems is to construct routes that optimize an objective function, the most common being cost minimization. A route is defined as an ordered set of locations to be visited. The Transportation Problem, the Shortest Path Problem, the Traveling Salesman Problem, and the Vehicle Routing Problem are classic problems in the routing literature. Most of the routing problems studied today have arisen from them.

The Traveling Salesman Problem (TSP), the Vehicle Routing Problem (VRP), and the Location-Routing Problem (LRP) have been extensively studied. Given a graph, the TSP identifies a minimum cost tour visiting every vertex of the graph (Dantzig et al., 1954). A tour is a route that starts and ends at the same vertex. Likewise, the VRP finds multiple tours starting and ending at a depot that minimize the total cost of visiting every vertex of the graph (Dantzig and Ramser, 1959). The LRP, on the other hand, is a combination of classical Facility Location Problems and the VRP. It is a variant of the VRP in which the depot locations are also decision variables (Perl and Daskin, 1985). For the TSP, we refer the interested readers to the surveys by Jünger et al. (1995), Hoffman and Padberg (2001), Laporte (2010), and the books by Lawler et al. (1985), Punnen (2007), and Applegate et al. (2006). There exist many studies on different variants of the VRP, and there have been several surveys and taxonomies devoted to the VRP, e.g., Desrochers et al. (1990), Cordeau et al. (2007), Laporte (2009), Eksioglu et al. (2009), Lahyani et al. (2015) and Braekers et al. (2016) along with the books by Golden et al. (2008) and Toth and Vigo (2002). The LRP combines location analysis with the underlying issues of vehicle routing. For the interested readers, we refer to the book chapters by Laporte (1988) and Berman et al. (1995) and to the surveys by Balakrishnan et al. (1987), Min et al. (1998), Nagy and Salhi (2009), Prodhon and Prins (2014) and Drexl and Schneider (2015). The TSP, the VRP and the LRP, share two common features: i) every customer has to be served, and ii) the profit associated with the service is not considered.

When classifying routing problems, different aspects can be considered: the characteristics of the vehicles (such as the depot, the vehicle capacity, the number of vehicles, the types of goods to be delivered, the costs associated with vehicle utilization, and the arc of the road network), the characteristics of the customers (such as the demand, the period of the day, the vertex of the road network, and the time required to serve the customer), types of objective functions (such as the minimization of transportation costs, the minimization of the number of vehicles, and the minimization of the penalties associated with the service of the customers), and assumptions on uncertainty (deterministic, stochastic, and robust settings).

In the aforementioned classical problems, routing through every vertex may be restrictive and is not always applicable in every context. In SRPs, on the other hand, service is provided to satisfy the demand by visiting the corresponding vertex directly or by visiting another vertex that "covers" it. In this study, we focus on SRPs.

1.1. Our Motivation

SRPs include several variants, and several strongly connected problems have been defined and formulated differently based on similar concepts. Beasley and Nascimento (1996) and Laporte and Rodríguez-Martín (2007) have classified and modeled some of these problems within a unified framework. This paper extends their studies by providing a unifying framework for a wider range of SRPs. Through extensive analysis, we observe that these problems have the same purpose: *"servicing the customers"*, and they use the same tool for achieving it: *"routing"*. In this paper, we use the terms "coverage" for the former and "visiting" for the latter.

We classify SRPs according to their *selectiveness degree*. In order to uncover and formalize the connection among these problems, we present a generic mathematical model and illustrate how each problem considered in this survey can be modeled using this generic model. Our work identifies problems with similar structures that share many commonalities but have sometimes been introduced under different names. It also covers SRPs that have not yet been considered, thus identifying research gaps and offering potential research directions.

1.2. Review Methodology and Problems Considered

Our framework paper aims to provide the reader with a classification and point out several research directions for future work. This paper aims to present a general framework and does not focus on computations.

We have used three bibliographic sources: (1) the Web of Science (Web of Science, 2022) as an academic database by using the following keywords: "selective routing problem", "covering routing problem", "traveling salesman problem with profits", "vehicle routing problem with profits", "selective traveling salesman problem", "selective vehicle routing problem", "selective location routing problem", "generalized routing problem", "generalized traveling salesman problem", "generalized vehicle routing problem", "generalized location routing problem"; (2) bibliographies of articles found by the first source, and (3) bibliographies of book chapters and survey papers on SRPs identified by the first source. Bibliographical entries that refer to studies in languages other than English were eliminated. We did not restrict our search to any year or category since our aim is to provide the basic versions of SRPs rather than their extensions. Moreover, we restricted our search to the following categories of the Web of Science: "Operations Research Management Science", "Telecommunications", "Transportation Science Technology", "Engineering Industrial", "Management", "Mathematics Applied", "Transportation", "Mathematics", "Multidisciplinary Sciences" and "Mathematics Interdisciplinary Applications".

We have used a broad range of terms to cover as many problems as possible and we have identified 3,126 related articles. We have eliminated the papers irrelevant to SRPs by analyzing the abstracts and keywords. We have also excluded selective variations of Arc Routing Problems such as the Maximum Benefit Chinese Postman Problem, the Prize-Collecting Rural Postman Problem, the Arc Orienteering Problem, the Team Orienteering Arc Routing Problem, and the Profitable Arc Tour Problem. Moreover, the Shortest Path Problem and its variants such as the Shortest Covering Path Problem, the Maximum Covering/Shortest Path Problem, the Median Shortest Path Problem, and the tree problems such as the Generalized Minimum Spanning Tree Problem, the k-Minimum Spanning Tree Problem, the Prize-Collecting Generalized Minimum Spanning Tree Problem, the Prize-Collecting Steiner Tree Problem, and the Capacitated Ring Arborescence Problems with Profits are not considered within the context of the survey. Due to the assumptions on the connection of the facilities to each other and the flow exchange of clients, Hub Location-Routing Problems and their selective variants are also excluded. As a result of these eliminations, we have identified 300 works relevant to our context. Furthermore, rather than providing the basic extensions of SRPs such as time windows and vehicle capacity, we focus on problems highlighting the *selectiveness degree*. It is possible to miss or not cover all the basic variants while searching the academic database. Therefore, in the next step, we reviewed the introduction sections, problem definitions, and literature reviews of each paper to identify the basic extensions of problems or to find more relevant problems.

Table 1 displays the combination of searched keywords and their results and the number of articles eliminated for failing to match the aforementioned inclusion rules. In Table 2, the SRPs considered within the framework are presented. This list is not meant to be exhaustive but covers several problems defined over the years with similar concepts. There are 77 papers considering 49 different variants of SRPs. Some problems have the same definitions but are presented in different years under different names. We have combined them under one common name and presented them as a single problem. Each problem is assigned a reference number for the remainder of this paper. For instance, the Orienteering Problem, the Maximum Collection Problem, the Selective Traveling Salesman Problem, and the Bank Robber Problem are assigned the same reference number since these problems aim to find a maximum profit route within a prespecified budget with certain starting and ending vertices. The Multiple Tour Maximum Collection Problem and the Team Orienteering Problem aim to find a profitable tour within a specified budget with certain starting and ending vertices. As another example, both the Traveling Circus Problem and the Median Tour Problem were defined as the problem of finding a least cost tour that visits a predetermined number of vertices while covering all vertices. These problems are combined under one name and presented as a single problem. In Table 2, the names of the same problems and articles in which these names are mentioned are given.

Furthermore, in Table 2, the single vehicle and multi-vehicle versions of the problems are numbered consecutively, and they are not separated with horizontal lines, whereas the different SRPs are separated

Table 1: Screening and selection of papers

Item	Description
Keywords	"selective routing problem" OR "covering routing problem" OR "selective location routing problem" OR "selective traveling salesman problem" OR "selective vehicle routing problem" OR "routing problem with profits" OR "traveling salesman with profits" OR "vehicle routing with profits" OR "location routing problem with profits" OR "generalized routing problem" OR "generalized location routing problem" OR "generalized traveling salesman problem" OR "generalized vehicle routing problem"
Databases	ISI Web of Science
Search fields	Title, abstract, keywords
Time windows	No limit
Query result	3,126
Exclusion criteria	Papers not considering any selectiveness
Excluded based on abstract	2686
Excluded based on entire paper	140
Final sample from query	300
Included via bibliography	151
Exclusion criteria	Papers not considering basic variants of the selective routing problem
Final sample of basic variants	77

with lines. For example, the Generalized TSP and the Generalized VRP are numbered 1 and 2, respectively, as they are single vehicle and multi-vehicle versions of the same problem. Therefore, there is no horizontal line between these problems in the table. However, the School Bus Routing Problem has different dynamics from the previous problems; thus, it has been separated with a horizontal line.

