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Optimization of Assortment Breadth and Allocation of the Selected Product Groups to the Two-Dimensional Shelves

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Abstract. This paper addresses the integrated optimization of assortment breadth, shelf assignment, shelf space allocation, and positioning of the selected product groups on the multi-level shelves to maximize store profitability. To improve the tidiness and findability of the product groups along the shelves, following the merchandising rules, the allocated space to each product group should be rectangular and enclosed within the region of the larger group it belongs to. We formulate the problem as a mixed-integer linear programming model. A two-phased matheuristic algorithm is proposed to solve the problem. In the first phase, a simplified version of the problem is solved by a column generation heuristic. An optimization-based algorithm provides the initial columns by which the efficiency of the column generation heuristic is improved. The second phase uses the output of the first phase and solves a set of independent single-shelf problems. The numerical studies show that the proposed algorithm yields high-quality solutions for problem instances with up to 40 multilevel shelves and more than 1000 product groups with a relative optimality criterion of less than 3.8% in a reasonable time. Further, we demonstrate the usefulness of the proposed methodology by using a case study.

Keywords: Assortment breadth optimization, merchandising rules, macro-level shelf space allocation, two-phased matheuristic, column generation heuristic.

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1. Introduction

The breadth of the assortment provided by a store is the main driver to attract customers, while the conversion rate of the customers is mainly influenced by how the space allocation is conducted. During a visit to a physical store, customers are exposed to items which are not on their shopping list. For example, a customer visiting a store to buy staple items might end up with unplanned and impulsive purchases. Accordingly, retailers should offer an assortment of high-sales and impulsive product groups and arrange them along the shelves in a way that increases the visibility of both fast-moving and impulsive ones (Flamand et al., 2018).

The customers' in-store experience is also heavily influenced by how convenient it is to find the products they are searching for. The following merchandising rules can enhance the tidiness and findability of product groups along the shelves (Bianchi-Aguiar et al., 2021). Firstly, the area allocated to each product group should be enclosed within the area of its corresponding larger product group. For example, a retailer may place subcategories of classic and drinkable yogurt in the area assigned to the yogurt category. Secondly, to improve the findability of the areas allocated to various product groups within a shelf, it is recommended that these areas be rectangular (Geismar et al., 2015; Bianchi-Aguiar et al., 2018; Hübner et al., 2020).

Shelf space planning is divided into two hierarchical levels. At the macro-level, the product groups form the basis of decisions on shelf space planning, while at the micro-level, the individual products within each group are allocated to the shelves (Bianchi-Aguiar et al., 2018; Hübner et al., 2020). Similarly, assortment planning includes hierarchical decisions about (i) the variety of product groups which are offered in the store, i.e., breadth of the assortment, and (ii) the variety of items within each product group, i.e., depth of the assortment (Kök et al., 2015). A broad variety of product groups is in conflict with the limited space of store shelves. Consequently, determining a subset of product groups to be carried in the store, i.e., the breadth of assortment, and shelf space planning are two interrelated problems of great importance to retailers.

This paper addresses assortment breadth planning and allocation of the selected product groups to the multi-level shelves considering the merchandising rules. We develop a two-phased matheuristic algorithm, which provides high-quality solutions to instances with up to 40 multi-level shelves and more than 1000 product groups. We also apply the proposed approach to a real case study and analyze the obtained results.

The remainder of the paper is organized as follows. Section 2 provides a review of relevant literature. Section 3 presents the formal problem statement. Section 4 is dedicated to describing the mixed-integer linear programming model for the problem. Section 5 delineates the solution approach. Section 6 presents the computational experiments. Section 7 investigates the usefulness of the proposed approach in practice using real data. Finally, Section 8 concludes the paper and summarizes the findings.

2. Literature review

One main stream of shelf space allocation studies focuses on the micro-level, specifically addressing individual products. Corstjens and Doyle (1981), as one of the first researchers in this area, developed a model to find the optimal space allocated to each product considering the effects of space and cross-space elasticity on demand. Hwang et al. (2009) extended this model by taking into account the location effects on demand as well as deciding on the shelf design and item allocation. Geismar et al. (2015) proposed a model for two-dimensional shelf space allocation that allows the display area of a product to spread across multiple levels of a shelf. Hübner and Schaal (2017) developed an integrated model to determine the depth of assortment and the space allocation decisions for listed items by considering both space and substitution effects. Hübner et al. (2020) extended this problem by taking into account tilted two-dimensional shelves. Bianchi-Aguiar et al. (2018) formulated a novel two-dimensional shelf space allocation model which includes additional features related to the merchandising rules. The authors considered a specific product group whose location within the store has been determined in advance. Düsterhöft et al. (2020) developed an integrated optimization model for shelf management and replenishment decision that considers shelf segments with different height, depth, and width. Kim and Moon (2021) proposed an integrated model for product selection, two-dimensional shelf space allocation, and replenishment decisions considering the effects of space and location on demand. Gencosman and Begen (2022) presented an exact method using logic-based Benders decomposition to solve real-size instances of a two-dimensional shelf space allocation problem with rectangular arrangement. Gecili and Parikh (2022) formulated a model that jointly determines the design of a two-dimensional shelf and optimizes the number of assigned faces per item within each product family.

These studies primarily focused on optimizing facing decisions to determine the details of planograms, i.e., blueprints of the shelves, at an operational level. In most cases, it is assumed that the assignment of product groups to the shelves was given by preceding decisions at macro-level planning. Therefore, these studies investigated a set of shelves independently without taking into account the store layout considerations such as the customer traffic density in different store areas.

The other stream of shelf space planning studies addresses the macro-level and deals with the product groups. Irion et al. (2011) developed an optimization model to determine the assortment breadth and the amount of space allocated to each selected product group. Ghoniem et al. (2016b) devised an optimization framework based on a large-scale local search method to assign product groups to the shelves and simultaneously determined the horizontal location and the amount of space allocated to each product group. In another study, Ghoniem et al. (2016a) provided an exact solution to the problem using a branch-and-price algorithm which could solve the instances with up to 60 product groups and 10 shelves. Flamand et al. (2016) developed an optimization model to maximize impulse purchase profit by allocating shelf space to the product groups. Flamand et al. (2018) proposed an integrated optimization model for assortment planning and store-wide shelf space allocation problem by considering a store that includes a set of single-level shelves. To maximize store profitability, this model jointly optimized the breadth of the assortment, the assignment of product groups to the shelves, the horizontal location of product groups within shelves, and the amount of space allocated to each product group.

