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# The Two-Echelon Location-Routing Problem: A Comparative Analysis of Novel and Existing Compact Formulations

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**Abstract.** The two-echelon location-routing problem (2E-LRP) is a well-known problem in the literature that is commonly used to address applications in which deliveries occur at two levels. It concerns the location of facilities and the routing of vehicle fleets. Most studies addressing this problem and its variants rely on mixed-integer programming (MIP) formulations that are compact (i.e., have a polynomial number of variables and constraints). Although the formulations with two-index arc variables tend to perform better than those with vehicle index variables in vehicle routing problems, most of the literature on the 2E-LRP is based on the latter. In this paper, we present a comparative analysis of three compact formulations for the 2E-LRP: a literature-based formulation with vehicle index variables, and two novel formulations with two-index arc variables. Additionally, we propose enhancements for the literature-based formulation and polynomial valid inequalities for all of them. The linear programming relaxations of these formulations are compared, showing that those of the two-index formulations are stronger. Extensive computational experiments evaluate the formulations' performances on a general-purpose MIP solver. The results show that the formulations with vehicle index variables, despite being the standard approach in the literature, lead to poor solver performance, failing to find feasible solutions even for instances with only 50 customers. In fact, the best performance comes from the novel formulations, one of which leads to feasible solutions for all benchmark instances evaluated. Valid inequalities can be used to improve this performance even further. These experiments resulted in the discovery of 125 new best-known lower bounds and 55 new optimal solutions (out of 131 benchmark instances evaluated).

**Keywords**: location-routing; two-echelon location-routing problem; compact formulation; vehicle index variables; two-index arc variables

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# 1 Introduction

The continuous worsening of traffic conditions in urban centers has prompted many municipalities to impose restrictions on the traffic of large vehicles in the cities (Enthoven et al., 2020; Friedrich and Elbert, 2022). This, associated with a growing demand for urban deliveries, has led many companies to develop new logistics schemes. In city logistics, one particularly popular approach is the delivery in two echelons (Cuda et al., 2015; Senna et al., 2024). This way, larger vehicles transport goods from central depots (platforms) to smaller facilities (satellites) closer to the urban centers, in the first echelon (FE). From these satellites, smaller vehicles serve the customers in the second echelon (SE). Thus, the long distances are traveled by more cost-efficient vehicles while complying with constraints on urban traffic.

From an operational standpoint, it is important to determine the most efficient vehicle routes, which is the concern of the well-known two-echelon vehicle routing problem (2E-VRP), as reviewed by Sluijk et al. (2023). From a strategic and tactical perspective, one must consider both location and routing decisions, which leads to the so-called location-routing problems (Prodhon and Prins, 2014). In particular, the two-echelon location-routing problem (2E-LRP) studies the decisions regarding the opening of facilities (platforms and satellites) and the routes of FE and SE vehicles, with the objective of minimizing overall costs (Drexl and Schneider, 2015).

The idea of integrating location and routing decisions in a two-echelon scheme can be traced back to the works of Jacobsen and Madsen (1980) and Madsen (1983). However, it was only in 2011 that Boccia et al. formally defined and formulated the 2E-LRP (Boccia et al., 2011). The authors introduced three mixed-integer programming (MIP) formulations, two of which were compact (i.e., with polynomial numbers of variables and constraints) and one was extensive (i.e., with an exponential number of variables). The results of computational experiments indicated that the best compact formulation (CF) was based on arc variables with a vehicle index, clearly outperforming the one based on two-index arc variables. Results for the extensive formulation were not presented. Since then, most papers dealing with CFs for the 2E-LRP and its variants have relied on this vehicle index-based formulation.

The main difficulty in designing two-index arc variables CFs for the 2E-LRP is ensuring that the vehicles return to the facility they left from. In formulations with a vehicle index, this is simply made by flow conservation constraints, which ensure that the vehicle flow arriving at a node in one vehicle must leave this node in the same vehicle. Hence, the vehicles must make closed loops. In formulations without the vehicle index, this constraint does not work anymore in this sense because, although the vehicle flow should be maintained, it is possible that the vehicle arriving at a facility is not the same that left it. Thus, additional variables and constraints are required to guarantee that a vehicle starts and ends its route at the same facility.

In this paper, we propose two novel CFs with two-index arc variables for the 2E-LRP. The main difference between them is exactly the variables and constraints used to ensure that vehicles return to the facilities they departed from. Both of these formulations outperform the original formulation with vehicle index variables in general-purpose MIP solvers. Because the majority of papers addressing the 2E-LRP rely on CFs, this may be a significant development, as it will allow future researchers and practitioners to work with simple yet more powerful options. The contributions of this paper are fivefold:

- A vehicle index-based formulation adapted from Boccia et al. (2011) by revising some minor inaccuracies and two novel formulations based on two-index arc variables;

- New valid inequalities for all of the proposed formulations;
- A theoretical comparison of the linear programming relaxations (LRs) of the different formulations;
- Extensive computational experiments to assess which is the best CF for the 2E-LRP when relying on a general-purpose MIP solver;
- 125 best known lower bounds for benchmark instances and 55 new optimal solutions (out of 131 instances evaluated).

The remainder of this paper is organized as follows. Section 2 provides an overview of the literature on the 2E-LRP. In Section 3, we formally define the problem, present the different formulations, and introduce the valid inequalities. Section 4 provides a theoretical comparison of the LRs of the formulations. In Section 5, we discuss the results of the computational experiments. Finally, Section 6 presents concluding remarks.

## 2 Literature review

This section presents a review of the literature on the 2E-LRP and its variants, with a particular focus on the formulations used in these publications. We restrict our review to papers that present MIP formulations. For comprehensive reviews, we refer the reader to the works of Prodhon and Prins (2014), Cuda et al. (2015), and Drexl and Schneider (2015).

Boccia et al. (2011) were the first to formally define and formulate the 2E-LRP. They presented three different MIP formulations. The first is a CF that considers binary arc variables with a vehicle index (three-index formulation). The second one is also compact and avoids the vehicle index by using only two-index arc variables. The third one is an extensive formulation with an exponential number of variables representing all the feasible routes for the problem. Their computational experiments only provide results for the CFs and demonstrate empirically that the formulation with vehicle index variables is better than the alternative, which had a significant impact on subsequent literature.

Nguyen et al. (2012a) were the first to develop metaheuristics for the 2E-LRP, while also presenting the formulation with vehicle index variables introduced by Boccia et al. (2011). Contardo et al. (2012) and Nguyen et al. (2012b) also worked on the 2E-LRP as defined by Boccia et al. (2011) by proposing metaheuristics and extensive two-index arc variables-based formulations with an exponential number of constraints. Govindan et al. (2014) extended the problem to encompass time windows in a multi-objective approach to design a sustainable perishable food supply chain. They presented a CF based on the three-index formulation proposed by Boccia et al. (2011). Breunig et al. (2016) extended the problem by considering split deliveries in the FE and provided an extensive formulation with an exponential number of variables.

Rahmani et al. (2016) and Wang et al. (2018) adapted the 2E-LRP to two beverage distribution applications, also presenting CFs based on arc variables with a vehicle index. Pichka et al. (2018) extended the problem for an open routing situation and Zhao et al. (2017) looked at the particularities of heterogeneous fleets. Both papers propose CFs based on variables with a vehicle index. Darvish et al. (2019) incorporated the notion of flexibility into the 2E-LRP, modeling it with a CF and presenting valid inequalities and an exact method. Dai et al. (2019) addressed the 2E-LRP as well as two other extensions considering three and four echelons, modeling them with variables with a vehicle index.