Table 2: Selective Routing Problems considered in this framework

Problem number	Problem Names	Proposed by
0-i	TSP	Dantzig et al. (1954)
0-ii	VRP	Dantzig and Ramser (1959)
0-iii	LRP	Laporte et al. (1986)
1	Generalized TSP	Henry-Labordére (1969)
		Srivastava et al. (1969)
		Saskena (1970)
		Laporte and Nobert (1983)
2	Generalized VRP	Ghiani and Improta (2000)
3	School Bus Routing Problem	Newton and Thomas (1969)
		Bennett and Gazis (1972)
		Bodin and Berman (1979)
		Dulac et al. (1980)
4	Traveling Purchaser Problem	Ramesh (1981)
5	Traveling Purchaser VRP	Choi and Lee (2011)
6	Covering Tour Problem/Covering TSP	Current (1981)
		Gendreau et al. (1997)
7	Multi-vehicle Covering Tour Problem	Hachicha et al. (2000)
8	Orienteering Problem	Tsiligirides (1984)
		Golden et al. (1987)
	Multi-objective Vending Problem	Keller (1985)
		Keller and Goodchild (1988)
	Maximum Collection Problem	Kataoka and Morito (1988)
	Selective TSP	Laporte and Martello (1990)
	Bank Robber Problem	Awerbuch et al. (1998)
9	Multiple Tour Maximum Collection Problem	Butt and Cavalier (1994)

continues on next page

	Team Orienteering Problem	Chao et al. (1996)
10	Prize Collecting TSP	Balas and Martin (1985) Balas (1989)
11	Prize Collecting VRP	Tang and Wang (2006)
12	Single VRP with Selective Backhauls Single VRP with Deliveries and Selective Pickups (Single VRP with Fixed Delivery and Optional Collections) One-commodity TSP with Selective Pickup and Delivery	Süral and Bookbinder (2003) Gribkovskaia et al. (2008) Gutiérrez-Jarpa et al. (2009) Falcon et al. (2010)
13	VRP with Deliveries and Selective Pickups (VRPDSP) Selective Pickup and Delivery Problem	Yano et al. (1987) Privé et al. (2006) Gutiérrez-Jarpa et al. (2010) Ting and Liao (2013)
14	Covering Salesman Problem	Current and Schilling (1989)
15	Covering VRP	Buluc et al. (2022)
16	Generalized Orienteering Problem	Ramesh and Brown (1991)
17	Generalized K-Peripatetic Salesman Problem	Kort and Volgenant (1994)
18	Profitable Tour Problem	Bienstock et al. (1993) Dell'Amico et al. (1995)
19	Multi-vehicle Profitable Tour Problem	Archetti et al. (2009)
20	Traveling Circus Problem Median Tour Problem	ReVelle and Laporte (1993) Current and Schilling (1994)
21	Maximal Covering Tour Problem	Current and Schilling (1994)
22	Geometric Covering Salesman Problem/TSP with Neighbourhood	Arkin and Hassin (1994)
23	Single Vehicle Routing Allocation Problem	Beasley and Nascimento (1996)
24	Orienteering Problem with Variable Profits	Orienteering Problem with Arrival Time Dependent Profits Erkut and Zhang (1996) Traveling Repairman Problem with Profits Coene and Spieksma (2008) Orienteering Problem with Service Time Dependent Profits Erdoğan and Laporte (2013)
25	Team Orienteering Problem with Variable Profits	Team Orienteering Problem with Arrival Time Dependent Profits Tang et al. (2007) Team Orienteering Problem with Service Time Dependent Profits Gunawan et al. (2017)
26	Quota TSP	Awerbuch et al. (1998)
27	Median Cycle Problem	Labbé et al. (2005)
28	Time Dependent Orienteering Problem	Fomin and Lingas (2002)
29	Time Dependent Team Orienteering Problem	Li (2011)
30	Ring Star Problem	Labbé et al. (2004)
31	m-Ring Star Problem	Baldacci et al. (2007)
32	Close-Enough TSP	Gulczynski et al. (2006)
33	Close-Enough VRP	Mennell (2009)
34	Attractive TSP	Erdoğan et al. (2010)
35	Location and Selective Routing Problem	Aras et al. (2010)
36	Min-Max Selective Vehicle Routing	Valle et al. (2011)
37	Generalized LRP with Profits	Ahn et al. (2008) Ahn et al. (2012)

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38	Generalized Covering Salesman Problem	Golden et al. (2012)
39	Single Vehicle Cyclic Inventory Routing Problem	Aghezzaf et al. (2012)
40	Share-a-Ride Problem	Li et al. (2014)
41 42	Clustered Orienteering Problem Clustered Team Orienteering Problem	Angelelli et al. (2014) Yahiaoui et al. (2019)
43	Family TSP	Morán-Mirabal et al. (2014)
44	Selective Vehicle Routing with Integrated Tours	Şahinyazan et al. (2015)
45	Selective Assessment Routing Problem	Balcik (2016) Balcik (2017)
46	The Set Orienteering Problem	Archetti et al. (2018)
47	Generalized Median Tour Problem	Obreque et al. (2020)
48	Location-or-Routing Problem	Arslan (2021)
49	The Covering VRP with Integrated Tours	Buluc et al. (2022)

This paper is organized as follows. In the following section, applications of SRPs are described. The concept of the framework and the classification in the framework are provided in Sections 3 and 4, respectively. We then introduce a unifying mathematical model in Section 5 and briefly map the problems to this model in Section 6. We present potential future research directions in Section 7 and conclude the study in Section 8.

2. Applications

There exist several SRPs, and they span a wide variety of applications. In this section, we briefly mention some of these. Golden et al. (1987) discuss the Orienteering Problem in the home heating oil delivery application. An oil tanker is routed to service ships at various locations. The main goal is to select a subset of customers who require urgent service. Golden et al. (1981), and Golden et al. (1986) discuss the application of the Orienteering Problem in customer and vehicle assignment, inventory, and routing. Balas and Martin (1985) model the Prize Collecting Traveling Salesman Problem for scheduling the daily operation of a steel rolling mill. Tang and Wang (2006) consider the scheduling of hot rolling production, where aim is to minimize costs while maximizing the profits of scheduled orders.

The multi-vehicle case of the Orienteering Problem and of the Profitable Tour Problem applies to choosing potential customers through an electronic auction. Through the Internet, the carriers decide to serve potential customers by evaluating the capacity restrictions of vehicles (Archetti et al., 2009). Jiang et al. (2022) discuss applications in last mile delivery for online shopping. In their problem, the courier departing from the central depot visits customers to deliver parcels. The profit is collected with the service, and the courier can also deliver parcels to a parcel locker that covers the customers. Gutiérrez-Jarpa et al. (2009) also consider parcel delivery by modeling it as a vehicle routing problem with selective backhauls. Privé et al. (2006) describe a complex real-life problem for the distribution of soft drinks as an SRP. In this context, customers pay a deposit on cans and bottles, and return them to the retailers for a refund, and the retailers send them for recycling. Yano et al. (1987) solve a similar transportation problem for a company operating in a retail chain. The company owns a fleet of vehicles to perform routes between distribution centers and stores and between vendors and distribution centers. Due to capacity limitations, a full vendor coverage may not be possible. In this case, carriers are outsourced to pick up the material from the remaining vendors. The use of backhauls increased the profit of the company.

SRPs are also used in medical care services (e.g., mobile clinics). In such applications, all customers cannot be visited by vehicles, and some customers in unvisited areas should travel to the area visited by a vehicle (Hachicha et al., 2000; Hodgson et al., 1998). Hachicha et al. (2000) introduce the Multi-vehicle Covering Tour Problem in the context of routing mobile healthcare delivery teams. Doerner et al. (2006)

extends the studies of Hachicha et al. (2000) and Hodgson et al. (1998) by evaluating mobile healthcare facilities on the economic efficiency criterion, the average distance criterion, and the coverage criterion. Applications of mobile healthcare facilities are mentioned in Current and Schilling (1989) and Current and Schilling (1994).

In humanitarian logistics, Nolz et al. (2010) discuss the Multi-vehicle Covering Tour Problem for providing medicine, food, and shelter for an area affected by a disaster. Naji-Azimi et al. (2012) study humanitarian aid supply to affected people by locating satellite temporary relief centers and routing relief vehicles. The supplies are provided by several satellite distribution centers located within a predefined distance from their domiciles. Allahyari et al. (2015) discuss the distribution of goods to the people in an area affected by a disaster. Due to time limitations, all customers cannot be visited; therefore, the coverage concept is used to allocate unvisited customers to the nearest visited customer. Erdoğan and Laporte (2013) discuss the application of search and rescue operations by a helicopter in order to increase the number of survivors found. The longer the duration or the higher the number of helicopter passes at a target location, the higher the profit. Alinaghian et al. (2019) consider facilitating air relief operations, and in this context, they locate temporary relief centers and route air rescue vehicles while distributing relief supplies. Afsar et al. (2014) discuss Generalized Vehicle Routing Problems in disaster relief operations. Due to the damaged road network, the delivery of resources such as medical staff or equipment to damaged sites is disrupted, and planes are used instead. In this setting, the resources are dropped off at one of the operational airports in each region. The local authorities are then responsible for distributing resources to the affected areas. Buluc et al. (2022) consider the problems of refugee children. Their main focus is to increase the efficiency of the education services provided to refugee children. They model it as a service provided via mobile trucks, which are called child-friendly spaces. The mobile trucks aim to provide education and psychological support to children. In order to reach as many children as possible, the authors model the situation as Covering VRP with Integrated Tours.

Another interesting application of SRPs is in planning the tours of a circus or a theater company (Erdoğan et al., 2010; Erdoğan and Laporte, 2013). The visit to a facility generates profit, and customers find the facilities with extra services that are closer to the larger population centers more attractive. In this context, visiting such facilities generates more profit, which is an application of the Attractive TSP. From another perspective, entertainment vehicles can collect more profits if there are multiple shows or if they stay longer at a location. This problem is then an application of the Orienteering Problem with Variable Profits (Erdoğan and Laporte, 2013). Erdoğan et al. (2010) propose the same problem in routing military reconnaissance vehicles. In this application, there are also customer and facility sites. However, the hostile installations are taken as customer sites, and potential observation points are taken as facility sites. The attractiveness of candidate observation points depends on their visibility range and concealment factor. The objective function is the maximization of the information gathered. Similarly, Erdoğan and Laporte (2013) state that this application can be modeled as the Orienteering Problem with Variable Profits if there is the option of staying longer at a location in order to gather more information.

Applications based on unmanned air vehicles can also be modeled as an SRP. Margolis et al. (2022) route unmanned vehicles to surveil a set of targets based on applications in homeland security and public safety. Karaoğlan et al. (2018) report the use of UAVs in a wide range of applications beyond military and security, such as agriculture and forestry, disaster management, and ecosystem management.

Gulczynski et al. (2006) describe the use of radio frequency identification (RFID) to read water, gas, or electricity meters. They aim to minimize the distance the meter reader travels while servicing each customer on their route. As another interesting application, SRPs can be used for the exploration of planetary bodies (Ahn et al., 2008, 2012). NASA identifies a set of potential surface locations to serve as exploration bases, and routes are constructed based on different routing strategies. Ahn et al. (2012) also suggest the application to military surface operations and gas extraction problems.

3. Concepts of the Framework

There exist many variants of SRPs, and these problems cover similar concepts with several applications. In order to obtain a broad perspective on SRPs, it is instructive to study them within a unifying framework.