Table 1 summarizes the most related works to the present study. To the best of our knowledge, there is no study that jointly determines the breadth of the assortment and

Table 1. Summary of relevant studies

Reference	Shelf space planning scope	Assortment decision	Shelf space allocation decisions	Merchandising rules	Shelves dimension	Store wide	Solution approach
Hwang et al. (2009)	Micro		S/V/H	R	2D		Genetic algorithm
Geismar et al. (2015)	Micro		S/V/H	R	2D	✓	Decomposition heuristic
Bianchi-Aguiar et al. (2018)	Micro		S/V/H	R/P	2D		Optimization-based heuristic
Hübner et al. (2020)	Micro	D	S/V/H	R	2D		Genetic-based heuristic
Kim & Moon (2021)	Micro	D	S/V/H		2D		Tabu search, Genetic
Gencosman & Begen (2022)	Micro		S/V/H	R	2D	✓	Two-stage exact model
Gecili & Parikh (2022)	Micro		S/V/H	R/P	2D		Decomposition-based approach
Irion et al. (2011)	Macro	B	S		1D		Hierarchical decomposition
Ghoniem et al. (2016a)	Macro		S/H		1D	✓	Branch-and-price
Ghoniem et al. (2016b)	Macro		S/H		1D	✓	Large-scale neighborhood search
Flamand et al. (2016)	Macro		S/H		1D	✓	Mixed-integer programing
Flamand et al. (2018)	Macro	B	S/H		1D	✓	Optimization-based heuristic
This paper	**Macro**	**B**	**S/V/H**	**R/P**	**2D**	✓	**Two-phased matheuristic**

D = depth of the assortment; B = breadth of the assortment. S = space allocation; V = vertical location within a shelf; H = horizontal location within a shelf.
R = rectangular arrangement; P = product hierarchy. 1D = one-dimensional; 2D = two-dimensional.

macro-level shelf space allocation which takes into account rectangular arrangement and product hierarchies. The studies investigating shelf space planning at the macro-level assume that the store is composed of a set of single-level shelves, while the current study considers a store that consists of multi-level shelves. According to this table, the current study extends the problem studied by Flamand et al. (2018) with the consideration of product hierarchies and rectangular arrangement of product groups along multi-level shelves.

3. Problem definition

In this study, we consider a store consisting of K shelves. Each shelf $k \in \mathcal{K}$ consists of $V_k \times H_k$ segments, where $v, v' \in \mathcal{V}_k$ are the indices of shelf levels and $h, h' \in \mathcal{H}_k$ are the indices of horizontal segments. For example in Fig. 1, the displayed shelf consists of 4×3 segments. The intersection of shelf level v and horizontal segment h of the shelf k is denoted by segment $(v, h)_k$. Moreover, the horizontal coordinate of the leftmost and rightmost corners of segment $(v, h)_k$ are denoted by B_h^k and E_h^k, respectively. Segment $(v, h)_k$ has attractiveness $\gamma_{vh}^k \in (0, 1]$, which indicates the impact of its location on the likelihood of visiting by the customers.

The retailer has pre-clustered the individual products based on a set of common characteristics into U product groups (indexed by $u, m, n \in \mathcal{U}$). The problem comprises selecting a subset of product groups and allocating two-dimensional space of the shelves to them in order to maximize the store profitability. If a product group is selected in the assortment, it can be located on a number of consecutive shelf levels and its area can spread horizontally over multiple contiguous segments. However, its total allocated space should be between lower bound LO_u and upper bound UP_u and form a rectangle.

Furthermore, product groups are embedded in a product hierarchy with three levels ($L = 3$, indexed by $l \in \mathcal{L}$) of families (e.g., dairy), categories (e.g., milk), and subcategories (e.g., low-fat milk), represented by sets $\mathcal{F}$, $\mathcal{C}$, and $\mathcal{S}$, respectively (see Fig. 2). We specify product groups available at level l using set $\mathcal{N}_l$. The parent of all product families is a dummy node at the root of the product hierarchy ($\mathcal{N}_0$). We

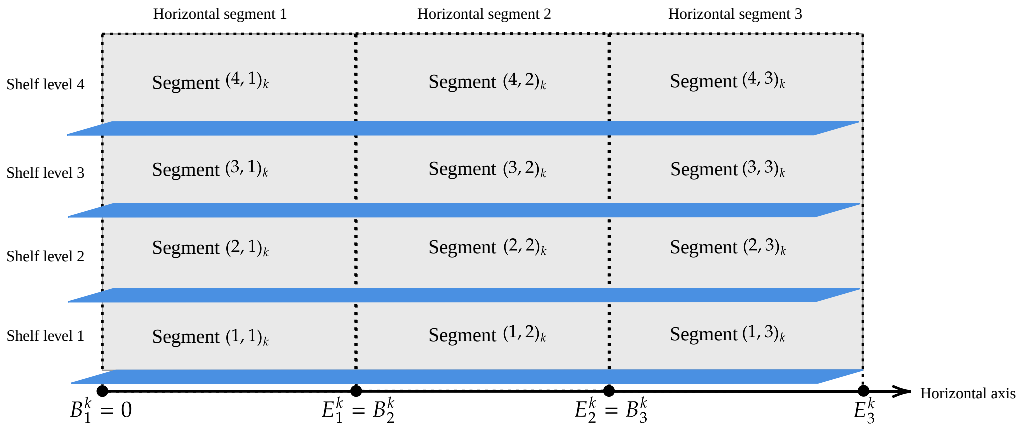

Figure 1. Front view of a shelf k consisting of 4×3 segments

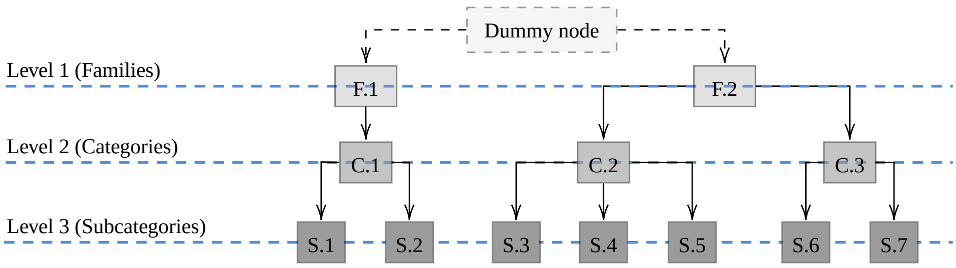

Figure 2. Example of a product hierarchy with three levels

denote the product groups belonging to product group u by $\mathcal{M}_u$. For example, in Fig. 2, $\mathcal{M}_{F.1} = \{C.1\}$ and $\mathcal{M}_{C.1} = \{S.1, S.2\}$.

The area allocated to each product group must be enclosed within the area of its parent. For instance, the area allocated to the toothpaste subcategory should be enclosed within the area of the oral care category. In the same way, the area allocated to the oral care category must be enclosed within the area allocated to the personal care family. In contrast to product groups that should be placed together, specific pairs of product groups, denoted by set $\mathcal{A}$, should not be displayed on the same shelf due to a lack of affinity. For example, suppose that $\mathcal{A} = \{(\text{Bleach}, \text{Juice})\}$. If these two product groups are selected simultaneously in the store assortment, they will not be assigned to the same shelf.