The 2E-LRP has also been applied to model off-shore oil and gas supply chains (Amiri et al., 2019), postal services (Mirhedayatian et al., 2021), electric vehicles applications (Wang et al., 2021), disaster waste clean-up in humanitarian contexts (Cheng et al., 2022), cold supply chains (Wang et al., 2023), and other city logistics situations (Agnimo et al., 2023). Sutrisno and Yang (2023) looked at the problem with mobile satellites instead of fixed ones, and Escobar-Vargas and Crainic (2024) dealt with synchronization constraints. All of them used variables with a vehicle index.

Yıldız et al. (2023) discussed a variant of the 2E-LRP with pickup and delivery. They relied on a formulation with two-index arc variables based on assignment variables used for the 2E-VRP (Belgin et al., 2018). They adapted this formulation to the 2E-LRP, but without including the platforms' capacity constraints since the way it was modeled would create a non-linearity. In Section 3.2, we introduce a formulation that is based on what they proposed while including these capacity constraints by adopting a commodity flow-based formulation. We also present an improvement to this formulation.

Tian and Hu (2023) and Ben Mohamed et al. (2023) proposed branch-and-price algorithms, considering extensive formulations with an exponential number of variables. The first study considered a variant of the 2E-LRP with satellite recommendations whereas the second analyzed a multi-period stochastic variant.

Table 1 provides a summary of the information presented, analyzing the formulation characteristics of each work. Of the 23 works presented, 18 of them (78%) defined their problems with CFs. Of those, 16 (89%) had vehicle index variables in their formulations. Moreover, Amiri et al. (2019) worked with vehicle index variables despite having an extensive formulation. Only four of the works presented two-index arc variables. Of those, only two presented CFs and Boccia et al. (2011) showed that their two-index variables formulation performed worse than the vehicle index one, while Yıldız et al. (2023) ignored the platforms capacity constraints. It is important to note that, in the context of this discussion, we do not refer to echelon-related indices because, in many works, the FE and SE arc variables have different notations.

This outcome indicates the importance of CFs for the 2E-LRP because, even though most of the papers present tailored optimization methods (exact and heuristic) for their problems, they usually apply CFs to formally define the addressed variants and compare the performance of their methods with that of the CF. Hence, the better the formulation, the fairer the comparison. Moreover, given that the vast majority of formulations include vehicle index variables, it is important to assess whether this is the best approach. The present paper aims at solving this issue by presenting novel formulations without vehicle index variables and by comparing all of them theoretically and computationally.

# 3 Problem definition and mathematical formulations

The 2E-LRP is defined over a graph $G = (\mathcal{N}, \mathcal{A})$. The node set is $\mathcal{N} = \mathcal{P} \cup \mathcal{S} \cup \mathcal{C}$, with $\mathcal{P}$ being the set of potential platforms, $\mathcal{S}$ the set of potential satellites, and $\mathcal{C}$ the set of customers. Platforms and satellites are also called facilities. The set of echelons is $\mathcal{E} = \{1, 2\}$, with $e = 1$ representing the FE and $e = 2$ the SE. We define sets $\mathcal{N}^1 = \mathcal{P} \cup \mathcal{S}$ and $\mathcal{N}^2 = \mathcal{S} \cup \mathcal{C}$. To improve notation, we shall denote by $\mathcal{O}^e$ and $\mathcal{D}^e$ the sets of origins and destinations in echelon $e$, i.e., $\mathcal{O}^1 = \mathcal{P}$, $\mathcal{D}^1 = \mathcal{S}$, $\mathcal{O}^2 = \mathcal{S}$, and $\mathcal{D}^2 = \mathcal{C}$. The set of arcs is $\mathcal{A} = \mathcal{A}^1 \cup \mathcal{A}^2$, with $\mathcal{A}^1 = \{(i,j) | (i \in \mathcal{P}, j \in \mathcal{S}) \vee (i \in \mathcal{S}, j \in \mathcal{P}) \vee (i \in \mathcal{S}, j \in \mathcal{S}, i \neq j)\}$ and $\mathcal{A}^2 = \{(i,j) | (i \in \mathcal{S}, j \in \mathcal{C}) \vee (i \in \mathcal{C}, j \in \mathcal{S}) \vee (i \in \mathcal{C}, j \in \mathcal{C}, i \neq j)\}$.

| Reference | Compact formulation | Vehicle index variables | Two-index arc variables |
|---|---|---|---|
| Boccia et al. (2011) | ✓ | ✓ | ✓ |
| Contardo et al. (2012) | | | ✓ |
| Nguyen et al. (2012a) | ✓ | ✓ | |
| Nguyen et al. (2012b) | | | ✓ |
| Govindan et al. (2014) | ✓ | ✓ | |
| Breunig et al. (2016) | | | |
| Rahmani et al. (2016) | ✓ | ✓ | |
| Zhao et al. (2017) | ✓ | ✓ | |
| Pichka et al. (2018) | ✓ | ✓ | |
| Wang et al. (2018) | ✓ | ✓ | |
| Darvish et al. (2019) | ✓ | | |
| Dai et al. (2019) | ✓ | ✓ | |
| Amiri et al. (2019) | | ✓ | |
| Mirhedayatian et al. (2021) | ✓ | ✓ | |
| Wang et al. (2021) | ✓ | ✓ | |
| Cheng et al. (2022) | ✓ | ✓ | |
| Wang et al. (2023) | ✓ | ✓ | |
| Agnimo et al. (2023) | ✓ | ✓ | |
| Tian and Hu (2023) | ✓ | ✓ | |
| Ben Mohamed et al. (2023) | | | |
| Yıldız et al. (2023) | ✓ | | ✓ |
| Sutrisno and Yang (2023) | ✓ | ✓ | |
| Escobar-Vargas and Crainic (2024) | ✓ | ✓ | |

Table 1: A summary of the main formulations found for the 2E-LRP in the literature.

In each echelon, there is an unlimited and homogeneous fleet (FE and SE vehicles may be different). FE vehicles take goods from the platforms to the satellites, where they are transshipped and delivered to the customers by the SE vehicles. Every vehicle route starts and ends at the same facility. In the 2E-LRP, each facility $i \in \mathcal{N}^1$ has a fixed cost $H_i$ associated with opening it and a capacity $B_i$. FE and SE vehicles have capacities $Q^1$ and $Q^2$ and fixed costs $f^1$ and $f^2$, respectively. Each customer $i \in \mathcal{C}$ has a demand $q_i$. The cost of traveling in arc $(i,j) \in \mathcal{A}^e$ in echelon $e \in \mathcal{E}$ is $c^e_{ij}$. It is worth noting that superindex $e$ in $c^e_{ij}$ could be suppressed, since each arc only belongs to one echelon. However, we opted to keep it since it makes notation clearer both in this parameter and in some variables.

The goal of the 2E-LRP is to determine the optimal subset of facilities to open, along with the least-cost FE and SE routes that can serve all customers. Figure 1 illustrates the problem by presenting a feasible solution to an instance with three potential platforms, three potential satellites, and four customers. This solution uses a single platform and two satellites (the shaded ones are potential facilities that are not selected in this solution). In the FE, the vehicle serves the two satellites from a single platform, whereas in the SE, two vehicles serve the customers.