Feillet et al. (2005) presented a detailed classification of some of these problems as a special case of Traveling Salesman Problems with Profits and provided a literature review. These problems are also included in previous frameworks by Beasley and Nascimento (1996) and Laporte and Rodríguez-Martín (2007), but variations of SRP are not limited to them. There are SRPs in which vertices can be served without directly visiting the vertices. Therefore, in addition to decisions regarding which vertices to visit, SRPs include decisions regarding which vertices to cover. We present the concepts of "*service*" and "*selectiveness degree*", which constitute the basis of our framework.

3.1. Concept of Service

We first introduce the concept of *service.* Service refers to the act of providing a certain level of attention to a vertex. In SRPs, service can be provided to a vertex by visiting it or by covering it without visiting it. When deciding where to locate facilities, we consider locating a facility at a certain vertex to be the same as visiting that vertex. As a result, "visiting" includes the actions of visiting by vehicle and locating a facility. We consider vertices that are assigned to a visited vertex, are in the same cluster as a visited vertex, or are within a coverage radius of a visited vertex to be the same as covering that vertex. Then, "covering" includes the actions of allocation, clustering, and being in the coverage radius. In this paper, we distinguish between these two concepts: "visiting" and "covering". We relate service to the concept of coverage: a vertex is served if it is covered. On the other hand, visiting is considered a tool for coverage. We can cover vertices through the visited ones: a vertex is served if it is *covered* by a visited vertex. We summarize these concepts as follows:

- In order to serve a vertex, it should be covered. A visited vertex covers a vertex according to the coverage type specified.
- In order to visit a vertex, it should be on the route.

We define coverage as the *capacity of serving a vertex.* SRPs can be described using a coverage type, which specifies how a visited vertex can cover another unvisited vertex. In this paper, we incorporate the coverage concept into all SRPs and illustrate how these problems can be presented from the same perspective. Various coverage types in SRPs can be expressed as follows: In the first case, a vertex must be visited to be served; therefore, the condition of coverage is defined as *visiting.* In the second case, a tour is constructed by including a subset of the vertices, and this tour is considered to serve the vertices within a coverage radius of the visited vertices. Therefore, the coverage condition is to be within a coverage *radius* of a visited vertex. In the third case, a vertex is considered to be served if a certain number of vertices in its cluster are visited. Therefore, the coverage condition is related to the *clusters* of vertices. Finally, a vertex is served by being allocated to one of the visited vertices. This coverage condition is referred to as an *allocation.* The numbers of the problems based on their coverage types are given in Figure 1. Three problems stand out as being particularly interesting. They will be discussed in much more detail in the following sections. We will briefly mention them here in order to give a better understanding of the service and coverage concepts. There are only three problems whose coverage types exclude visiting. These problems are: the Attractive TSP, the Multi-vehicle Covering Tour Problem, and the Covering Tour Problem. The visiting coverage type provides service to a vertex in most SRPs. These three problems exclude visiting to provide service to a vertex. Nevertheless, visits are employed to choose the locations where services will be provided. In other words, facilities that provide service are visited rather than customer vertices. We will next formally define the coverage types. SRPs are defined on a graph with two types of vertex sets. Let V be the set of vertices that can be visited by a vehicle or be the location of a facility. Let W be the set of vertices that can be served by covering the vertices. These sets will be detailed in Section 4.

Let S_i be the set of vertices that can cover vertex $i \in W$. The set S_i is a subset of V since a vertex can only be covered by a visited vertex. This enables us to present the coverage concept in a generic mathematical formulation, regardless of the coverage type defined in each specific problem. The possible sets for S_i are summarized in Table 3.

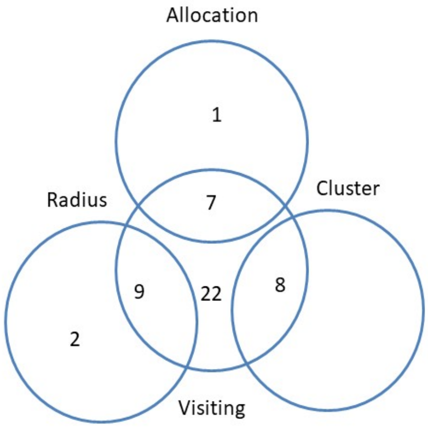

Figure 1: Coverage types of problems within Selective Routing Problems

Table 3: Coverage types

S_i **sets with respect to the coverage types:**	
Coverage by visit (V)	$S_i = \{i\}$
Coverage by coverage radius (R)	$S_i = \{j \in V \| d_{ij} \leq r\}$
Coverage by clusters (C)	$S_i = \{j \in V \| j$ is in the same cluster of $i\}$
Coverage by allocation (A)	$S_i = V$

We also use the selectiveness degree for classifying the Selective Routing Problems, which are presented next.

3.2. Concept of Selectiveness

In the previous section, we defined two sets: *vertices to be covered* and *vertices to be visited*. Observe that the set of vertices to be covered may include the set of vertices to be visited, or these two sets may be disjoint. In the latter case, the sole purpose of visiting vertices is to cover other vertices. SRPs may include vertices that *must be visited*, which we refer to as $\overline{V}$, or that *must be covered*, which we refer to as $\overline{W}$. We refer to $\overline{V}$ as a must-visit set and $\overline{W}$ as a must-cover set.

Even though the routing problems we investigate are selective, they may include constraints that *limit* the problem's selectiveness. In particular, there may exist sets of vertices that must be visited or that must be covered. These sets restrict the problem's selective nature. Regarding vertices to cover and visit, problems may have full, partial, or zero freedom. In terms of vertices to cover, $\overline{W} = \varnothing$ denotes full selectiveness, $\varnothing \neq \overline{W} \subset W$ denotes partial selectiveness, and $\overline{W} = W$ denotes zero selectiveness. Regarding vertices to visit, $\overline{V} = \varnothing$ denotes full selectiveness, while $\varnothing \neq \overline{V} \subset V$ presents partial selectiveness. For notational convenience, $\overline{W} \subset W$ and $\overline{V} \subset V$ imply non-empty sets.

Routing problems may be classified as Classical (non-selective) Routing Problems and SRPs. This paper extends this idea by further classifying SRPs according to their selectiveness degree in a unified framework. We also show how the classical routing problems may be presented as a *selective routing problem with zero selectiveness degree*.

Sets and their notations are presented in Table 4, and their relationship is illustrated as a Venn diagram in Figure 2.

Table 4: The sets for selectiveness degree

Sets	
N	$V \cup W$ as the set of all vertices
V	set of vertices that can be visited
W	set of vertices that can be covered
$\overline{V}$	set of vertices that must be visited
$\overline{W}$	set of vertices that must be covered

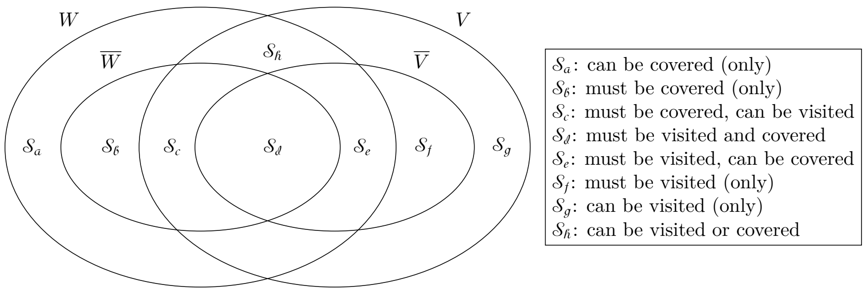

Figure 2: The relationship between V, $\overline{V}$, W and $\overline{W}$ sets as a Venn Diagram

4. Classification

We now provide a classification based on the selectiveness degree. This will allow us to better understand the relationships between different concepts and how they relate to each other within the framework. Additionally, by categorizing the problems considered within the framework based on this classification, we can gain insights into how different types of problems can be examined using the framework. The classification of SRPs based on their selectiveness degree is illustrated in Figure 3. Our framework divides SRPs into three classes in terms of selectiveness degree for W and divides each class into three categories in terms of selectiveness degree for V. Each category is further divided into three concerning the relationship between the W and V sets.

We investigate SRPs according to these nine classes. The corresponding SRPs are listed in each class in Figure 3 using the reference numbers of Table 2. For every class, we identify the most general problem. These *main* problems have less restrictive properties and broader definitions. The relationship between SRPs in each class and the hierarchy of SRPs variants are illustrated with Figure 4. Class 1's main problem is identified as the Location and Selective Routing Problem. In order to obtain the Generalized Location-Routing with Profits, different routing strategies should be integrated into the Location and Selective Routing Problems. Similarly, Class 2's main problem is identified as the Maximal Covering Tour Problem. The Orienteering Problem can be obtained from the Maximal Covering Tour Problem by discarding the coverage aspect and limiting the number of vertices to visit. In Class 2, the other problems are obtained by integrating some properties into the Orienteering Problem. Those properties include different objective functions and bounds, integrated tours, purchase cost, inventory routing, selective backhauling, and time variants. For each class, the hierarchy between the problems is presented. Moreover, the hierarchy of problems from different classes is illustrated in Figure 4 between each class. The TSP and the VRP can be obtained from the LRP by discarding the location decision. The Location-or-Routing Problem can be illustrated by integrating the coverage aspect of the facility and selective routing into the LRP. In contrast, the Location and Selective Routing Problem can be obtained by integrating the property of selective routing into the Location-Routing Problem.

The most general Selective Routing Problem is the *Single Vehicle Routing Allocation Problem*, which defines the sets W and V as two arbitrary sets and considers $\overline{W}$ and $\overline{V}$ to be any subsets of W and V, respectively. The mathematical formulation offered in this paper can be adapted to every SRP by modifying the input sets and parameters. Tables 5 to 10 show the sets V, W, $\overline{V}$ and $\overline{W}$ according to the classification framework and specify the class of each problem. According to the selectiveness degree, decisions made on each problem vary. These main decisions are on vertices to visit (including visits by vehicles and the location of facilities) and on vertices to cover and to allocate. Note that the routing problems included in the classification also cover selective location-routing problems.