Following Flamand et al. (2018), the objective function is the total possible profit of the selected product groups. The possible profit of each product group is a function of its expected demand, profit margin, conversion rate, and the visibility of the region in which it is displayed. It is worth mentioning that the substitution effect typically exists between similar individual products within the same subcategory. Since this study deals with product groups at the macro-level, here, this effect can be disregarded and tackled in a downstream micro-level problem.

4. Model formulation

This section introduces the mathematical formulation proposed for the problem, referred to as the General Problem (GP) model. The decision variables of the proposed model are as follows:

- $x_u^k \in \{0,1\}$: $x_u^k = 1$ if and only if product group u is assigned to shelf k.
- $z_{uvh}^k \in \{0,1\}$: $z_{uvh}^k = 1$ if and only if product group u is assigned to segment $(v,h)_k$.
- $r_{um} \in \{0,1\}$: $r_{um} = 1$ if and only if product group m is displayed on the right side of product group u in the same segment of a shelf.
- s_{uvh}^k: length of space allocated to product group u along segment $(v,h)_k$.
- c_{uh}^k: horizontal coordinate of the left side of the area allocated to product group u in horizontal segment h of shelf k.

Fig. 3 illustrates the definition of the last two variables for a product group u which is located on the horizontal segments 1, 2, and 3 of shelf levels 2 and 3. The detailed description of the GP is provided in Sections 4.1 to 4.3.

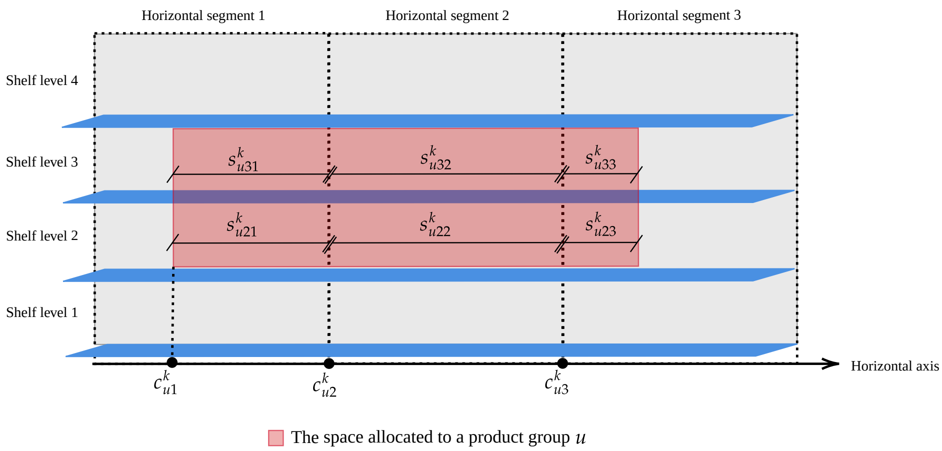

Figure 3. Illustration of variables s_{uvh}^k and c_{uh}^k

4.1. Objective function

The objective function maximizes the total possible profit, benefiting from the same idea of Flamand et al. (2018):

$$\text{Maximize} \sum_{k\in\mathcal{K}} \sum_{u\in\mathcal{S}} \sum_{v\in\mathcal{V}_k} \sum_{h\in\mathcal{H}_k} \phi_u \frac{\gamma_{vh}^k s_{uvh}^k}{E_h^k - B_h^k}, \tag{1}$$

where $\phi_u = \rho_u \nu_u \eta_u$ is a measure of the highest possible profit for the product group u in which ρ_u, ν_u, and η_u represent the profit margin, the expected demand volume, and the conversion rate of this product group, respectively. It is assumed that the demand of each product group is deterministic and computed as an average of all possible scenarios. The conversion rate of product group u, $\eta_u \in (0,1]$, represents the likelihood of it being purchased when noticed by a customer. The term $\gamma_{vh}^k s_{uvh}^k/(E_h^k - B_h^k)$ calculates the visibility of product group u along segment $(v,h)_k$. This term expresses the likelihood of product group u to be spotted by a customer (Flamand et al., 2016).

γ^k_{vh} represents the visibility of segment $(v,h)_k$ and $s^k_{uvh}/(E^k_h - B^k_h)$ calculates the proportion of this segment occupied by product group u.

4.2. Assignment constraints and space requirements

The constraints corresponding to the assignment of product groups to the store shelves along with their space requirements are as follows:

$$\sum_{k\in\mathcal{K}} x^k_u \leq 1, \qquad \forall u \in \mathcal{F} \tag{2}$$

$$\sum_{v\in\mathcal{V}_k}\sum_{h\in\mathcal{H}_k} z^k_{uvh} \leq M x^k_u, \qquad \forall u \in \mathcal{U}, k \in \mathcal{K} \tag{3}$$

$$s^k_{uvh} \leq (E^k_h - B^k_h) z^k_{uvh}, \qquad \forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{4}$$

$$\sum_{k\in\mathcal{K}}\sum_{v\in\mathcal{V}_k}\sum_{h\in\mathcal{H}_k} s^k_{uvh} \leq UP_u \sum_{k\in\mathcal{K}} x^k_u, \quad \forall u \in \mathcal{U} \tag{5}$$

$$\sum_{k\in\mathcal{K}}\sum_{v\in\mathcal{V}_k}\sum_{h\in\mathcal{H}_k} s^k_{uvh} \geq LO_u \sum_{k\in\mathcal{K}} x^k_u, \quad \forall u \in \mathcal{U} \tag{6}$$

$$x^k_u + x^k_m \leq 1, \qquad \forall u, m \in \mathcal{U} : (u,m) \in \mathcal{A}, k \in \mathcal{K} \tag{7}$$

Constraints (2) ensure that each product family can be assigned to no more than one shelf. Constraints (3) enforce that a product group can be displayed on segments of a shelf only if it is assigned to that shelf. Constraints (4) state that the amount of space allocated to any product group in each segment cannot exceed the segment capacity. Constraints (5) and (6) guarantee that the total space allocated to each product group should not violate its corresponding lower and upper bounds. Constraints (7) prevent pairs of product groups having a lack of affinity to be assigned on the same shelf.