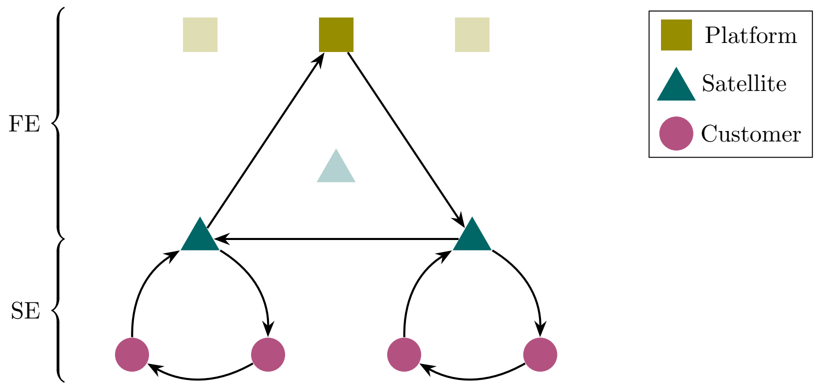



Figure 1: An illustrative example of the 2E-LRP.

We present three CFs for this problem. Section 3.1 presents a formulation with vehicle index variables proposed for the 2E-LRP (CF1) based on the one introduced by Boccia et al. (2011), and discusses an improvement of CF1 by considering commodity flow constraints (ICF1). We do not present the other formulations proposed by Boccia et al. (2011) since their experiments proved that these formulations performed worse. Section 3.2 introduces a formulation with two-index arc variables based on binary assignment variables (CF2), adapted from what is proposed by Yıldız et al. (2023), and a possible enhancement (ICF2). In Section 3.3, another formulation with two-index arc variables is presented, without binary assignment variables (CF3). Valid inequalities (VIs) are discussed for all formulations. As discussed in Section 1, the main difference between the two formulations with two-index arc variables (CF2 and CF3) is the variables and constraints that are used to ensure that each vehicle returns to the facility it left from. In what follows, the binary variable $y_i$ is common to all proposed formulations and indicates whether a facility $i \in \mathcal{N}^1$ is opened.

## 3.1 Formulation with vehicle index variables (CF1)

In this section, we present the formulation with three-index variables for the 2E-LRP as introduced by Boccia et al. (2011), but we fix minor errors of their presentation. This formulation requires additional sets $\mathcal{K}^e$ of vehicles in echelon $e \in \mathcal{E}$.

The binary variable $w^2_{si}$ indicates whether customer $i \in \mathcal{C}$ is assigned to satellite $s \in \mathcal{S}$. Also, the binary variable $x^e_{ijk}$ indicates whether a vehicle $k \in \mathcal{K}^e$ travels through arc $(i,j) \in \mathcal{A}^e$ in echelon $e \in \mathcal{E}$. Another binary variable $z_k$ is required to indicate whether vehicle $k \in \mathcal{K}^1 \cup \mathcal{K}^2$ is used. Load flow from platform $p \in \mathcal{P}$ to satellite $s \in \mathcal{S}$ in vehicle $k \in \mathcal{K}^1$ is controlled by the continuous and non-negative variable $g_{psk}$. Finally, $u^e_i$ is an auxiliary variable for subtour elimination that indicates the position of node $i \in \mathcal{N}^e$ in a route in echelon $e \in \mathcal{E}$.

The formulation introduced by Boccia et al. (2011) for the 2E-LRP is:

$$\text{(CF1) min } \sum_{i \in \mathcal{N}^1} H_i y_i + \sum_{e \in \mathcal{E}} \sum_{k \in \mathcal{K}^e} f^e z_k + \sum_{e \in \mathcal{E}} \sum_{k \in \mathcal{K}^e} \sum_{(i,j) \in \mathcal{A}^e} c^e_{ij} x^e_{ijk} \tag{1}$$

$$\text{s.t. } \sum_{i:(i,j) \in \mathcal{A}^e} x^e_{ijk} = \sum_{i:(j,i) \in \mathcal{A}^e} x^e_{jik}, \ \forall\ j \in \mathcal{N}^e, k \in \mathcal{K}^e, e \in \mathcal{E} \tag{2}$$

$$u^e_j \geq u^e_i + 1 - |\mathcal{D}^e| \left(1 - \sum_{k \in \mathcal{K}^e} x^e_{ijk}\right), \ \forall\ (i,j) \in \mathcal{A}^e, e \in \mathcal{E} \tag{3}$$

$$\sum_{j \in \mathcal{O}^e} \sum_{i:(i,j) \in \mathcal{A}^e} x^e_{ijk} \leq 1, \ \forall\ k \in \mathcal{K}^e, e \in \mathcal{E} \tag{4}$$

$$\sum_{k \in \mathcal{K}^1} \sum_{j:(s,j) \in \mathcal{A}^1} x^1_{sjk} = y_s, \ \forall\ s \in \mathcal{S} \tag{5}$$

$$\sum_{k \in \mathcal{K}^2} \sum_{j:(i,j) \in \mathcal{A}^2} x^2_{ijk} = 1, \ \forall\ i \in \mathcal{C} \tag{6}$$

$$\sum_{s \in S} w^2_{si} = 1, \ \forall\ i \in \mathcal{C} \tag{7}$$

$$\sum_{j:(i,j) \in \mathcal{A}^2} x^2_{ijk} + \sum_{j:(s,j) \in \mathcal{A}^2} x^2_{sjk} - w^2_{si} \leq 1, \ \forall\ i \in \mathcal{C}, s \in \mathcal{S}, k \in \mathcal{K}^2 \tag{8}$$

$$\sum_{k \in \mathcal{K}^1} \sum_{p \in \mathcal{P}} g_{psk} = \sum_{i \in \mathcal{C}} q_i w^2_{si}, \ \forall\ s \in \mathcal{S} \tag{9}$$

$$\sum_{k \in \mathcal{K}^1} \sum_{s \in \mathcal{S}} g_{psk} \leq B_p y_p, \ \forall\ p \in \mathcal{P} \tag{10}$$

$$\sum_{k \in \mathcal{K}^1} \sum_{p \in \mathcal{P}} g_{psk} \leq B_s y_s, \ \forall\ s \in \mathcal{S} \tag{11}$$

$$Q^1 \sum_{j:(s,j) \in \mathcal{A}^1} x^1_{sjk} \geq g_{psk}, \ \forall\ p \in \mathcal{P}, s \in \mathcal{S}, k \in \mathcal{K}^1 \tag{12}$$

$$Q^1 \sum_{j:(p,j) \in \mathcal{A}^1} x^1_{pjk} \geq g_{psk}, \ \forall\ p \in \mathcal{P}, s \in \mathcal{S}, k \in \mathcal{K}^1 \tag{13}$$

$$\sum_{p \in \mathcal{P}} \sum_{s \in \mathcal{S}} g_{psk} \leq Q^1 z_k, \ \forall\ k \in \mathcal{K}^1 \tag{14}$$

$$\sum_{i \in \mathcal{C}} \sum_{j:(i,j) \in \mathcal{A}^2} q_i x^2_{ijk} \leq Q^2 z_k, \ \forall\ k \in \mathcal{K}^2 \tag{15}$$

$$x^e_{ijk} \in \{0,1\}, \ \forall\ (i,j) \in \mathcal{A}^e, k \in \mathcal{K}^e, e \in \mathcal{E} \tag{16}$$

$$y_i \in \{0,1\}, \ \forall\ i \in \mathcal{N}^1 \tag{17}$$

$$w^2_{si} \in \{0,1\}, \ \forall\ s \in \mathcal{S}, i \in \mathcal{C} \tag{18}$$

$$z_k \in \{0,1\}, \ \forall\ k \in \mathcal{K}^1 \cup \mathcal{K}^2 \tag{19}$$

$$g_{psk} \geq 0, \ \forall\ p \in \mathcal{P}, s \in \mathcal{S}, k \in \mathcal{K}^1 \tag{20}$$

$$u^e_i \in [1, |\mathcal{D}^e|], \ \forall\ i \in \mathcal{N}^e, e \in \mathcal{E}. \tag{21}$$