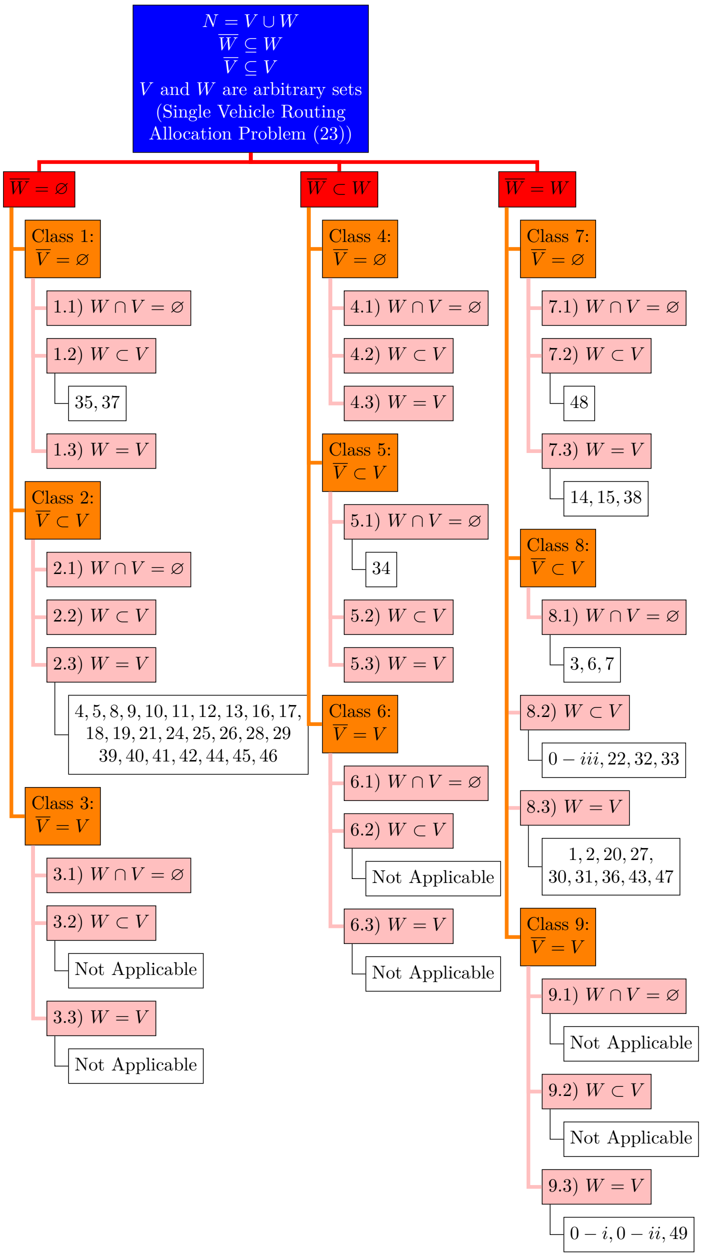

Figure 3: Classification Diagram for Selective Routing Problems

Class 1: $\overline{W} = \varnothing$ and $\overline{V} = \varnothing$

This class presents a full degree of freedom in terms of vertices to visit and cover. We have identified two problems in this class. In Table 5, these two problems with $\overline{W} = \varnothing$ and $\overline{V} = \varnothing$ are presented. $\overline{W} = \varnothing$

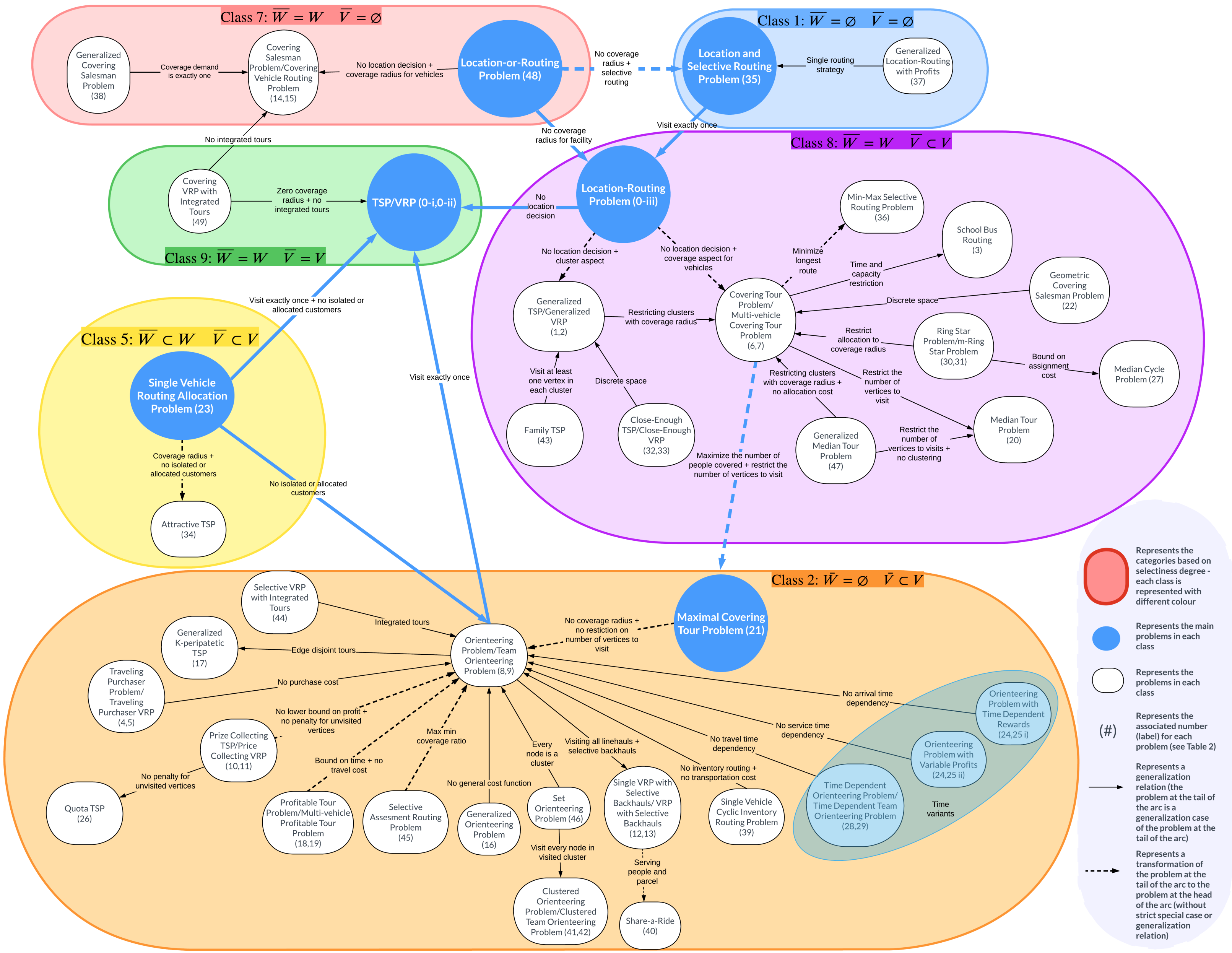

Figure 4: Relationships between the Selective Routing Problems

implies that the must-cover set is empty. Note that there are other problems with $\overline{W} = \varnothing$, such as the Orienteering Problems, where profits are gained to determine the vertices to be covered. However, these problems have predetermined starting and ending vertices, so they include a must-visit set. This class has applications when the initial and terminal vertices are also decided. SRPs without predefined initial and terminal vertices are presented in this class. We have not identified any problems with $W = V$ and $W \cap V = \varnothing$; however, they are valid applications.

Table 5: Framework of Selective Routing Problems with sets $\overline{W} = \varnothing$ and $\overline{V} = \varnothing$ in Class 1

#	Problem	Sets	Objective	Coverage type			
				V	R	C	A
35	Location and Selective Routing Problem	$W \subset V$	S	✓			✓
37	Generalized Location-Routing Problem with Profits	$W \subset V$	S	✓			

The problems within this class are “Location and Selective Routing Problem” and “Generalized Location Routing Problem with Profits”. When $W \subset V$, the problem can be considered a location and selective routing problem, and the set V includes both candidate location and routing vertices. However, the set W only includes the vertices that can be routed for service.

Class 2: $\overline{W} = \varnothing$ and $\overline{V} \subset V$

Table 6 presents 26 problems with $\overline{W} = \varnothing$ and $\overline{V} \subset V$. This class presents full selectiveness regarding vertices to cover, meaning there is no obligation to cover certain vertices. Problems in this class include decisions on the vertices to cover and visit. These problems are introduced in this class with the sets $W = V$.

The Maximal Covering Tour Problem is the main problem of this class as it has a broader scope, including two coverage types: visiting and coverage radius. The definition of this problem is more generalized and reduces to the Orienteering Problem when the coverage radius is zero. This class investigates the Orienteering Problem and its variations, as mandatory initial and terminal vertices are included in the set $\overline{V}$. These problems are differentiated through their objective functions, and all include visiting as a coverage type. The rest of the problems in this class and their coverage types are identified in Table 6. This class is a valid research area in cases with $W \subset V$ and $W \cap V = \varnothing$. Because in both of these subclasses, partial selectiveness in terms of vertices to visit does not restrict selectiveness in terms of vertices to cover.