4.3. Merchandising rules constraints

In each segment, different product groups should not overlap horizontally. Further, the space allocated to each product group in each shelf level should not be composed of separate areas. Eqs. (8) to (15) guarantee these requirements.

$$r_{mn} + r_{nm} \geq z^k_{nvh} + z^k_{mvh} - 1,$$
$$\forall l \in \mathcal{L} \cup \{0\}, u \in \mathcal{N}_l, n, m \in \mathcal{M}_u : n < m, (n,m) \notin \mathcal{A}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{8}$$

$$c^k_{nh} + s^k_{nvh} \leq c^k_{mh} + M(1 - r_{nm}),$$
$$\forall l \in \mathcal{L} \cup \{0\}, u \in \mathcal{N}_l, n, m \in \mathcal{M}_u : n \neq m, (n,m) \notin \mathcal{A}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{9}$$

$$c^k_{uh} + s^k_{uvh} \leq E^k_h, \quad \forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{10}$$

$$c^k_{u,h+1} \leq B^k_{h+1} + M(2 - z^k_{uvh} - z^k_{uv,h+1}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \{1, \ldots, H_k - 1\} \tag{11}$$

$$c^k_{u,h+1} \geq B^k_{h+1} - M(2 - z^k_{uvh} - z^k_{uv,h+1}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \{1, \ldots, H_k - 1\} \tag{12}$$

$$c^k_{u,h-1} + s^k_{uv,h-1} \leq E^k_{h-1} + M(2 - z^k_{uvh} - z^k_{uv,h-1}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \{2, \ldots, H_k\} \tag{13}$$
$$c^k_{u,h-1} + s^k_{uv,h-1} \geq E^k_{h-1} - M(2 - z^k_{uvh} - z^k_{uv,h-1}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \{2, \ldots, H_k\} \tag{14}$$
$$s^k_{uv,h+1} \geq (E^k_{h+1} - B^k_{h+1})(z^k_{uvh} + z^k_{uv,h+2} - 1),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \{1, \ldots, H_k - 2\} \tag{15}$$

Constraints (8) require that if two product groups are on the same shelf segment, one must be on the right side of the other. Constraints (9) ensure that the product groups on the right side of product group n are horizontally positioned after the location of product group n plus the length of the area occupied by this product group. Constraints (10) limit the space for each product group on a segment to the right border of that segment. Constraints (11) to (14) determine the left and right coordinates for a group when it is assigned to adjacent horizontal segments. Constraints (15) allow a group to be assigned to two nonadjacent segments if it spans the segment between them.

On each shelf, the space allocated to each product group should be enclosed in the allocated space of its parent and rectangular. Eqs. (16) to (22) assure these requirements.

$$c^k_{mh} \geq c^k_{uh}, \quad \forall l \in \mathcal{L}, u \in \mathcal{N}_l, m \in \mathcal{M}_u, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{16}$$
$$c^k_{uh} + s^k_{uvh} \leq c^k_{mh} + s^k_{mvh}, \quad \forall l \in \mathcal{L}, u \in \mathcal{N}_l, m \in \mathcal{M}_u, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{17}$$
$$s^k_{u,v+1,h} \leq s^k_{uvh} + M(2 - z^k_{uvh} - z^k_{u,v+1,h}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \{1, \ldots, V_k - 1\}, h \in \mathcal{H}_k \tag{18}$$
$$s^k_{u,v+1,h} \geq s^k_{uvh} - M(2 - z^k_{uvh} - z^k_{u,v+1,h}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \{1, \ldots, V_k - 1\}, h \in \mathcal{H}_k \tag{19}$$
$$\sum_{v' \in \mathcal{V}_k} \sum_{\substack{h' \in \mathcal{H}_k, \\ h' \geq h+1}} z^k_{uv'h'} \leq M(1 - z^k_{uvh} + z^k_{uv,h+1}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \{1, \ldots, H_k - 1\} \tag{20}$$
$$\sum_{\substack{v' \in \mathcal{V}_k, \\ v' \geq v+1}} \sum_{h' \in \mathcal{H}_k} z^k_{uv'h'} \leq M(1 - z^k_{uvh} + z^k_{u,v+1,h}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \{1, \ldots, V_k - 1\}, h \in \mathcal{H}_k \tag{21}$$
$$\sum_{\substack{v' \in \mathcal{V}_k, \\ v' \leq v-1}} \sum_{h' \in \mathcal{H}_k} z^k_{uv'h'} \leq M(1 - z^k_{uvh} + z^k_{u,v-1,h}),$$
$$\forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \{2, \ldots, V_k\}, h \in \mathcal{H}_k \tag{22}$$
$$x^k_u, z^k_{uvh} \in \{0, 1\}, \ \forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{23}$$
$$r_{um} \in \{0, 1\}, \qquad \forall u, m \in \mathcal{U} : u \neq m \tag{24}$$
$$s^k_{uvh}, c^k_{uh} \geq 0, \qquad \forall u \in \mathcal{U}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{25}$$

Constraints (16) and (17) enforce that, in each segment, the space allocated to each product group is contained within the space allocated to its parent. Constraints (18) and (19) guarantee that if product group u is simultaneously assigned to segments $(v,h)_k$ and $(v+1,h)_k$, its allocated space in these two segments should be equal. Constraints (20)-(22) preclude the assignment of any product group to the sets of segments when it contradicts the requirement that the whole space of each group must form a rectangle. Finally, constraints (23) to (25) indicate the domains of the decision variables.

5. Solution approach

This section proposes a two-phased matheuristic, called TPM, to solve GP. In the first phase, a simplified version of the GP, called SGP, is solved using a column generation heuristic. In the second phase, GP is solved independently for each shelf, considering only the product families which have been assigned to that shelf in the solution of the SGP.

5.1. Phase 1: SGP model

SGP is formulated by replacing merchandising constraints (8)-(22) in GP with constraints (26) and (27):

$$s^k_{uvh} \geq \sum_{m \in \mathcal{M}_u} s^k_{uvh}, \qquad \forall l \in \mathcal{L}, u \in \mathcal{N}_l, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{26}$$

$$\sum_{u \in \mathcal{N}_l} s^k_{uvh} \leq E^k_h - B^k_h, \quad \forall l \in \mathcal{L} \cup \{0\}, k \in \mathcal{K}, v \in \mathcal{V}_k, h \in \mathcal{H}_k. \tag{27}$$

In the absence of constraints (8)-(22), constraints (26) guarantee that the space allocated to a product group along any segment is at least as large as the total space allocated to its included product groups. Moreover, constraints (27) restrict the total amount of allocated space of a segment to its capacity.

To solve this problem, we first reformulate SGP to a column-oriented form and then decompose it into a master problem (MP), and a pricing sub-problem (PSP). The former is a 0-1 integer programming model that selects the family-to-shelf assignment patterns (columns) from a given set of available patterns. The latter is a mixed-integer linear programming model, searching for the columns with negative reduced costs to improve the solution of MP.

Let $\mathcal{J}_k$ denote the set of distinct columns corresponding to feasible family-to-shelf assignment patterns for shelf k. Then, the column vector D^k_j, which represents the jth column for shelf k, is defined as follows:

$$D^k_j = \begin{array}{c} u=1 \\ u=2 \\ \vdots \\ u \\ \vdots \\ u=|\mathcal{F}| \end{array} \begin{bmatrix} \\ \\ \left(d^k_{uj}\right) \\ \\ \\ \end{bmatrix},$$

where $d^k_{uj} = 1$ implies that pattern j of shelf k includes family u. In other words, in this pattern $x^k_u = 1$.