The objective function (1) aims to minimize facilities and vehicles fixed costs as well as distance-related costs. Constraints (2) are vehicle flow conservation constraints for both echelons. Constraints (3) are Miller-Tucker-Zemlin (MTZ) subtour elimination constraints for both echelons (Miller et al., 1960). Constraints (4) ensure that each vehicle performs a single route. Constraints (5) define that a satellite is opened if and only if an FE vehicle leaves it. Constraints (6) state that every customer is visited exactly once. Constraints (7) define that each customer is assigned to exactly one satellite. Constraints (8) ensure that if a customer is assigned to a satellite, the vehicle that serves it leaves the corresponding satellite. Constraints (9) define that the amount of load transferred from a platform to a satellite is equal to the demand of the customers assigned to this satellite. Constraints (10) and (11) ensure that the capacities of the platforms and satellites are respected. Constraints (12) and (13) define that there is a load flow from a platform to a satellite only if they are both served by the same vehicle. Constraints (14) and (15) make sure that the vehicles' capacities are respected. Constraints (16)–(21) define the variables' domains. This formulation has $O((|\mathcal{P}| + |\mathcal{S}|)^2|\mathcal{K}^1| + (|\mathcal{S}| + |\mathcal{C}|)^2|\mathcal{K}^2|)$ variables and $O((|\mathcal{P}| + |\mathcal{S}|)^2 + (|\mathcal{S}| + |\mathcal{C}|)^2 + (|\mathcal{P}| + |\mathcal{S}|)|\mathcal{K}^1| + (|\mathcal{S}| + |\mathcal{C}|)|\mathcal{K}^2| + |\mathcal{P}||\mathcal{S}||\mathcal{K}^1| + |\mathcal{S}||\mathcal{C}||\mathcal{K}^2|)$ constraints.

It is worth noting that the original formulation has two minor issues that are corrected in CF1. First, Boccia et al. (2011) do not consider the echelon related index of variable $u_i^e$. Hence, for the satellites, these variables become poorly defined, since they appear in the constraints of both echelons. Additionally, in their paper, constraints (13) use $Q^2$ instead of $Q^1$, which is incorrect since they are related to the FE.

### 3.1.1 Improved formulation

A possible improvement to this formulation is the substitution of constraints (3), (7)–(15), and (18)–(21) by a commodity flow based formulation (Gavish and Graves, 1978). To this extent, we define continuous variables $g_{ij}^e$ that represent the flow of commodities in arc $(i,j) \in \mathcal{A}^e, e \in \mathcal{E}$. The new formulation (ICF1) becomes:

$$\text{(ICF1)}\ \min \sum_{i \in \mathcal{N}^1} H_i y_i + \sum_{e \in \mathcal{E}} \sum_{k \in \mathcal{K}^e} \sum_{i \in \mathcal{O}^e} \sum_{j \in \mathcal{D}^e} f^e x_{ijk}^e + \sum_{e \in \mathcal{E}} \sum_{k \in \mathcal{K}^e} \sum_{(i,j) \in \mathcal{A}^e} c_{ij}^e x_{ijk}^e \tag{22}$$

s.t. (2), (4)–(6), (16)–(17)

$$\sum_{i:(i,s) \in \mathcal{A}^1} g_{is}^1 - \sum_{i:(s,i) \in \mathcal{A}^1} g_{si}^1 = -\sum_{j \in \mathcal{C}} g_{js}^2,\ \forall\ s \in \mathcal{S} \tag{23}$$

$$\sum_{i:(i,j) \in \mathcal{A}^2} g_{ij}^2 - \sum_{i:(j,i) \in \mathcal{A}^2} g_{ji}^2 = -q_j,\ \forall\ j \in \mathcal{C} \tag{24}$$

$$\sum_{i \in \mathcal{D}^e} g_{ij}^e \leq B_j y_j,\ \forall\ j \in \mathcal{O}^e, e \in \mathcal{E} \tag{25}$$

$$0 \leq g_{ij}^e \leq Q^e \sum_{k \in \mathcal{K}^e} x_{ijk}^e,\ \forall\ (i,j) \in \mathcal{A}^e, e \in \mathcal{E}. \tag{26}$$

The objective function (22) is equivalent to (1) but uses a different form for calculating vehicle fixed costs. Constraints (23) define that the difference between the load arriving and leaving a satellite is the load transshipped through this satellite. Constraints (24) do the same for the customers. Constraints (25) ensure that the facilities capacities are respected. Constraints (26) define the domain of the new decision variables. The number of variables in ICF1 is of the same order as in CF1, but the number of constraints is significantly reduced to $O((|\mathcal{P}| + |\mathcal{S}|)|\mathcal{K}^1| + (|\mathcal{S}| + |\mathcal{C}|)|\mathcal{K}^2|)$.

### 3.1.2 Valid inequalities

It is well-known that vehicle index formulations for routing problems exhibit solution symmetries, which may negatively impact the performance of branch-and-bound-based methods (Furtado et al., 2017; Munari and Savelsbergh, 2022). To mitigate this issue, one could add the following valid inequalities (VIs):

$$\sum_{i \in \mathcal{O}^e} \sum_{j \in \mathcal{D}^e} x^e_{ijk} \geq \sum_{i \in \mathcal{O}^e} \sum_{j \in \mathcal{D}^e} x^e_{ij(k+1)}, \ \forall\ k \in \mathcal{K}^e \setminus \{|\mathcal{K}^e|\}, e \in \mathcal{E} \tag{27}$$

$$\sum_{(i,j) \in \mathcal{A}^2 : i < h} x^2_{ij(k-1)} \geq \sum_{j:(h,j) \in \mathcal{A}^2} x^2_{hjk}, \ \forall\ h \in \mathcal{C} \setminus \{1\}, k \in \mathcal{K}^2 \setminus \{1\}. \tag{28}$$

Constraints (27) state that, if a vehicle is used, a vehicle with a smaller index is also used. Constraints (28) ensure that, if a vehicle serves a customer, a vehicle with a smaller index serves another customer with a smaller index.

In addition to these, the following VIs could be used to tighten the LR of the formulations:

$$\sum_{i \in \mathcal{O}^e} y_i \geq o^e_{min}, \ \forall\ e \in \mathcal{E} \tag{29}$$

$$\sum_{i \in \mathcal{O}^e} \sum_{j \in \mathcal{D}^e} \sum_{k \in \mathcal{K}^e} x^e_{ijk} \geq \left\lceil \frac{1}{Q^e} \sum_{i \in \mathcal{C}} q_i \right\rceil, \ \forall\ e \in \mathcal{E} \tag{30}$$

$$\sum_{k \in \mathcal{K}^e} \sum_{j:(i,j) \in \mathcal{A}^e} x^e_{ijk} \geq y_i, \ \forall\ i \in \mathcal{O}^e, e \in \mathcal{E} \tag{31}$$

$$2 \sum_{k \in \mathcal{K}^1} x^1_{psk} \leq y_p + y_s, \ \forall\ p \in \mathcal{P}, s \in \mathcal{S} \tag{32}$$

$$2 \sum_{k \in \mathcal{K}^1} x^1_{spk} \leq y_p + y_s, \ \forall\ p \in \mathcal{P}, s \in \mathcal{S} \tag{33}$$

$$\sum_{k \in \mathcal{K}^2} x^2_{sjk} \leq y_s, \ \forall\ s \in \mathcal{S}, j \in \mathcal{C} \tag{34}$$