Table 6: Framework of Selective Routing Problems with sets $\overline{W} = \varnothing$ and $\overline{V} \subset V$ in Class 2

#	Problem	Sets	Objective	Coverage type			
				V	R	C	A
21	Maximal Covering Tour Problem	$W = V$	M	✓	✓		
4	Traveling Purchaser Problem	$W = V$	S	✓			
5	Multi-vehicle Traveling Purchaser Problem	$W = V$	S	✓			
8	Orienteering Problem	$W = V$	S	✓			
9	Team Orienteering Problem	$W = V$	M	✓			
10	Prize Collecting TSP	$W = V$	M	✓			
11	Multi-vehicle Prize Collecting TSP	$W = V$	S	✓			
12	Single Vehicle Routing with Selective Backhauls	$W = V$	S	✓			
13	Vehicle Routing with Selective Backhauls	$W = V$	S	✓			
16	Generalized Orienteering Problem	$W = V$	S	✓			
17	Generalized K-peripatetic Salesman Problem	$W = V$	S	✓			
18	Profitable Tour Problem	$W = V$	S	✓			
19	Multi-vehicle Profitable Tour Problem	$W = V$	M	✓			
24	Orienteering Problem with Variable Profits	$W = V$	S	✓			
25	Team orienteering Problem with Variable Profits	$W = V$	S	✓			
26	Quota TSP	$W = V$	M	✓			
28	Time Dependent Orienteering Problem	$W = V$	S	✓			
29	Time Dependent Team Orienteering Problem	$W = V$	S	✓			

continues on next page

39	Single Vehicle Cyclic Inventory Routing Problem	$W = V$	S	✓			
40	Share-a-Ride Problem	$W = V$	S	✓			
41	Clustered Orienteering Problem	$W = V$	S	✓		✓	
42	Clustered Team Orienteering Problem	$W = V$	S	✓		✓	
44	Selective Vehicle Routing Problem with Integrated Tours	$W = V$	M	✓			
45	Selective Assessment Routing Problem	$W = V$	M	✓			
46	Set Orienteering Problem	$W = V$	S	✓			

Class 3: $\overline{W} = \varnothing$ and $\overline{V} = V$

This class presents full selectiveness in terms of vertices to cover and zero selectiveness in terms of vertices to visit. We have yet to identify any problems in this class; however, given the existence of coverage by allocation and capacities, this class is a valid research area for $W \cap V = \varnothing$. In this problem class, the set of vertices to be visited is predetermined as it equals the complete set. However, the allocation of vertices in the set W to the vertices visited can be a decision of a mathematical model when vertices in the set V have capacities. Therefore, even when the problem does not include selectiveness regarding vertices to visit, it can include selectiveness regarding vertices to cover.

Class 4: $\overline{W} \subset W$ and $\overline{V} = \varnothing$

This class presents full selectiveness regarding vertices to visit and partial selectiveness in terms of vertices to cover. While there are no existing problems in this class, it is a valid research area. When there is a set of must-cover vertices, a problem can be defined with full selectiveness regarding the vertices to visit. Therefore, it becomes a problem to serve specific vertices with *any* route.

This classification is valid for $W = V$ because we distinguish between the concepts of coverage and of visit. Therefore, a covered vertex is not necessarily visited, and $\overline{W} \subset W$ does not imply $\overline{V} \subset V$. The same idea applies to $W \subset V$ as the resulting route might entirely consist of vertices in the $V \backslash W$ set. The class is also valid for $W \cap V = \varnothing$ since this subclass presents independent sets of $\overline{V}$ and $\overline{W}$. In this case, the problem is to select, with full selectiveness, an optimal set of vertices of V that covers the set W.

Class 5: $\overline{W} \subset W$ and $\overline{V} \subset V$

Table 5 presents problems with sets $\overline{W} \subset W$ and $\overline{V} \subset V$. The problems in this class have partial selectiveness in terms of vertices to cover and visit. This means that decisions regarding vertices to cover and visit are incorporated into the problems but are constrained by the sets $\overline{W}$ and $\overline{V}$. Two problems are introduced in this class; one is presented with the sets $W \cap V = \varnothing$.

The Single Vehicle Routing Allocation Problem is the main problem of this class due to its broad definition and generalized form. Within the same class, the Attractive TSP is given as a subproblem. In the Single Vehicle Routing Allocation Problem, the sets V and W are not necessarily the same; however, these sets are distinctly separate in the Attractive TSP. Moreover, although both problems have a single objective, cost minimization is stressed in the former, and profit maximization is highlighted in the latter.

This class is still a proper study area concerning selectiveness in cases with $W = V$ and $W \subset V$. We consider the Orienteering Problems in Class 2, as by the problem definition, there is no need to serve the initial and terminal vertices, but there is an obligation to visit them. If this problem is defined with a must-cover set, for example, with contracted customers and other customers, it will present $\overline{W} \subset W$, $\overline{V} \subset V$, and $W = V$.

Table 7: Framework of Selective Routing Problems with sets $\overline{W} \subset W$ and $\overline{V} \subset V$ in Class 5

#	Problem	Sets	Objective	Coverage type			
				V	R	C	A
	Our Generalized Formulation		S	✓	✓	✓	✓
23	Single Vehicle Routing Allocation Problem		S	✓			✓
34	Attractive TSP	$V \cap W = \varnothing$	S				✓

Class 6: $\overline{W} \subset W$ and $\overline{V} = V$

This class presents partial selectiveness in terms of vertices to cover and no selectiveness in terms of vertices to visit. As in Class 3, when vertices in V have capacities and allocation is the coverage type, this class is a valid research area for $W \cap V = \varnothing$.

Class 7: $\overline{W} = W$ and $\overline{V} = \varnothing$

This class presents zero and full selectiveness regarding vertices to cover and to visit, respectively. In Table 8, four problems with sets $\overline{W} = W$ and $\overline{V} = \varnothing$ are presented. Three of these problems are introduced with the sets $W = V$. The Generalized Covering Salesman Problem (38) has a broader definition since it incorporates the definition of coverage demand (k_i) and utilizes k_i values greater than one. This problem can be reduced to the Covering Salesman Problem when $k_i = 1$.

Moreover, there exists one problem with sets $W \subset V$. The Location-or-Routing Problem (LoRP) (48) is considered the main problem of this class. This problem has a broader definition than the Generalized Covering Salesman Problem since it includes the location decision. In the LoRP, all users are covered when a vehicle visits a location. Also, the uncovered vertices are visited by vehicles. Therefore, both facilities and vehicles can be used to cover a vertex. Since candidate facility vertices should be visited and uncovered customer vertices should be visited by vehicles, we define $W \subset V$.

This class is still a valid research area regarding selectiveness in cases with $W \subset V$ and $W \cap V = \varnothing$. Since we consider different coverage types, $W \subset V$ does not imply $W \subset \overline{V}$. Therefore, vertices to be visited can entirely consist of $V \backslash W$. When $W \cap V = \varnothing$, the problem can be viewed as the Covering Tour Problem without any must-visit vertices, i.e., $\overline{V} = \varnothing$. When W and V are disjoint sets, it is possible to have full selectiveness in terms of vertices to visit and zero selectiveness in terms of vertices to cover.

Table 8: Framework of Selective Routing Problems with sets $\overline{W} = W$ and $\overline{V} = \varnothing$ in Class 7

#	Problem	Sets	Objective	Coverage type			
				V	R	C	A
48	The Location-or-Routing Problem	$W \subset V$	S	✓	✓		
38	Generalized Covering Salesman Problem	$W = V$	S	✓	✓		
14	Covering Salesman Problem	$W = V$	M	✓	✓		
15	Covering Vehicle Routing Problem	$W = V$	S	✓	✓		

Class 8: $\overline{W} = W$ and $\overline{V} \subset V$

Table 9 presents ten problems with sets $\overline{W} = W$ and $\overline{V} \subset V$. The problems in this class do not include decisions on vertices to cover, as it is a mandatory condition to cover all. In this case, the main decision concerns which vertices to visit to ensure all vertices are covered.

The Covering Tour Problem is the most generalized in this class, with disjoint sets of V and W. Disjoint sets of V and W present a broader definition as set definitions $W = V$ and $W \subset V$ can be deducted from this definition by simply including the same vertices in both sets and by including the vertices of W in the set V, respectively. This problem involves one coverage type: coverage radius. Vertices to visit are selected from the set V and are utilized to cover the vertices in the disjoint set W.

The Close-Enough TSP can also be considered within the class since the tour must start and end at a given depot. As the Close-Enough TSP is defined on a continuous graph and solved through discretization, a set of vertices to be visited is constructed by including the intersection points of coverage radii as well as the vertices to be covered. Therefore, we identify W to be a subset of V. The Close-Enough TSP has a broader definition than the Generalized Median Tour Problem, as the set definition of $W = V$ in the Generalized Median Tour Problem can be deducted from $W \subset V$.

The Location-Routing Problem belongs to this class. Even though it is one of the well-known classical routing problems, it falls under this classification. The reason is due to location decisions. Despite not being a type of coverage, location decisions are still regarded as visits. Customers receiving service are represented by V in the problem and candidate facility locations that will provide service. Due to the requirement that at least one depot be found and that any candidate locations may be chosen as depots, $\overline{V} \subset V$. Furthermore,

$\overline{W} = W$ since all customer vertices should be served or covered. Additionally, the service is offered using only visiting as a type of coverage.

Table 9: Framework of Selective Routing Problems with sets $\overline{W} = W$ and $\overline{V} \subset V$ in Class 8

#	Problem	Sets	Objective	Coverage type			
				V	R	C	A
0-iii	Location-Routing Problem	$W \subset V$	S	✓			
6	Covering Tour Problem	$W \cap V = \varnothing$	S		✓		
7	Multi-vehicle Covering Tour Problem	$W \cap V = \varnothing$	S		✓		
22	Geometric Covering Salesman Problem	$W \subset V$	S	✓	✓		
32	Close-Enough TSP	$W \subset V$	S	✓		✓	
33	Close-Enough VRP	$W \subset V$	S	✓		✓	
1	Generalized TSP	$W = V$	S	✓		✓	
2	Generalized VRP	$W = V$	S	✓		✓	
3	School Bus Routing	$W \cap V = \varnothing$	S	✓			✓
20	Median Tour Problem (Traveling Circus Problem)	$W = V$	M	✓			✓
27	The Median Cycle Problem	$W = V$	S	✓			✓
30	Ring Star Problem	$W = V$	S	✓			✓
31	m-Ring Star Problem	$W = V$	S	✓			✓
36	Min-Max Selective Vehicle Routing Problem	$W = V$	S	✓	✓		
43	Family TSP	$W = V$	S	✓		✓	
47	Generalized Median Tour Problem	$W = V$	S	✓		✓	

Class 9: $\overline{W} = W$ and $\overline{V} = V$

Lastly, problems with sets $\overline{W} = W$ and $\overline{V} = V$ correspond to classical routing problems. These problems generally do not include a selective feature. The problems are presented in Table 10 to demonstrate the unifying nature of this framework. When explored from a broader perspective, classical routing problems can be identified as SRPs with zero degrees of selectiveness. Additionally, by including the classical routing problems in a single class of this problem, we emphasize the broadness of the characteristics of this literature and the necessity of classifying routing literature further than an already existing selective/non-selective classification.