The formulation of the MP model is stated as follows:

$$\text{MP : Maximize} \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}_k} \delta_j^k y_j^k \tag{28}$$

$$\text{subject to : } \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}_k} d_{uj}^k y_j^k \leq 1, \quad \forall u \in \mathcal{F} \tag{29}$$

$$\sum_{j\in\mathcal{J}_k} y_j^k \leq 1, \quad \forall k \in \mathcal{K} \tag{30}$$

$$y_j^k \in \{0,1\}, \quad \forall k \in \mathcal{K}, j \in \mathcal{J}_k, \tag{31}$$

where y_j^k is a binary variable representing whether or not column D_j^k is selected. Further, let $\bar{s}_{uvh}^k$ be the space allocated to product group u along segment $(v,h)_k$ in the PSP optimal solution, corresponding to column D_j^k. Then, the objective coefficient δ_j^k is calculated by setting $s_{uvh}^k = \bar{s}_{uvh}^k$ and $\mathcal{K} = \{k\}$ in Eq. (1). Constraints (29) guarantee the assignment of each product family to at most one shelf. Constraints (30) ensure that each shelf will be represented at most by one column. Constraints (31) enforce the logical binary restrictions on the decision variables.

By relaxing constraint (31) to $0 \leq y_j^k \leq 1$, the resulting linear master problem (LMP) can be efficiently solved by column generation (CG). This procedure starts by solving a restricted linear master problem (RLMP), with a subset of feasible columns. At each iteration, PSP generates a new column for each shelf, based on dual solutions associated with RLMP constraints. The generated column for a shelf is added to RLMP if it has a negative reduced cost and updated RLMP is solved again. This procedure continues until no column with negative reduced cost is found; in this case, the optimal solution of LMP is obtained (Chen et al., 2009; Wolsey, 2020).

Let π and μ represent the dual variables associated with constraints (29) and (30), respectively, where $\bar{\pi}$ and $\bar{\mu}$ indicate their values. For each shelf $k \in \mathcal{K}$, we solve the following pricing sub-problem:

$$\text{PSP}(\bar{\pi},\bar{\mu},k) : \text{Minimize} \left\{ \bar{\mu}_k + \sum_{u\in\mathcal{F}} \bar{\pi}_u x_u - \sum_{u\in\mathcal{S}} \sum_{v\in\mathcal{V}_k} \sum_{h\in\mathcal{H}_k} \phi_u \frac{\gamma_{vh}^k s_{uvh}}{E_h^k - B_h^k} \right\} \tag{32}$$

$$\text{subject to : } \sum_{v\in\mathcal{V}_k} \sum_{h\in\mathcal{H}_k} z_{uvh} \leq M x_u, \quad \forall u \in \mathcal{U} \tag{33}$$

$$s_{uvh} \leq (E_h^k - B_h^k) z_{uvh}, \quad \forall u \in \mathcal{U}, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{34}$$

$$\sum_{v\in\mathcal{V}_k} \sum_{h\in\mathcal{H}_k} s_{uvh} \leq UP_u x_u, \quad \forall u \in \mathcal{U} \tag{35}$$

$$\sum_{v\in\mathcal{V}_k} \sum_{h\in\mathcal{H}_k} s_{uvh} \geq LO_u x_u, \quad \forall u \in \mathcal{U} \tag{36}$$

$$x_u + x_m \leq 1, \quad \forall u, m \in \mathcal{U} : (u,m) \in A \tag{37}$$

$$s_{uvh} \geq \sum_{m\in\mathcal{M}_u} s_{uvh}, \quad \forall l \in \mathcal{L}, u \in \mathcal{N}_l, v \in \mathcal{V}_k, h \in \mathcal{H}_k \tag{38}$$

$$\sum_{u\in\mathcal{N}_l} s_{uvh} \leq E_h - B_h, \quad \forall l \in \mathcal{L} \cup \{0\}, v \in \mathcal{V}_k, h \in \mathcal{H}_k, \tag{39}$$

where x_u is a binary variable indicating the selection of product group u in the column of shelf k. Additionally, z_{uvh} is a binary variable representing whether product group u is assigned to segment $(v,h)_k$, and s_{uvh} is the length of space allocated to product group u along segment (v,h) in the column of shelf k. PSP generates a column by employing x-variables related to the families, minimizing the reduced cost, i.e., objective function (32). Constraints (33)-(39) are similar to constraints (3)-(7) and (26)-(27) of the SGP, but they are specifically applied to single-shelf pricing subproblems where $\mathcal{K} = \{k\}$. Furthermore, we add the following valid inequalities to speed up the PSP solution time.

$$x_m \leq x_u, \quad \forall l \in \mathcal{L}, u \in \mathcal{N}_l, m \in \mathcal{M}_u, \tag{40}$$

which state that each product group can be selected for shelf k if its parent product group is also selected.

5.1.1. Generating initial columns

In this section, we present an optimization-based algorithm to create initial columns for RLMP. To this end, let Cap_k represent the capacity of each shelf $k \in \mathcal{K}$ and a_k be a measure of its relative attractiveness, which are computed as follows:

$$Cap_k = V_k \times \left[\sum_{h \in \mathcal{H}_k} (E_h^k - B_h^k) \right], \tag{41}$$

$$a_k = \frac{\sum_{v \in \mathcal{V}_k} \sum_{h \in \mathcal{H}_k} \gamma_{vh}^k (E_h^k - B_h^k)}{Cap_k}. \tag{42}$$

Additionally, let $\sigma = (\sigma_1, \ldots, \sigma_K)$ be a permutation of the shelves, sorted in non-increasing order of their relative attractiveness. In the proposed method, the initial columns are generated respectively for $\sigma_1, \ldots, \sigma_K$. The initial column for each shelf is generated by consecutively solving two optimization problems. Firstly, a set of families are assigned to shelf k^* by solving the following knapsack problem using the algorithm proposed by Dantzig (1957):

$$\begin{aligned}
\text{KP}(k^*) : \text{Maximize} & \sum_{u \in \mathcal{F} \setminus S_f} \phi_u x_u \\
\text{subject to :} & \sum_{u \in \mathcal{F} \setminus S_f} UP_u x_u \leq Cap_{k^*} \\
& 0 \leq x_u \leq 1, \forall u \in \mathcal{F} \setminus S_f.
\end{aligned}$$

Secondly, we solve model S-SGP(k^*) for shelf k^*, which is SGP with only shelf $\mathcal{K} = \{k^*\}$ and its objective function is denoted by $\delta_0^{k^*}$. The pseudocode of this method is described in Algorithm 1.

As a simple alternative approach, the RLMP could be initialized with columns where all elements are set to zero, indicating that all shelves are empty and no product group is selected in the assortment. The performance of the CG procedure under these alternative initializations is compared in Section 6.