$$\sum_{k \in \mathcal{K}^2} x^2_{jsk} \leq y_s, \ \forall\ s \in \mathcal{S}, j \in \mathcal{C} \tag{35}$$

$$\sum_{i \in \mathcal{C}} w^2_{si} \geq y_s, \ \forall\ s \in \mathcal{S} \tag{36}$$

$$w^2_{si} \leq y_s, \ \forall\ s \in \mathcal{S}, i \in \mathcal{C}. \tag{37}$$

Constraints (29) define lower bounds on the number of platforms and satellites (Yıldız et al., 2023). In these VIs, $o^e_{min}$ are lower bounds on the number of facilities opened and can be defined by ordering the corresponding facilities in decreasing order of capacity and taking the smallest number of them that can serve all the customers' demands. Constraints (30) are lower bounds on the number of vehicles needed in each echelon. Constraints (31) define that if a facility is opened, at least one vehicle leaves it. Constraints (32) and (33) forbid vehicles from traveling between a platform and a satellite if one of them is not opened. Constraints (34) and (35) state that a vehicle can only leave from or return to a satellite if it is opened. Constraints (36) state that, if a satellite is opened, at least one customer is assigned to it. Constraints (37) forbid customers to be assigned to satellites that are not opened. It is worth noticing that VIs (36) and (37) cannot be used with ICF1 because variables $w^2$ are not defined in this formulation.

## 3.2 Formulation with two-index arc variables and binary assignments (CF2)

We introduce a novel formulation for the 2E-LRP with two-index routing variables. This formulation is based on a 2E-VRP formulation (Belgin et al., 2018) that has been adapted to the 2E-LRP by Yıldız et al. (2023). However, when adapting it to the 2E-LRP, they did not include the platforms' capacity constraints since it would create nonlinearities. We have adapted it by considering commodity flow variables in both echelons to ensure that these capacities are respected.

In this formulation, the binary variable $x^e_{ij}$ indicates whether a vehicle traverses arc $(i, j) \in \mathcal{A}^e$ in echelon $e \in \mathcal{E}$. As mentioned in Section 1, when avoiding the variables with vehicle index, additional variables and constraints must be used to ensure that the vehicles end their routes in the facilities they started from. In this formulation, this is made by the variable $w^2_{si}$ already employed in CF1 and the binary variable $w^1_{ps}$ that indicates whether satellite $s \in \mathcal{S}$ is assigned to platform $p \in \mathcal{P}$.

The first formulation with two-index arc variables is defined as:

$$\text{(CF2) min} \sum_{i \in \mathcal{N}^1} H_i y_i + \sum_{e \in \mathcal{E}} \sum_{i \in \mathcal{O}^e} \sum_{j \in \mathcal{D}^e} f^e x^e_{ij} + \sum_{e \in \mathcal{E}} \sum_{(i,j) \in \mathcal{A}^e} c^e_{ij} x^e_{ij} \tag{38}$$

$$\text{s.t. (7), (17), (23)–(25)}$$

$$\sum_{i:(i,j) \in \mathcal{A}^e} x^e_{ij} = \sum_{i:(j,i) \in \mathcal{A}^e} x^e_{ji}, \ \forall \ j \in \mathcal{N}^e, e \in \mathcal{E} \tag{39}$$

$$\sum_{j:(s,j) \in \mathcal{A}^1} x^1_{sj} = y_s, \ \forall \ s \in \mathcal{S} \tag{40}$$

$$\sum_{j:(i,j) \in \mathcal{A}^2} x^2_{ij} = 1, \ \forall \ i \in C \tag{41}$$

$$\sum_{p \in P} w^1_{ps} = y_s, \ \forall \ s \in \mathcal{S} \tag{42}$$

$$x^e_{ij} \leq w^e_{ij}, \ \forall \ i \in \mathcal{O}^e, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{43}$$

$$x^e_{ji} \leq w^e_{ij}, \ \forall \ i \in \mathcal{O}^e, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{44}$$

$$x^e_{ij} + w^e_{hi} + \sum_{h' \in \mathcal{O}^e \setminus \{h\}} w^e_{h'j} \leq 2, \ \forall \ i, j \in \mathcal{D}^e, i \neq j, h \in \mathcal{O}^e \tag{45}$$

$$x^e_{ij} \in \{0, 1\}, \ \forall \ (i, j) \in \mathcal{A}^e, e \in \mathcal{E} \tag{46}$$

$$w^e_{ij} \in \{0, 1\}, \forall \ i \in \mathcal{O}^e, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{47}$$

$$0 \leq g^e_{ij} \leq Q^e x^e_{ij}, \ \forall \ (i, j) \in \mathcal{A}^e, e \in \mathcal{E}. \tag{48}$$

The objective function (38) and constraints (39)–(41) are the two-index variables equivalent of (22), (2), (5), and (6), respectively. Constraints (42) define that if a satellite is opened, it is assigned to a platform. Constraints (43) and (44) state that if a vehicle travels between an origin and a destination, this destination is assigned to this origin. Constraints (45) ensure that a vehicle can only travel between two destinations assigned to the same origin. Constrains (46)–(48) define the domain of variables. This formulation has $O((|\mathcal{P}|+|\mathcal{S}|)^2+(|\mathcal{S}|+|\mathcal{C}|)^2)$ variables and $O(|\mathcal{P}||\mathcal{S}|^2+|\mathcal{S}||\mathcal{C}|^2+(|\mathcal{P}|+|\mathcal{S}|)^2+(|\mathcal{S}|+|\mathcal{C}|)^2)$ constraints.

### 3.2.1 Improved formulation

The first possible improvement to CF2 concerns constraints (45). They were presented this way since it is the common approach in the literature (Belgin et al., 2018; Yıldız et al., 2023). However, they can be improved to become sparser and provide a tighter LR. For the SE, from constraints (7), we have that $\sum_{h' \in \mathcal{O}^2 \setminus \{h\}} w^2_{h'j} = 1 - w^2_{hj}$ and this can be substituted in constraints (45) to make them sparser. For the FE, we would have $\sum_{h' \in \mathcal{O}^1 \setminus \{h\}} w^1_{h'j} = y_h - w^1_{hj}$ from constraints (42), but it is possible to use $1 - w^1_{hj}$ because constraints (45) are redundant for $y_h = 0$. Moreover, given that constraints (45) define that a vehicle may travel between two destinations only if they are both assigned to the same origin, we can add $x^e_{ji}$ to their left-hand side, tightening the LR. This way, we obtain the following formulation ICF2:

$$\text{(ICF2)} \ \min \ (38)$$
$$\text{s.t.} \ (7), (17), (23)\text{–}(25), (39)\text{–}(44), (46)\text{–}(48)$$
$$x^e_{ij} + x^e_{ji} + w^e_{hi} - w^e_{hj} \leq 1, \ \forall \, i, j \in \mathcal{D}^e, i \neq j, h \in \mathcal{O}^e. \quad (49)$$