However, there exists one problem that has a selective nature and also fits into this class. The Covering Vehicle Routing Problem is in Class 8 with $\overline{W} = W$ and $\overline{V} \subset V$. With the integrated tour property, they generate smaller tours for covered vertices originating at the visited ones. Therefore, the problem revolves around routing by the vehicles and the team while ensuring that all vertices are visited by either vehicles or teams.

Table 10: Framework of Selective Routing Problems with sets $\overline{W} = W$ and $\overline{V} = V$ in Class 9

#	Problem	Sets	Objective	Coverage type			
				V	R	C	A
0-i	TSP	$W = V$	S	✓			
0-ii	VRP	$W = V$	S	✓			
49	The Covering VRP with Integrated Tours	$W = V$	S	✓	✓		

Note that our general framework is unifying, so routing problems can be classified within the framework when decomposed with the given sets.

5. A Generic Mathematical Model

We now develop a mathematical model for SRPs, which allows us to link the problems considered in this paper. We present the concept of coverage using the set S_i. It yields a modeling structure that interconnects visiting and covering and presents SRPs independently of the coverage type identified in each problem. Using this set, which can express every coverage type in our generic mathematical model, we offer a formulation that also generalizes the Single Vehicle Routing Allocation Problem.

*5.1. **Single Vehicle Model:***

This model is defined on a directed graph $G = (N, A)$, where $N = V \cup W$ is the set of all vertices and A is the set of arcs. Recall that SRPs are identified on a graph of V and W. Accordingly, sets that describe the selectiveness of the problems $\overline{W}$ and $\overline{V}$ are included in the formulation. Furthermore, this framework has four coverage types denoted by $S_i \subset V, i \in W$. All vertices in set W are candidates to be covered by different coverage types. The sets are given in Table 3 for each coverage type. Moreover, when integrating location and routing decisions, we define V_1 and V_2 as the sets of vertices that can be located in a facility and visited by a vehicle. We have $V = V_1 \cup V_2$. Similarly, sets $\overline{V_1}$ and $\overline{V_2}$ are the counterparts of V_1 and V_2, respectively. The definitions of sets are given in Table 11.

Table 11: Sets for location and routing decisions

Sets for location and routing decisions	
V_1	set of vertices that can be located a facility (set of facilities)
V_2	set of vertices that can be visited via routing a vehicle (set of customers)
$\overline{V_1}$	set of vertices that must be visited via locating a facility (set of facilities that must be located)
$\overline{V_2}$	set of vertices that must be visited via routing a vehicle (set of customers that must be on the route)

Table 12: Parameters

Parameters	
c_{ij}	travel cost from vertex $i \in V$ to vertex $j \in V$
f_l	cost of opening a facility at vertex $l \in V_1$
a_{ij}	cost of allocating vertex $i \in W$ to vertex $j \in V$
p_i	cost of not covering vertex $i \in W$
t_i	demand of vertex $i \in W$
e_i	profit of covering vertex $i \in W$
k_i	coverage demand of vertex $i \in W$ (the number of times that vertex i needs to be covered)
n	upper bound on the number of vertices to visit
$\mathcal{B}$	upper bound on the total route cost
$\mathcal{P}$	lower bound on the total profit gained by the covered vertices
q_i	stay over cost of vertex $i \in V$
λ_i	attractiveness (weight for profit) of vertex $i \in W$
α_d	weight of the performance measure $d \in \{1, ..., 5\}$

Table 13: Decision variables

Decision variables	
v_{1l}	$\begin{cases} 1 & \text{if a facility is located at vertex } l \in V_1 \\ 0 & \text{otherwise} \end{cases}$
v_{2i}^{l}	$\begin{cases} 1 & \text{if vertex } i \in V_2 \text{ is visited by vehicle from } l \in V_1 \\ 0 & \text{otherwise} \end{cases}$
w_i	$\begin{cases} 1 & \text{if vertex } i \in W \text{ is covered} \\ 0 & \text{otherwise} \end{cases}$
z_i	$\begin{cases} 1 & \text{if vertex } i \in W \text{ is not covered} \\ 0 & \text{otherwise} \end{cases}$
x_{ij}^{l}	$\begin{cases} 1 & \text{if vehicle travels from vertex } i \in V_2 \text{ to } j \in V_2 \text{ departing from } l \in V_1 \\ 0 & \text{otherwise} \end{cases}$
y_{ij}	$\begin{cases} 1 & \text{if vertex } i \in W \text{ is allocated to vertex } j \in V \\ 0 & \text{otherwise} \end{cases}$
u_i^l	the visiting order of vertices $i \in V$ by a vehicle departing from $l \in V_1$

$$\text{minimize} \quad \alpha_1 \sum_{l \in V_1} \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij}^{l} + \alpha_2 \sum_{i \in W} \sum_{j \in V} a_{ij} t_i y_{ij} + \alpha_3 \sum_{i \in W} p_i z_i$$

$$- \alpha_4 \sum_{i \in W} e_i w_i + \alpha_5 \sum_{l \in V_1} f_l v_{1l} \tag{1}$$

subject to

$$\sum_{j \in V_2} x_{lj}^{l} = \sum_{j \in V_2} x_{jl}^{l} = v_{1l} \qquad l \in V_1 \tag{2}$$

$$\sum_{j \in V} x_{ij}^{l} = \sum_{j \in V} x_{ji}^{l} = v_{2i}^{l} \qquad i \in V, l \in V_1 \tag{3}$$

$$v_{2i}^{l} \leq v_{1l} \qquad i \in V_2, l \in V_1 \tag{4}$$

$$\sum_{l \in V_1} \sum_{j \in S_i} v_{2j}^{l} \geq w_i \times k_i \qquad i \in W \tag{5}$$

$$\sum_{j \in S_i} y_{ij} = w_i \times k_i \qquad i \in W \tag{6}$$

$$w_i + z_i = 1 \qquad i \in W \tag{7}$$

$$w_i = 1 \qquad i \in \overline{W} \tag{8}$$

$$v_{1l} = 1 \qquad l \in \overline{V_1} \tag{9}$$

$$\sum_{l \in V_1} v_{2i}^{l} = 1 \qquad i \in \overline{V_2} \tag{10}$$

$$\sum_{l \in V_1} v_{2i}^{l} \leq 1 \qquad i \in V_2 \tag{11}$$

$$u_i^l - u_j^l + |V| \, x_{ij}^{l} \leq |V| - 1 \qquad i, j \in V, l \in V_1, i \neq j \neq l \tag{12}$$

$$\sum_{l \in V_1} \sum_{i \in V} \sum_{j \in V} d_{ij} x_{ij}^{l} + \sum_{l \in V_1} \sum_{i \in V} q_i v_{2i}^{l} \leq \mathcal{B} \tag{13}$$

$$\sum_{i \in W} e_i w_i \geq \mathcal{P} \tag{14}$$

$$\sum_{l \in V_1} \sum_{i \in V} v_{2i}^{l} \leq n \tag{15}$$

$$y_{ij} \leq \sum_{l \in V_1} v_{2j}^l \qquad i \in W, j \in V \qquad (16)$$

$$v_{1l} \in \{0,1\} \qquad l \in V_1 \qquad (17)$$

$$v_{2i}^l \in \{0,1\} \qquad i \in V, l \in V_1 \qquad (18)$$

$$w_i, z_i \in \{0,1\} \qquad i \in W \qquad (19)$$

$$x_{ij}^l \in \{0,1\} \qquad i,j \in V, l \in V_1 \qquad (20)$$

$$y_{ij} \in \{0,1\} \qquad i \in W, j \in V \qquad (21)$$

$$u_i^l \geq 0 \qquad i \in V, l \in V_1. \qquad (22)$$

The objective function minimizes the total cost, which consists of the routing cost, the allocation cost, the isolation cost, and the fixed cost of opening a facility, minus the profit gained from covering a vertex. Constraints (2) are flow conservation equations. Constraints (3) consist of the same balance equations for the visited vertices by vehicles. Constraints (4) ensure that if a facility is not located at a vertex, then other vertices cannot be visited by a vehicle departing from that vertex. Constraints (5) state that a vertex is covered if vertices that can cover it are visited as many times as the required coverage demands. Constraints (6) allocate a vertex to the vertices that cover it. Constraints (7) mean that a vertex in the set W is either covered or isolated. Constraints (8), (9), and (10) ensure that the must-cover and must-visit sets are covered and visited, respectively. Constraints (11) mean that a vertex can be visited by a vehicle from at most one facility. Constraints (12) are the Miller-Tucker-Zemlin subtour elimination constraints. Constraint (13) defines the maximum allowable routing budget for a route. Constraint (14) sets a lower bound on the total profit gained, and constraint (15) sets an upper bound on the number of vertices visited. Moreover, constraints (16) enable an allocation only to visited vertices. Finally, constraints (17) to (22) define the domains of the variables.

5.2. ***Multi-vehicle Model:***

The general formulation provided for a single vehicle case is extended to a multi-vehicle case. The sets, parameters, and decision variables are accordingly extended in Tables 14–16.