Algorithm 1 Initialization procedure

1: Input σ, $k \leftarrow 1, S_f \leftarrow \emptyset$
2: $k^* \leftarrow \sigma_k$
3: Solve KP(k^*)
4: $\mathcal{F}_{k^*} \leftarrow$ set of all families with positive value of x_u in the optimal solution of KP(k^*)
5: Solve S-SGP(k^*), regarding product families in $\mathcal{F}_{k^*}$ and their included product groups
6: $\mathcal{F}_{k^*} \leftarrow$ set of all product families with $x_u = 1$ in the optimal solution of S-SGP(k^*), $S_f \leftarrow S_f \cup \mathcal{F}_{k^*}$
7: $\delta_0^{k^*} \leftarrow$ the calculated objective function of S-SGP(k^*)
8: Generate $D_0^{k^*}$ regarding $\mathcal{F}_{k^*}$
9: Add $D_0^{k^*}$ to RLMP.
10: If $(k < K)$ and $(S_f \neq \mathcal{F})$ then $k \leftarrow k+1$ and go to step 2
11: Stop

5.1.2. Column generation heuristic

The column generation procedure solves the LMP. In order to obtain the integral values for y_j^ks, CG procedure is equipped with the the following heuristic. Let $\bar{Y}_t$ be the set of optimal values of variables associated with the RLMP columns (i.e., y_j^k) at iteration t of the CG procedure. We denote the largest value of this set by $\bar{y}_t(k^*, j^*)$. Further, consider Ω_t as the set of columns with negative reduced cost at iteration t of the CG procedure. At any iteration, if RLMP is non-degenerated for the last τ iterations or no column is identified with a negative reduced cost, i.e., $\Omega_t = \emptyset$, the j^*th column of shelf k^* is accepted as an assignment for this shelf, i.e., $y_{j^*}^{k^*} = 1$. The set of product families selected and assigned to shelf k is denoted by $\mathcal{F}_k$. This procedure continues until an assignment is obtained for every shelf. The flowchart of Phase 1 is shown in Fig. 4.

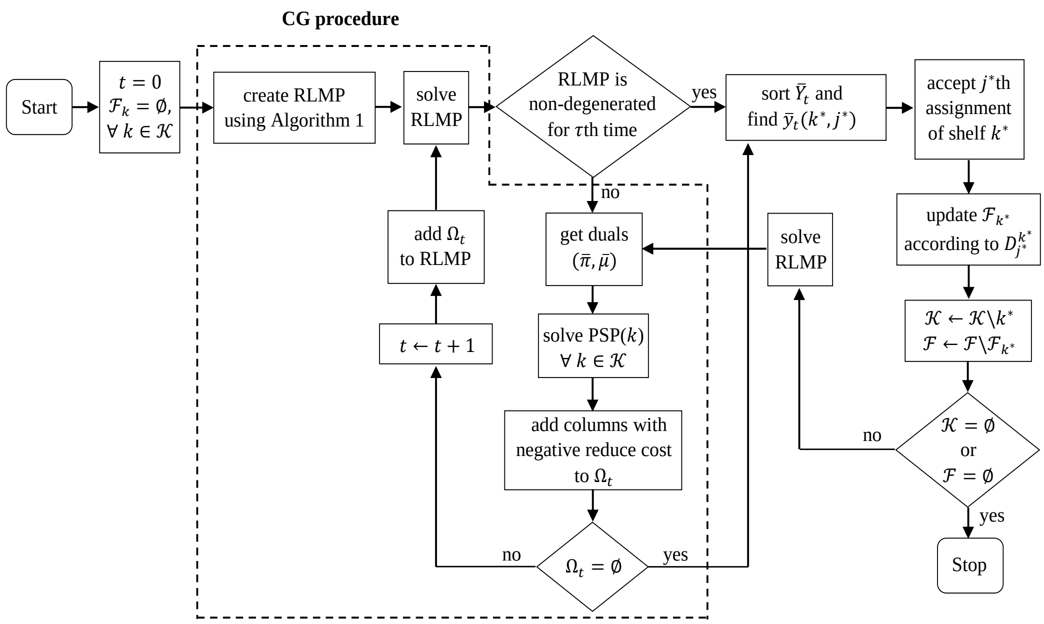

Figure 4. Phase 1

5.2. Phase 2: repairing algorithm

The previous phase obtains an integer solution for SGP. However, it is possible that the space allocated to some selected product groups does not follow the relaxed merchandising rules, e.g., not organized in rectangular shapes. In this phase, we solve the single-shelf variant of the GP model for each shelf $k \in \mathcal{K}$, wherein the selectable set of product groups for shelf k is limited to the families assigned to this shelf in Phase 1 together with their corresponding categories and subcategories. Consequently, the formulation of the single-shelf problem for shelf k can be obtained by setting $\mathcal{K} = \{k\}$, $\mathcal{F} = \mathcal{F}_k$, $\mathcal{C} = \mathcal{C}_k = \{u | u \in \mathcal{M}_{\mathcal{F}_k}\}$, $\mathcal{S} = \mathcal{S}_k = \{u | u \in \mathcal{M}_{\mathcal{C}_k}\}$, and $\mathcal{U} = \mathcal{U}_k = \{u : u \in \mathcal{F}_k \vee \mathcal{C}_k \vee \mathcal{S}_k\}$ in the GP model.

6. Computational studies

In this section, we evaluate the performance of the TPM algorithm by comparing the obtained solutions to the upper bounds achieved by CG as well as the solutions of GP found by the CPLEX solver. The algorithms were coded in `C#` and all mathematical programming models were solved using CPLEX 12.8. The numerical tests were conducted on a computer having an Intel® Core™ i7-9700K processor and a CPU at 3.60 GHz with 25 GB of RAM.

6.1. Problem instances

To evaluate the proposed algorithm, we use four sets of random instances with different sizes. To enhance diversity among the instances, product groups are distributed randomly within the hierarchical structure for each instance. Each family or category contains one to five immediate subgroups. Other features of the problem instances are as follows:

- Each shelf has five levels and includes three horizontal segments, each of which has six units of length.
- The attractiveness of each segment $(v, h)_k$, γ^k_{vh}, is randomly determined by a uniform distribution based on the ranges specified in Table 2. α represents the shelf attractiveness which is selected with equal probability from $\{0.05, 0.25, 0.45, 065, 0.85\}$.
- For each subcategory u, LO_u is randomly selected from $\{1, 2, \ldots, 6\}$. Moreover, for each category or family m, $LO_m = \min_{n \in \mathcal{M}_m}\{LO_n\}$.
- For each subcategory u, UP_u is randomly selected from $\{LO_u, \ldots, 6\}$. Moreover, for each category or family m, $UP_m = \sum_{n \in \mathcal{M}_m} UP_n$.
- For each subcategory u, ϕ_u is randomly selected from $\{1, \ldots, 25\}$.