### 3.2.2 Valid inequalities

Both CF2 and ICF2 can be enhanced by the following VIs:

$$(29), (36)\text{–}(37)$$
$$\sum_{i \in \mathcal{O}^e} \sum_{j \in \mathcal{D}^e} x^e_{ij} \geq \left\lceil \frac{1}{Q^e} \sum_{c \in \mathcal{C}} q_c \right\rceil, \ \forall \, e \in \mathcal{E} \quad (50)$$
$$\sum_{j \in \mathcal{D}^e} x^e_{ij} \geq y_i, \ \forall \, i \in \mathcal{O}^e, e \in \mathcal{E} \quad (51)$$
$$2x^1_{ps} \leq y_p + y_s, \ \forall \, p \in \mathcal{P}, s \in \mathcal{S} \quad (52)$$
$$2x^1_{sp} \leq y_p + y_s, \ \forall \, p \in \mathcal{P}, s \in \mathcal{S} \quad (53)$$
$$x^2_{sj} \leq y_s, \ \forall \, s \in \mathcal{S}, j \in \mathcal{C} \quad (54)$$
$$x^2_{js} \leq y_s, \ \forall \, s \in \mathcal{S}, j \in \mathcal{C} \quad (55)$$
$$\sum_{s \in \mathcal{S}} w^1_{ps} \geq y_p, \ \forall \, p \in \mathcal{P} \quad (56)$$
$$2w^1_{ps} \leq y_p + y_s, \ \forall \, p \in \mathcal{P}, s \in \mathcal{S}. \quad (57)$$

Constraints (50)–(55) are the two-index variables equivalent to (30)–(35). Constraints (56) define that if a platform is opened at least one satellite is assigned to it. Constraints (57) define that a satellite can only be assigned to a platform if both the satellite and the platform are opened. Constraints (50) and (52)–(56) can be found in Yıldız et al. (2023).

## 3.3 Formulation with two-index arc variables and continuous assignments (CF3)

In this section, we present a novel two-index formulation that does not require the assignment variables $w$ from CF2. Instead of binary assignments, this formulation is based on continuous variables $v^e_j$ that indicate from which origin the vehicle visiting destination $j \in \mathcal{D}^e, e \in \mathcal{E}$ departed. It is inspired by the index propagation formulation of Furtado et al. (2017) for the pickup and delivery routing problem. Formulation CF3 is defined as:

$$\text{(CF3)} \ \min \ (38)$$
$$\text{s.t.} \ (17), (23)\text{–}(25), (39)\text{–}(41), (46), (48)$$

$$v_j^e \geq \sum_{i \in \mathcal{O}^e} i x_{ij}^e, \ \forall\, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{58}$$

$$v_j^e \geq \sum_{i \in \mathcal{O}^e} i x_{ji}^e, \ \forall\, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{59}$$

$$v_j^e \leq M_1^e - \sum_{i \in \mathcal{O}^e} (M_1^e - i) x_{ij}^e, \ \forall\, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{60}$$

$$v_j^e \leq M_1^e - \sum_{i \in \mathcal{O}^e} (M_1^e - i) x_{ji}^e, \ \forall\, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{61}$$

$$v_j^e \geq v_i^e - M_2^e(1 - x_{ij}^e - x_{ji}^e), \ \forall\, i, j \in \mathcal{D}^e, i \neq j, e \in \mathcal{E} \tag{62}$$

$$1 \leq v_j^e \leq |\mathcal{O}^e|, \ \forall\, j \in \mathcal{D}^e, e \in \mathcal{E}. \tag{63}$$

Constraints (58)–(61) impose that if a vehicle travels between an origin $i$ and a destination $j$, then $v_j^e$ assumes the value of $i$, this way indicating the origin related to node $j$. Constraints (62) ensure that if a vehicle travels between two destinations $i$ and $j$, then these nodes are in the same route and, therefore, have the same origin (i.e., $v_i^e = v_j^e$). Constraints (63) define the domain of the new variables. $M_1^e$ and $M_2^e$ are sufficiently large numbers. Their tightest possible values are $M_1^e = |\mathcal{O}^e|$ and $M_2^e = |\mathcal{O}^e| - 1$. This formulation has $O((|\mathcal{P}| + |\mathcal{S}|)^2 + (|\mathcal{S}| + |\mathcal{C}|)^2)$ variables and $O((|\mathcal{P}| + |\mathcal{S}|)^2 + (|\mathcal{S}| + |\mathcal{C}|)^2)$ constraints.

Formulation CF3 can be enhanced by the following VIs:

(29), (50)–(55)

$$v_j^e \geq i \left( y_i - \sum_{i' \in \mathcal{O}^e : i' < i} y_{i'} \right), \ \forall\, i \in \mathcal{O}^e \setminus \{1\}, j \in \mathcal{D}^e, e \in \mathcal{E} \tag{64}$$

$$v_j^e \leq i + \sum_{i' \in \mathcal{O}^e : i' > i} i' y_{i'} + (|\mathcal{O}^e| - i)(1 - y_i), \ \forall\, i \in \mathcal{O}^e \setminus \{|\mathcal{O}^e|\}, j \in \mathcal{D}^e, e \in \mathcal{E}. \tag{65}$$

Constraints (64) define that destinations are assigned to an origin with an index at least equal to the smallest index of opened origins. Analogously, constraints (65) ensure that destinations are assigned to an origin with an index at most equal to the greatest index of opened origins.

# 4 Comparison of LRs

In this section, we discuss some relationships between the LRs of the different proposed formulations. Propositions 1 to 6 and Corollaries 1 to 4 enunciate and prove them.

**Proposition 1.** *The LR of ICF1 is not weaker than that of CF1.*

*Proof.* The optimal value of the LR of ICF1 for instance "100–10MN" from set *Nguyen*[1] is 156,294, higher than that of CF1, which is 111,867. □

**Proposition 2.** *Formulation ICF2 has a stronger LR than formulation CF2.*

*Proof.* The optimal value of the LR of ICF2 for instance "100–10MN" from set *Nguyen* is 160,148, which is higher than that of the LR of CF2 for the same instance (156,294). Hence, CF2 does not have a stronger LR than ICF2.

[1]The benchmark instances sets are properly presented in Section 5.

The fact that the LR of ICF2 is stronger than that of CF2 comes directly from the fact that, if constraints (49) are satisfied, constraints (45) are also satisfied. Indeed, from constraints (7) and (42), $\sum_{h' \in \mathcal{O}^e \setminus \{h\}} w^e_{h'j} - 1 = -w^e_{hj}$. Substituting this in (49) makes

$$1 \geq x^e_{ij} + x^e_{ji} + w^e_{hi} - w^e_{hj} \geq x^e_{ij} + w^e_{hi} + \sum_{h' \in \mathcal{O}^e \setminus \{h\}} w^e_{h'j} - 1, \ \forall\ i,j \in \mathcal{D}^e, i \neq j, h \in \mathcal{O}^e,$$

corresponding precisely to constraints (45). □

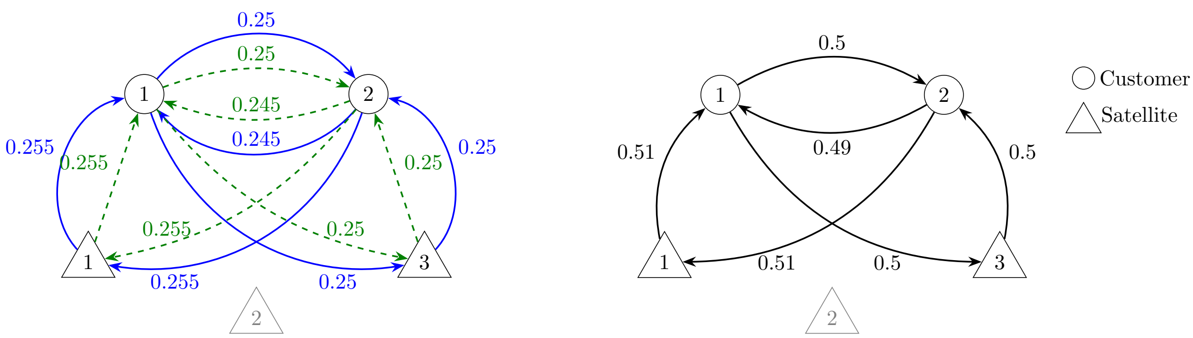
**Proposition 3.** *Formulation CF2 has a stronger LR than formulation ICF1.*