Table 14: Additional sets

Additional sets	
M	the set of vehicles that are to be used

Table 15: Additional parameters

Additional parameters	
h	total number of vehicles departing from a depot
$\mathcal{B}^m$	upper bound on the route cost for vehicle $m \in M$
n^m	upper bound on the number of vertices visited vehicle $m \in M$

Table 16: Decision variables

Decision variables	
w_i, z_i, v_1^l, y_{ij}	are defined as in single vehicle case
v_{2i}^{lm}	$\begin{cases} 1 & \text{if vertex } i \in V \text{ is visited by vehicle } m \in M \text{ departing from facility at } l \in V_1 \\ 0 & \text{otherwise} \end{cases}$
x_{ij}^{lm}	$\begin{cases} 1 & \text{if vehicle } m \in M \text{ departing from facility at } l \in V_1 \text{ travels from } i \in V_2 \text{ to } j \in V_2 \\ 0 & \text{otherwise} \end{cases}$
u_i^{lm}	visiting order of vertices $i \in V$ by vehicle $m \in M$ departing from $l \in V_1$

$$\text{minimize} \quad \alpha_1 \sum_{m\in M}\sum_{l\in L}\sum_{i\in V}\sum_{j\in V} c_{ij}x_{ij}^{lm} + \alpha_2 \sum_{i\in W}\sum_{j\in V} a_{ij}t_iy_{ij} + \alpha_3 \sum_{i\in W} p_iz_i$$

$$-\alpha_4 \sum_{i\in W} e_iw_i + \alpha_5 \sum_{l\in V_1} f_lv_{1l} \tag{1*}$$

subject to (6), (7), (8), (9), (14), (17), (19), (21)

$$\sum_{j\in V_2} x_{lj}^{lm} = \sum_{j\in V_2} x_{jl}^{lm} = v_{1l} \qquad l\in V_1, m\in M \tag{2*}$$

$$\sum_{j\in V} x_{ij}^{lm} = \sum_{j\in V} x_{ji}^{lm} = v_{2i}^{lm} \qquad i\in V, l\in V_1, m\in M \tag{3*}$$

$$v_{2i}^{lm} \leq v_{1l} \qquad i\in V_2, l\in V_1, m\in M \tag{4*}$$

$$\sum_{m\in M}\sum_{l\in V_1}\sum_{j\in S_i} v_{2j}^{lm} \geq w_i \times k_i \qquad i\in W \tag{5*}$$

$$\sum_{l\in V_1}\sum_{m\in M} v_{2i}^{lm} = 1 \qquad i\in \overline{V_2} \tag{10*}$$

$$\sum_{l\in V_1}\sum_{m\in M} v_{2i}^{lm} \leq 1 \qquad i\in V_2 \tag{11*}$$

$$u_i^{lm} - u_j^{lm} + |V|\,x_{ij}^{lm} \leq |V|-1 \qquad i,j\in V, l\in V_1, i\neq j\neq l, m\in M \tag{12*}$$

$$\sum_{m\in M}\sum_{i\in V}\sum_{j\in V} d_{ij}x_{ij}^{lm} + \sum_{m\in M}\sum_{i\in V} q_iv_{2i}^{lm} \leq \mathcal{B} \qquad l\in V_1 \tag{13*}$$

$$\sum_{i\in V}\sum_{j\in V} d_{ij}x_{ij}^{lm} + \sum_{i\in V} q_iv_{2i}^{lm} \leq \mathcal{B}^m \qquad l\in V_1, m\in M \tag{23}$$

$$\sum_{m\in M}\sum_{l\in V_1}\sum_{i\in V} v_{2i}^{lm} \leq n \tag{15*}$$

$$\sum_{l\in V_1}\sum_{i\in V} v_{2i}^{lm} \leq n^m \qquad m\in M \tag{24}$$

$$y_{ij} \leq \sum_{l\in V_1}\sum_{m\in M} v_{2j}^{lm} \qquad i\in W, j\in V \tag{16*}$$

$$v_{2i}^{lm} \in \{0,1\} \qquad l\in V_1, i\in V, m\in M \tag{18*}$$

$$x_{ij}^{lm} \in \{0,1\} \qquad i,j\in V, l\in V_1, m\in M \tag{20*}$$

$$u_i^{lm} \in \{0,1\} \qquad i\in V, l\in V_1, m\in M. \tag{22*}$$

This formulation is the multi-vehicle version; therefore, the constraints are modified through the vehicle index in the decision variables. The constraints with (*) are adaptations of constraints in the single-vehicle formulation to the multi-vehicle case. There are additional constraints needed to incorporate the dynamics of multiple vehicles. Constraints (13*) define the total routing budget, and (23) impose the routing budget for each vehicle. Constraints (15*) set an upper bound on the total number of vertices visited, and constraints (24) set an upper bound on the number of vertices visited by each vehicle. Constraints (16*) enable an allocation only to visited vertices. Finally, (18*), (20*), and (22*) define the domain of the variables.

6. Mapping the Generic Mathematical Model to Existing Problems of the Literature

In this section, we map each SRP to our unifying framework. We reformulate each problem according to the sets, parameters, and decisions used in the mathematical model. The first characteristic we investigate is the coverage set. Each problem explicitly or implicitly includes a coverage set that specifies which vertices can cover vertex $i \in W$. Our generic model presents this set as S_i for each $i \in W$. The vertices that this set can include are determined by the coverage type specified in the problem. We then provide parameter values that adapt the problem characteristics to the generic formulation. We identify the coverage demand k_i for vertex $i \in W$ as the number of times a vertex should be covered. Unless otherwise stated, the value of k_i equals one. The parameters $\mathcal{B}$, $\mathcal{P}$, and n define the bounds on the routing budget, the profit collected, and the number of vertices on the route, respectively. It should be noted that the routing budget may include time restrictions, capacity restrictions and any other resource restrictions. For notational convenience, if there is a bound on these parameters, $\tilde{\mathcal{B}}, \tilde{\mathcal{B}^m}, \tilde{\mathcal{P}}, \tilde{n}, \tilde{n^m}$ are used to present the bounds. For problems that do not involve them, the corresponding requirement can easily be relaxed by setting $\mathcal{B} = \infty$, $\mathcal{P} = 0$ and $n = |V|$.

The generic formulation includes several performance measures in its objective function to capture every problem's characteristics. While assignment or visit costs can be equal to zero by changing the parameters, α_d values are also used as binary coefficients indicating whether the corresponding performance measure d is incorporated in the specified problem.

Some problems using different names are presented as variants of a single problem and are presented only once to avoid duplication. Tables 17 to 22 show values that sets and parameters in the generic formulation should take to model each specific problem.

6.1. *Class 1:* $\overline{W} = \varnothing$ *and* $\overline{V} = \varnothing$

Table 17: The mapping table for problems with sets $\overline{W} = \varnothing$ and $\overline{V} = \varnothing$ in Class 1

#	Problem	Coverage set (S_i)	Coverage demand (k_i)	$\mathcal{B}$	$\mathcal{P}$	n	Objective function weights				
							α_1	α_2	α_3	α_4	α_5
35	Location and Selective Routing Problem	V	1	$\tilde{\mathcal{B}^m}$	0	$\|V\|$	1	-1	0	0	1
37	Generalized Location-Routing Problem	$\{i\}$	1	$\tilde{\mathcal{B}^m}$	0	$\|V\|$	0	0	0	1	0

6.2. *Class 2:* $\overline{W} = \varnothing$ *and* $\overline{V} \subset V$

Table 18: The mapping table for problems with sets $\overline{W} = \varnothing$ and $\overline{V} \subset V$ in Class 2

#	Problem	Coverage set (S_i)	Coverage demand (k_i)	$\mathscr{B}$	$\mathscr{P}$	n	Objective function weights				
							α_1	α_2	α_3	α_4	α_5
21	Maximal Covering Tour Problem	$\{j \in V \vert d_{ij} \leq r\}$	1	∞	0	$\tilde{n}$	$\min(\sum_{i\in V}\sum_{j\in V} c_{ij}x^0_{ij}, \quad \sum_{i\in W}\sum_{j\in V} a_{ij}t_iy_{ij})$ where $a_{ij} = -1 \quad i \in W$ and $j \in V$				
4	Traveling Purchaser Problem	$\{i\}$	1	∞	0	$\tilde{n}$	$(^*)\min\sum_{i\in V}\sum_{j\in V} c_{ij}x^0_{ij} + \sum_{k\in K}\sum_{i\in V} pur_k prod^k_i$				
5	Multi-vehicle Traveling Purchaser Problem	$\{i\}$	1	∞	0	$\tilde{n}$	$(^*)\min\sum_{i\in V}\sum_{j\in V} c_{ij}x^0_{ij} + \sum_{k\in K}\sum_{i\in V} pur_k prod^k_i$				
8	Orienteering Problem	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	0	0	0	1	0
9	Team Orienteering Problem	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	0	0	0	1	0
10	Prize Collecting TSP	$\{i\}$	1	∞	$\tilde{\mathscr{P}}$	$\vert V\vert$	1	0	1	0	0
11	Multi-vehicle Prize Collecting TSP	$\{i\}$	1	∞	$\tilde{\mathscr{P}}$	$\vert V\vert$	1	0	1	0	0
12	Single Vehicle Routing with Selective Backhauls	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	1	0	0	1	0
13	Vehicle Routing with Selective Backhauls	$\{i\}$	1	$\tilde{\mathscr{B}}^m$	0	$\vert V\vert$	1	0	0	1	0
16	Generalized Orienteering Problem	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	$(^\ddagger)\max(\sum_{j=1}^{g} W_j[\sum_{i\in V} w_i \cdot Score_j(i)]^k])$				
17	Generalized K-peripatetic Salesman Problem	$\{i\}$	1	∞	0	$\vert V\vert$	1	0	1	0	0
18	Profitable Tour Problem	$\{i\}$	1	∞	0	$\vert V\vert$	1	0	0	1	0
19	Multi-vehicle Profitable Tour Problem	$\{i\}$	1	∞	0	$\tilde{n}^m$	1	0	0	1	0
24	Orienteering Problem with Variable Profits	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	0	0	0	1	0
25	Team Orienteering Problem with Variable Profits	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	0	0	0	1	0
26	Quota TSP	$\{i\}$	1	∞	$\tilde{\mathscr{P}}$	$\vert V\vert$	1	0	0	0	0
28	Time Dependent Orienteering Problem	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	0	0	0	1	0
29	Time Dependent Team Orienteering Problem	$\{i\}$	1	$\tilde{\mathscr{B}}$	0	$\vert V\vert$	0	0	0	1	0

continues on next page

39	Single Vehicle Cyclic Inventory Routing Problem	$\{i\}$	1	$\tilde{\mathcal{B}}^m$	0	$\|V\|$	1	0	0	1	0
40	Share-a-Ride Problem	$\{i\}$	1	$\tilde{\mathcal{B}}^m$	0	$\tilde{n}^m$	1	0	0	1	0
41	Clustered Orienteering Problem	$\{j \in V \| j$ is in the same cluster of $i\}$	1	$\tilde{\mathcal{B}}$	0	$\|V\|$	0	0	0	1	0
42	Clustered Team Orienteering Problem	$\{j \in V \| j$ is in the same cluster of $i\}$	1	$\tilde{\mathcal{B}}$	0	$\|V\|$	0	0	0	1	0
44	Selective VRP with Integrated Tours	$\{i\}$	1	$\tilde{\mathcal{B}}$	$\tilde{\mathcal{P}}$	$\|V\|$	1	0	0	0	0
45	Selective Assessment Routing Problem	$\{i\}$	1	$\tilde{\mathcal{B}}$	0	$\|V\|$	(†) $\min\max_{c \in \mathcal{C}} \frac{\sum_{i \in V_2} \sum_{m \in M} \sigma_{i,c} v_{2i}^{0,m}}{\tilde{T}_c}$				
46	Set Orienteering Problem	$\{j \in V \| j$ is in the same cluster of $i\}$	1	$\tilde{\mathcal{B}}$	0	$\|V\|$	0	0	0	1	0