Table 2. The ranges of attractiveness for various segments of a shelf

Horizontal segments \ Shelf levels	Eye and hand level	Top or lowest
End segments	$[\alpha + 0.11, \alpha + 0.15]$	$[\alpha + 0.06, \alpha + 0.10]$
Middle segments	$[\alpha + 0.06, \alpha + 0.10]$	$[\alpha, \alpha + 0.05]$

6.2. Performance comparison and analysis

This section is dedicated to the analysis of the performance of the TPM algorithm for randomly generated instances. Table 3 compares the effectiveness and efficiency of the TPM algorithm against the CPLEX solver. The column generation procedure is launched under two policies based on our discussion in Section 5.1.1. In this table, Columns 10 and 12 report the relative optimality criterion. This criterion, which is called ROC, is calculated for each instance by $(BUB - OBJ)/BUB \times 100$, where OBJ represents the value of the objective function obtained by the algorithm and $BUB = \min\{up_1, up_2\}$, where up_1 is the best upper bound found by CPLEX for the GP after 18000 CPU seconds and up_2 is the optimal solution of LMP using the CG approach described in Section 5.1.

CPLEX failed to solve any of the generated instances to optimality within the preset time limit of 18,000 CPU seconds. This solver has an average relative optimality criterion of 30.93% for instances of Set 1. Furthermore, the solutions obtained by CPLEX for Sets 2 to 4 represent completely empty stores, where no product group is included in the assortment plan. TPM algorithm launched with the empty shelves as the initial columns found higher quality solutions for all instances than solutions provided by CPLEX. However, TPM launched with columns provided by Algorithm 1 achieved superior solutions with $ROC < 3.8\%$ for all problem instances. The average computational time of the proposed algorithm under the latter policy is practically reasonable, given that the GP problem is a medium to long range planning problem. To summarize this section, the proposed TPM algorithm equipped with Algorithm 1 can solve the GP both efficiently and effectively.

7. Case study

To demonstrate the usefulness of our proposed method, we implemented it for a case study based on real data for a new branch of an Iranian chain of retail stores called Ofoq Kourosh (OK). This store is composed of 13 shelves having a grid layout. Each shelf consists of three horizontal segments and five levels. The store merchandiser has estimated the attractiveness of each segment. In this regard, the segments located at the client's eye level and close to store entrances have high visibility and attractiveness. The product hierarchy of this store consists of 305 candidate product groups (20 families, 67 categories, and 218 subcategories). The candidate product families are Breakfast products, Poultry & Meats, Tea & Herbal tea, Oils, Cereals & Legumes, Dairy products, Condiments & Spices, Canned foods, Powders, Pasta & Noodles, Pickles & Olives, Dried fruit and nuts, Beverages, Snacks, Personal care, Household cleaners, Cosmetics, Sanitary paper products, Detergents, and Kitchen supplies. These families are numbered from 1 to 20. Due to our obligation to keep confidential information of the OK, a product family number is not necessarily equivalent to its introducing order. TPM algorithm obtained a solution with 3.4% of relative optimality criterion in less than 4 minutes. The assortment plan generated by the TPM algorithm included 75.1% of candidate product groups. All candidate product families were included in the assortment. Fig. 5 depicts how the proposed approach has assigned the families to the shelves of the new store. The illustration of all shelves violates the limited space of the paper. Hence, only the shelf that is allocated to the product groups of the Personal care family is displayed in Fig. 6. This figure depicts the mentioned hierarchical and rectangular space allocation rules for merchandising.

Table 3. Performance of the TPM algorithm vs. CPLEX.

Set	$(\lvert\mathcal{F}\rvert, \lvert\mathcal{C}\rvert, \lvert\mathcal{S}\rvert)$	K	Inst.	Model GP		TPM algorithm			
						Using the empty shelves		Using the Algorithm 1	
				CPU(s)	ROC(%)	CPU(s)	ROC(%)	CPU(s)	ROC(%)
Set 1	$(15, 60, 180)$	10	1	18000	19.56	281.2	17.89	429.8	2.41
			2	18000	44.88	141.8	34.42	415.7	2.61
			3	18000	31.53	256.9	18.39	359.2	3.40
			4	18000	32.40	414.3	7.36	376.4	3.40
			5	18000	27.73	395.5	10.00	444.7	2.88
			6	18000	42.08	391.1	10.59	435.0	3.38
			7	18000	30.03	248.4	24.02	479.7	2.74
			8	18000	34.30	354.0	15.04	378.9	2.44
			9	18000	26.13	272.2	24.71	387.4	3.02
			10	18000	20.69	360.9	8.08	464.9	2.37
Average				**18000**	**30.93**	**311.6**	**17.10**	**417.2**	**2.90**
Set 2	$(30, 120, 360)$	20	1	18000	-	738.9	18.29	988.3	2.82
			2	18000	-	384.3	29.95	837.2	2.49
			3	18000	-	453.8	39.94	1029.8	3.79
			4	18000	-	888.6	12.11	955.1	2.85
			5	18000	-	512.7	31.98	1056.1	3.23
			6	18000	-	409.6	35.99	943.7	3.43
			7	18000	-	448.3	32.07	1074.1	3.06
			8	18000	-	518.4	30.76	874.4	2.62
			9	18000	-	520.2	27.78	1007.2	3.49
			10	18000	-	573.2	26.87	958.7	2.83
Average				**18000**	**-**	**544.8**	**28.60**	**972.5**	**3.06**
Set 3	$(45, 180, 540)$	30	1	18000	-	760.8	33.96	1597.7	3.32
			2	18000	-	861.9	34.79	1634.0	2.44
			3	18000	-	767.0	36.58	1803.7	3.58
			4	18000	-	897.7	34.63	1824.3	3.26
			5	18000	-	713.0	39.65	1893.7	3.44
			6	18000	-	907.4	34.06	1662.5	2.65
			7	18000	-	1164.2	22.34	1650.9	3.43
			8	18000	-	1045.9	25.59	1775.5	3.09
			9	18000	-	1092.6	25.60	1690.7	3.10
			10	18000	-	639.4	45.08	1967.5	2.75
Average				**18000**	**-**	**885.0**	**33.20**	**1750.1**	**3.11**
Set 4	$(60, 240, 720)$	40	1	18000	-	1185.0	37.92	3054.5	2.99
			2	18000	-	2254.7	20.98	2513.4	2.83
			3	18000	-	1325.0	38.29	2854.7	2.89
			4	18000	-	1175.2	32.04	2333.9	2.69
			5	18000	-	1708.2	29.00	3273.2	3.13
			6	18000	-	1626.3	26.97	2608.1	3.00
			7	18000	-	1448.2	34.59	2943.8	2.81
			8	18000	-	1059.6	39.78	2356.0	3.15
			9	18000	-	1678.5	21.14	2815.7	2.52
			10	18000	-	1995.8	23.87	2347.7	2.88
Average				**18000**	**-**	**1545.7**	**30.50**	**2710.1**	**2.89**