*Proof.* Given a solution $\overline{x}^e_{ij}, (i,j) \in \mathcal{A}^e, e \in \mathcal{E}$, for the LR of CF2, it is possible to define a solution for the LR of ICF1 by making $\widetilde{x}^e_{ijk} = \frac{1}{|\mathcal{K}^e|}\overline{x}^e_{ij}, (i,j) \in \mathcal{A}^e, k \in \mathcal{K}^e, e \in \mathcal{E}$. This way, we have that constraints (39) $\Rightarrow$ (2), (40) $\Rightarrow$ (5), (41) $\Rightarrow$ (6), (46) $\Rightarrow$ (16), and (48) $\Rightarrow$ (26). Moreover, (40) and (41) $\Rightarrow$ (4). In fact, from (41),

$$\sum_{j \in \mathcal{C}\setminus\{i\}} \overline{x}^2_{ij} + \sum_{j \in \mathcal{S}} \overline{x}^2_{ij} = 1, \ \forall\ i \in \mathcal{C} \Rightarrow \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{S}} \overline{x}^2_{ij} = |\mathcal{C}| - \sum_{i,j \in \mathcal{C}: i \neq j} \overline{x}^2_{ij} \Rightarrow \sum_{i \in \mathcal{C}} \sum_{j \in \mathcal{S}} \widetilde{x}^2_{ijk} \leq \frac{|\mathcal{C}|}{|\mathcal{K}^2|} \leq 1, \ \forall\ k \in \mathcal{K}^2$$

since the fleet is unlimited, corresponding precisely to constraints (4) for the SE. For the FE, the derivation from (40) is analogous. Hence, it is proved that, if $\overline{x}$ is a solution to the LR of CF2, then $\widetilde{x}$ is a solution to the LR of ICF1.

Figure 2 represents an example of a solution of the LR of ICF1 that is not a solution of the LR of CF2. It presents an instance with three potential satellites and two customers (the FE is not presented since it is not needed in this demonstration). The customers in this instance have low demands, while vehicles and facilities have large enough capacities to ensure that load and capacity constraints are non-binding. Figure 2a portrays a solution of the LR of ICF1. The solid blue arrows represent the positive arc variables associated with one vehicle and the dashed green ones represent another vehicle. It is easy to see that these values respect all constraints of ICF1.




(a) Solution of the LR of ICF1. (b) Projection onto the two-index arc variables space.

Figure 2: An example for proving that ICF1 does not have a stronger LR than CF2.

The only way to project this solution onto the two-index arc variables space of CF2 while respecting constraints (41) is by defining $x^2_{ij} = \sum_{k \in \mathcal{K}^2} \overline{x}^2_{ijk}, \ \forall\ (i,j) \in \mathcal{A}^2$, yielding the solution shown in Figure 2b. However, constraints (43) and (44) would impose $w^2_{1,1} \geq 0.51$ and $w^2_{3,1} \geq 0.5$. This implies $\sum_{s \in \mathcal{S}} w^2_{s1} \geq 1.01$, violating constraints (7). Hence, this solution of the LR of ICF1 does not have a correspondent solution for the LR of CF2.

□

**Corollary 1.** *Formulation ICF2 has a stronger LR than formulation ICF1.*

**Corollary 2.** *The LR of CF2 is not weaker than the LR of CF1.*

**Corollary 3.** *The LR of ICF2 is not weaker than the LR of CF1.*

**Proposition 4.** *Formulation CF3 has a stronger LR than formulation ICF1.*

*Proof.* The proof that every feasible solution for the LR of CF3 has a corresponding feasible solution in the LR of ICF1 is the same as in the proof of Proposition 3 for CF2 and ICF1. Also, the optimal value of the LR of ICF1 for instance “100–10MN” from set *Nguyen* is 156,294, while for the LR of CF3 it is 160,146. □

**Corollary 4.** *The LR of CF3 is not weaker than the LR of CF1.*

**Proposition 5.** *The LRs of formulations CF2 and CF3 are not comparable.*

*Proof.* The optimal value of the LR of CF3 for instance “100–10MN” from set *Nguyen* is 160,146 while for the LR of CF2 it is 156,294. Hence, the LR of CF2 is not stronger than that of CF3.

Moreover, Figure 2b presents an example of a solution of the LR of CF3 that is not a solution for the LR of CF2. Once again, assume that the customers have low demands and the vehicles and facilities have high enough capacities. The arrows represent the value of the corresponding $x^2$ variables. It is easy to see that $v_1^2 = v_2^2 = 1.5$ is a solution to CF3. However, for CF2, constraints (43) and (44) would impose $w_{1,1}^2 \geq 0.51$ and $w_{3,1}^2 \geq 0.5$. This implies $\sum_{s\in\mathcal{S}} w_{s1}^2 \geq 1.01$, violating constraints (7). Hence, this solution of the LR of CF3 is not feasible for the LR of CF2. □

**Proposition 6.** *Formulation ICF2 has a stronger LR than formulation CF3 if $M_1^e = M_2^e = \frac{|\mathcal{O}^e|(|\mathcal{O}^e|+1)}{2}$.*

*Proof.* First, we prove that a solution in the LR of ICF2 has a corresponding solution in the LR of CF3 by defining $v_j^e = \sum_{i\in\mathcal{O}^e} i w_{ij}^e$, which automatically respects constraints (63). By multiplying constraints (43) and (44) by $i$ and summing over $\mathcal{O}^e$, we get

$$\sum_{i\in\mathcal{O}^e} i w_{ij}^e \geq \sum_{i\in\mathcal{O}^e} i x_{ij}^e, \ \forall\ j \in \mathcal{D}^e, e \in \mathcal{E}, \text{ and}$$
$$\sum_{i\in\mathcal{O}^e} i w_{ij}^e \geq \sum_{i\in\mathcal{O}^e} i x_{ji}^e, \ \forall\ j \in \mathcal{D}^e, e \in \mathcal{E},$$

which correspond to constraints (58) and (59), respectively.

From constraints (41), in the SE, we have

$$\sum_{s'\in\mathcal{S}} x_{is'}^2 + \sum_{j\in\mathcal{C}\setminus\{i\}} x_{ij}^2 = 1, \ \forall\ i \in \mathcal{C} \Rightarrow x_{is}^2 + \sum_{j\in\mathcal{C}\setminus\{i\}} x_{ij}^2 = 1 - \sum_{s'\in\mathcal{S}\setminus\{s\}} x_{is'}^2, \ \forall\ s \in \mathcal{S}, i \in \mathcal{C}.$$

From constraints (7) and (43),

$$x_{is}^2 + \sum_{j\in\mathcal{C}\setminus\{i\}} x_{ij}^2 \geq 1 - \sum_{s'\in\mathcal{S}\setminus\{s\}} w_{s'i}^2 = w_{si}^2, \ \forall\ s \in \mathcal{S}, i \in \mathcal{C}.$$

By multiplying these inequalities by $s$ and summing over $\mathcal{S}$, we get

$$\sum_{s\in\mathcal{S}} sw_{si}^2 \leq \sum_{s\in\mathcal{S}} sx_{is}^2 + \sum_{s\in\mathcal{S}} s\left(\sum_{j\in\mathcal{C}\setminus\{i\}} x_{ij}^2\right),\ \forall\, i\in\mathcal{C}$$
$$\Rightarrow v_i^2 \leq \sum_{s\in\mathcal{S}} sx_{is}^2 + \frac{|\mathcal{S}|(|\mathcal{S}|+1)}{2}\left(1-\sum_{s\in\mathcal{S}} x_{is}^2\right),\ \forall\, i\in\mathcal{C},$$

which yields constraints (61) for $e=2$. The derivations for constraints (60) and the FE are analogous.