$*$ where parameter pur_k is the purchasing cost of product k and variable $prod_i^k$ is the amount of product k, purchased at market i.
‡ where parameter $Score_j(i)$ is the score vector of a node $i \in N$ with a number of independent attributes denoted by j, and W_j is the weight for attribute j.
† where $\mathcal{C}$ is the set of characteristics, $\sigma_{i,c}$ is a binary parameter indicating if node i have characteristic $c \in \mathcal{C}$ or not, and $\tilde{T}_c$ is the number of sites that carry characteristic $c \in \mathcal{C}$.

6.3. *Class 5: $\overline{W} \subset W$ and $\overline{V} \subset V$*

Table 19: The mapping table for problems with sets $\overline{W} \subset W$ and $\overline{V} \subset V$ in Class 5

#	Problem	Coverage set (S_i)	Coverage demand (k_i)	$\mathcal{B}$	$\mathcal{P}$	n	Objective function weights				
							α_1	α_2	α_3	α_4	α_5
23	Single Vehicle Routing Allocation Problem	V	1	∞	0	$\|V\|$	1	1	1	0	0
34	Attractive TSP	V	1	∞	0	$\|V\|$	0	0	0	1	0

6.4. *Class 7:* $\overline{W} = W$ *and* $\overline{V} = \varnothing$

Table 20: The mapping table for problems with sets $\overline{W} = W$ and $\overline{V} = \varnothing$ in Class 7

#	Problem	Coverage set (S_i)	Coverage demand (k_i)	$\mathscr{B}$	$\mathscr{P}$	n	Objective function weights				
							α_1	α_2	α_3	α_4	α_5
48	Location-or-Routing Problem	$\{j \in V \mid d_{ij} \leq r\}$	1	$\tilde{\mathscr{B}}^m$	0	$\|V\|$	1	1	0	0	1
38	Generalized Covering Salesman Problem	$\{j \in V \mid d_{ij} \leq r\}$	k_i^*	∞	0	$\|V\|$	1	0	0	0	0
14	Covering Salesman Problem	$\{j \in V \mid d_{ij} \leq r\}$	1	∞	0	$\|V\|$	Bi-objective version: min $(\sum_{l \in V_1} \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij}^l, \sum_{i \in W} e_i w_i)$				
15	Covering VRP	$\{j \in V : d_{ij} \leq r\}$	1	∞	0	$\|V\|$	1	0	0	0	0

6.5. *Class 8:* $\overline{W} = W$ *and* $\overline{V} \subset V$

Table 21: The mapping table for problems with sets $\overline{W} = W$ and $\overline{V} \subset V$ in Class 8

#	Problem	Coverage set (S_i)	Coverage demand (k_i)	$\mathscr{B}$	$\mathscr{P}$	n	Objective function weights				
							α_1	α_2	α_3	α_4	α_5
0-iii	LRP	$\{i\}$	1	∞	0	$\|V\|$	1	0	0	0	1
6	Covering Tour Problem	$\{j \in V \mid d_{ij} \leq r\}$	1	∞	0	$\|V\|$	1	0	0	0	0
7	Multi-vehicle Covering Tour Problem	$\{j \in V \mid d_{ij} \leq r\}$	1	∞	0	$\|V\|$	1	0	0	0	0
1	Generalized TSP	$\{j \in V \mid j$ is in the same cluster of $i\}$	1	∞	0	$\|V\|$	1	0	0	0	0
2	Generalized VRP	$\{j \in V \mid j$ is in the same cluster of $i\}$	1	∞	0	$\|V\|$	1	0	0	0	0
3	School Bus Routing	V	1	$\tilde{\mathscr{B}}^m$	0	$\|V\|$	1	0	0	0	0
20	Median Tour Problem	V	1	∞	0	$\tilde{n}$	$\min(\sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij}^0 \quad \sum_{i \in W} \sum_{j \in V} a_{ij} y_{ij})$				
27	Median Cycle Problem	V	1	∞	0	$\|V\|$	1	1	0	0	0
30	Ring Star Problem	V	1	∞	0	$\|V\|$	1	1	0	0	0
31	m-Ring Star Problem	V	1	∞	0	$\|V\|$	1	1	0	0	0

continues on next page

36	Min-Max Selective VRP	$\{j \in V : d_{ij} \leq r\}$	1	∞	0	$\|V\|$	$\min \max_{m \in M} \sum i \in V \sum j \in V d_{ij} x_{ij}^{0,m}$				
43	Family TSP	$\{j \in V \| j$ is in the same cluster of $i\}$	1	∞	0	$\tilde{n}$	1	0	0	0	0
47	Generalized Median Tour Problem	$\{j \in V \| j$ is in the same cluster of $i\}$	1	∞	0	$\|V\|$	1	1	0	0	0

6.6. *Class 9:* $\overline{W} = W$ *and* $\overline{V} = V$

Table 22: The mapping table for problems with sets $\overline{W} = W$ and $\overline{V} = V$ in Class 9

#	Problem	Coverage set (S_i)	Coverage demand (k_i)	$\mathcal{B}$	$\mathcal{P}$	n	Objective function weights				
							α_1	α_2	α_3	α_4	α_5
1	TSP	$\{i\}$	1	∞	0	$\|V\|$	1	0	0	0	0
2	VRP	$\{i\}$	1	∞	0	$\|V\|$	1	0	0	0	0
49	Covering VRP with Integrated Tours	$\{j \in V : d_{ij} \leq r\}$	1	∞	0	$\|V\|$	1	0	0	0	0

7. Potential Future Research Areas

This research helps identify potential future research areas. Our classification in Figure 3 shows that valid SRPs have yet to be studied. In particular, according to our classification, Classes 1, 3, 4, 5 and 6 have not yet been explored and offers potential research areas. We have not identified any publications in these classes, and each class represents a valid research area.

In Class 1 (subclasses 1.1 with $W \cap V = \varnothing$ and 1.3 with $W = V$), consider an example with the route of a new bus line being designed with a set of candidate stations, where each station can cover certain regions. However, not all candidate stations may be on the route. Also, there may not be any initial or terminal vertices, as an arbitrary station on the resulting route can be selected as the initial point. If the problem is defined with school buses and houses where children are picked up from the nearest visited house, we have $W = V$. When this problem is defined with houses and separate bus stations, it yields the definition of $W \cap V = \varnothing$. If there is also an option to visit arbitrary stops or intersections of coverage radii of houses as in the discretization of continuous-space problems, then $W \subset V$. Therefore, these three relationships between sets W and V do not affect the full selectiveness of the problem and can be presented as a subclasses of Class 1. As another application in Class 1, consider for example mobile healthcare facilities maximizing their reach to the public in a humanitarian context. Such facilities may visit potential locations with medical staff and equipment to service the impacted population. Similar to the school bus routing example, while visiting a location, the same neighborhood may receive service by the same mobile unit, in which case we have $W = V$. If the stops of the healthcare facility and the households are distinct sets, we then have $W \cap V = \varnothing$. We also have the $W \subset V$ case when mobile facilities visit the intersections of coverage radii of households.

According to our classification, Class 4 regarding partial selectiveness in terms of vertices to cover ($\overline{W} \subset W$) and full selectiveness in vertices to visit ($\overline{V} = \varnothing$) has not been studied, even though it corresponds to a valid application area. For example, determining a bus route including the stations near the contracted malls yields the classification of $\overline{W} \subset W$ and $\overline{V} = \varnothing$. Similarly, there are problems in Class 5 that are not investigated. For example, $W \subset V$ can be defined for a bus routing problem with mandatory stops in certain schools ($\overline{W}$) and certain stations ($\overline{V}$), which represents an interesting application and a future research topic in this class. Moreover, Classes 3.1 and 6.1 regarding zero selectiveness in terms of vertices to visit ($\overline{V} = V$) and partial and full selectiveness in terms of vertices to cover ($\overline{W} \subset W$ and $\overline{W} = \varnothing$) have not been studied. When W and V are not identical sets, vertices in V have capacities and vertices in W are covered by allocation; visiting every vertex in V does not imply covering all vertices in W. Therefore, there would still be the question of which vertices to serve.

8. Concluding Remarks

The main purpose of routing problems is to serve the demand at the vertices. In this paper, we have associated service with coverage and considered routing a tool of coverage. This way, we have distinguished two widely used concepts: visiting and covering. We have connected these concepts with the idea of providing *coverage from the visited vertices.* We have elaborated on the *classical versus selective* distinction of the routing problems and have offered a framework that classifies problems in terms of their selectiveness degree. We have illustrated how the classical routing problems can be considered *Selective Routing Problems with zero degrees of selectiveness.*

We have offered a generic mathematical model, valid for every SRP, demonstrating the connection between all such problems. In order to examine different notions of service, we have considered four coverage types: visiting, radius, clusters, and allocation. We have presented all coverage types through the set S_i and have introduced a comprehensive formulation for the literature. We have demonstrated how each SRP can be formulated using the generic model. Finally, we have identified potential future research areas.

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