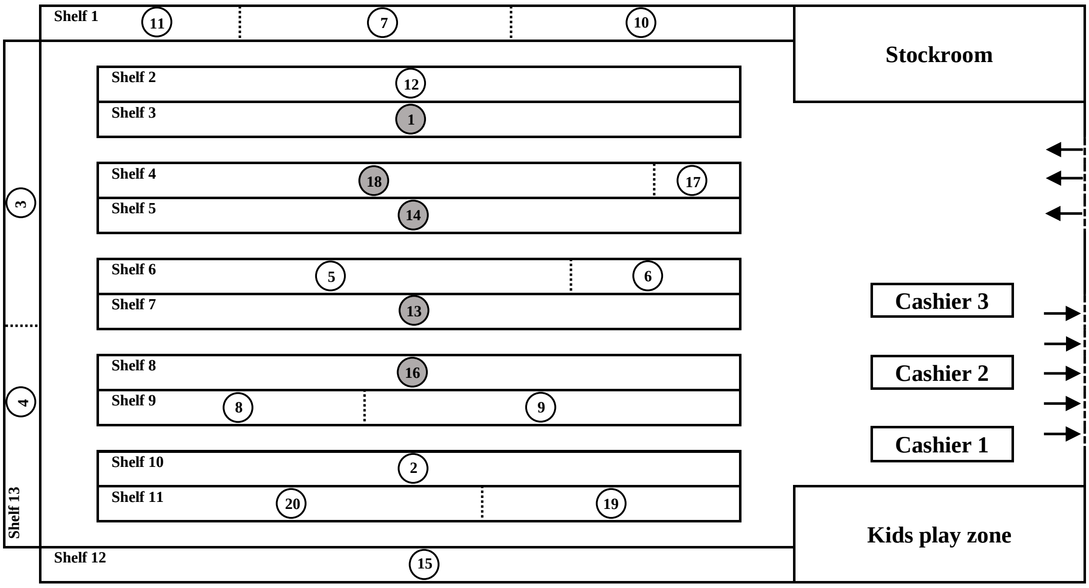

Figure 5. Bird's-eye view of the new store

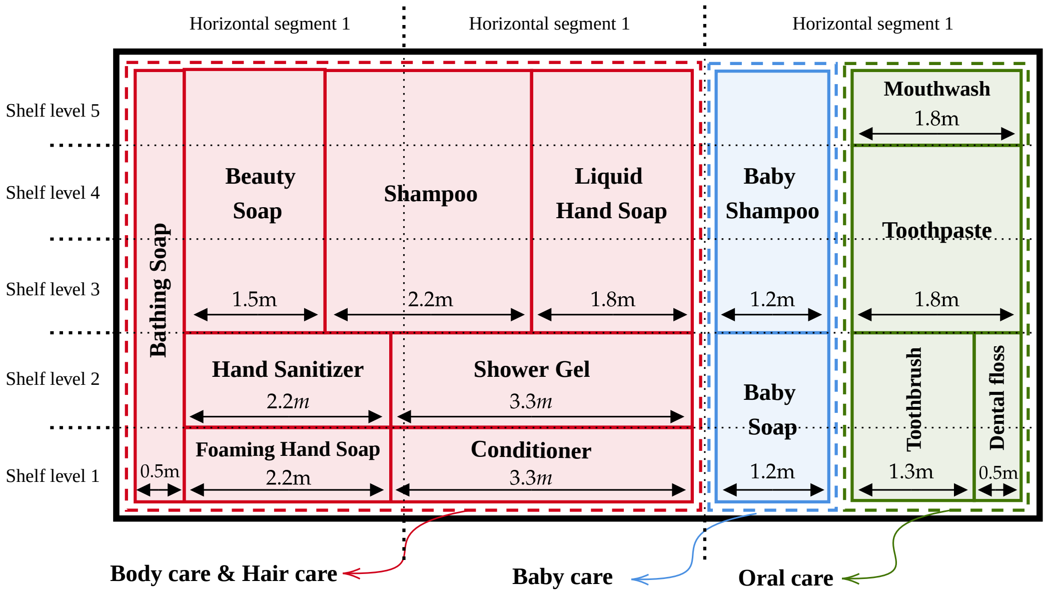

Figure 6. The shelf featuring Personal care family

Fig. 7 depicts the association between the visibility of product families and their highest possible profit in the optimized solution. The visibility of each product family is the summation of the allocated visibilities to its subcategories. It is worth reminding that the visibility of subcategory u located in segment $(v, h)_k$ is calculated through $\gamma_{vh}^k s_{uvh}^k/(E_h^k - B_h^k)$. According to this figure, generally, the higher values of visibility have been assigned to the product families which have higher values of ϕ_u. In this figure, the red-circled families have gained the highest visibility. According to the historical data, these families, which consist of fast-moving or impulsive products, can yield nearly 80% of the store's profit. These families are also highlighted in Fig. 5.

As shown, these profitable families occupy the maximum possible space (their upper bound or shelf capacity) and are located in aisles with the highest traffic density, both of which have led to their high level of visibility.

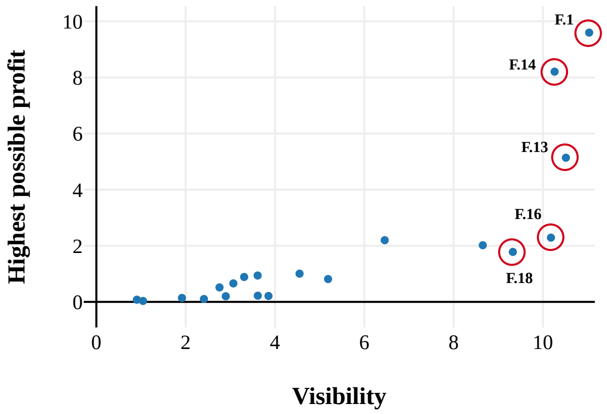

Figure 7. Visibility of families in the optimized solution

8. Conclusion

This paper addresses a joint problem of assortment breadth optimization and macro-level shelf space planning to maximize store profitability. To the best of our knowledge, this is the first study that considers merchandising rules, i.e., rectangular arrangement and product hierarchies, for an integrated problem of store-wide two-dimensional shelf space allocation and assortment breadth optimization. Due to the ineffectiveness of CPLEX to tackle the computational challenges of the problem, we developed a two-phased matheuristic algorithm that utilizes the column generation approach. In the computational study, 40 problem instances with up to 40 multi-level shelves and 1020 candidate product groups were randomly generated based on the literature. The numerical results showed that the proposed algorithm was able to obtain high-quality solutions with less than 3.8% of relative optimality criterion at acceptable times for all problem instances.

We also demonstrate the usefulness of the proposed optimization approach by examining the real data related to a real chain of stores. The optimized assortment and allocation plan was analyzed. In this case study, there was a positive correlation between the highest possible profits of the product groups and their assigned visibilities.

Declarations of interest: none

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