Finally, by multiplying constraints (49) by $h$ and summing over $\mathcal{O}^e$, we get

$$\frac{|\mathcal{O}^e|(|\mathcal{O}^e|+1)}{2}(x_{ij}^e + x_{ji}^e) + \sum_{h\in\mathcal{O}^e} h(w_{hi}^e - w_{hj}^e) \leq \frac{|\mathcal{O}^e|(|\mathcal{O}^e|+1)}{2},\ \forall\, i,j\in\mathcal{D}^e, i\neq j,$$

which are precisely constraints (62).

The example presented in Figure 2b along with the discussion in the proof of Proposition 5 also works in this proposition to show that the LR of CF3 is not stronger than the LR of ICF2. □

Figure 3 illustrates the properties presented in Propositions 1 to 6 and in Corollaries 1 to 4. This figure does not explicitly represent all of these relationships since the strength of the LR is a transitive property, i.e., if formulation A has a stronger LR than another formulation B and the LR of B is stronger than that of C, the LR of A dominates that of C.

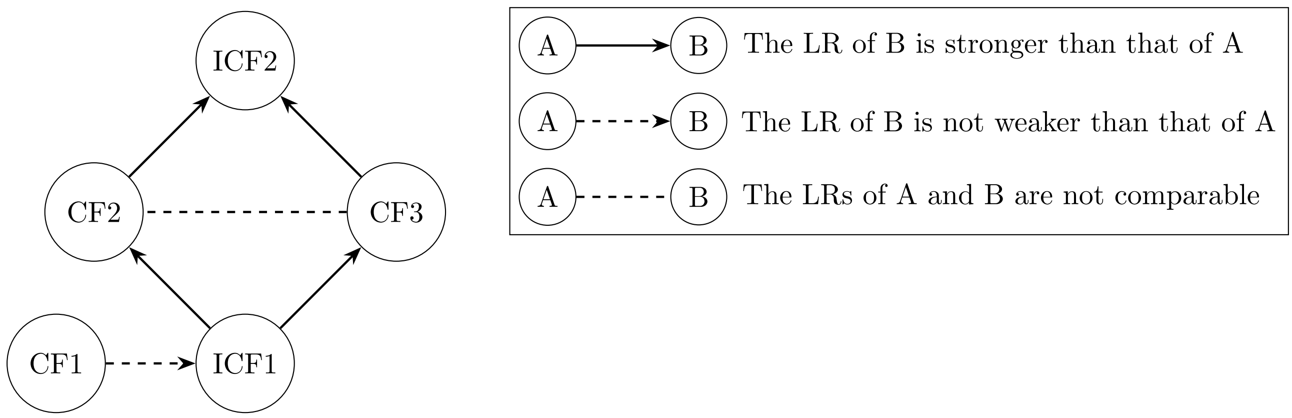



Figure 3: A visual representation of the relationships between different 2E-LRP formulations.

## 5 Computational experiments

This section presents the results of the extensive computational experiments developed to assess the performance of the presented formulations of the 2E-LRP in a general-purpose MIP solver. All experiments were run on a computing cluster from Compute Canada, where each node is equipped with 2xAMD Rome 7532 processors running at 2.4GHz. The formulations were implemented in C++ using Gurobi 11.0 as solver with an optimality tolerance of $10^{-7}$. All experiments were limited to one hour of runtime and 80GB of RAM, using up to eight threads.

We performed experiments with five benchmark instance sets of the 2E-LRP and its variants. The first of them is the *Prodhon* set, which contains 30 instances with the number of customers ranging from 20 to 200, the number of potential satellites being five or 10, and the number of potential platforms fixed as one. The second instance set is called *Nguyen* and contains 24 instances with one platform in each, five or ten potential satellites, and a range of customers that goes from 25 to 200. These two instances sets were introduced by Nguyen et al. (2012b).

The remaining three instance sets, named *I1*, *I2*, and *I3*, were generated by Contardo et al. (2012) following the procedure suggested by Boccia et al. (2011). These sets contain 31 instances each with the number of potential platforms ranging from two to five, the number of potential satellites going from three to 20, and the number of customers between eight and 200.

We ran our experiments using all instances in these sets, except those with 200 customers. These instances were excluded because they are too large for CFs to handle, as few formulations found feasible solutions and only for few instances of this size. The remaining instances were divided into three groups: small (from eight to 25 customers), medium (from 50 to 75 customers), and large (from 100 to 150 customers).

In Section 5.1, the presented formulations are compared in terms of their LR, their performance, and their number of constraints and variables. They are also compared with the best known solutions (BKS) from the literature. Section 5.2 assesses how the existence or absence of multiple potential platforms in an instance affects the performance of the formulations. Finally, Section 5.3 evaluates the benefits of including valid inequalities for each formulation.

## 5.1 Comparison of CFs

The first assessment to be made is on how the different formulations compare to each other empirically. We also confront these results with the BKS from the literature, considering both the best known lower bounds (BKLBs) and the best known upper bounds (BKUBs). The BKLBs have all been presented by Contardo et al. (2012), while the BKUBs have been reported by Contardo et al. (2012); Nguyen et al. (2012b); Schwengerer et al. (2012), and Breunig et al. (2016). For each instance, we computed the gap of the BKS as $\frac{BKUB-BKLB}{BKUB}$. Instances with this gap equal to zero were considered having an optimal solution found. Although no BKUB was improved, the presented formulations found many of the reported BKUBs and improved most of the BKLBs.

Table 2 presents the results of different metrics for each formulation. These results are aggregated by size (small, medium, and large) and also by all instances. Detailed results are presented as supplementary material. In this table, “Size” indicates the instance size, “Metric” presents the corresponding value, “BKS” is the best known solution, and “CF1”, “ICF1”, “CF2”, “ICF2”, and “CF3” indicate the corresponding formulation. “LR” represents the optimal value of the LR of the corresponding model, “LB” and “UB” are respectively the lower and upper bounds reported by the solver at the end of the runtime, “Gap (%)” corresponds to the optimality gap, “Time (s)” indicates the runtime in seconds, and “# of optimals” indicates the number of instances with proved optimality. Except for “# of optimals”, all reported values are averages. Moreover, for the gap, the presented value is the average of optimality gaps, not the gap computed with the average LB and UB. For each instance, the gap reported for the BKS is defined as $\frac{BKUB-BKLB}{BKUB}$, while for the five formulations it is the optimality gap reported by the solver $\left(\frac{UB-LB}{UB}\right)$. For the formulations that did not find feasible solutions to one or more instances of a given instance class, the corresponding UB and gap were reported as “N/A”, since it is impossible to define these values for these specific instances.

Figure 4 delves deeper into the performances of the MIP solver for different CFs. In the four charts, “# of optimals” indicates the number of optimal solutions found, “# no feasible sol.” corresponds to the number of instances for which the solver could not find a feasible solution, “# of BKLBs improved”

| Size | Metric | BKS | CF1 | ICF1 | CF2 | ICF2 | CF3 |
|---|---|---|---|---|---|---|---|
| Small (71 insts.) | LR | – | 5,139 | 6,715 | 6,715 | 6,924 | 6,921 |
| | LB | 8,506 | 7,880 | 8,741 | 8,930 | 8,930 | 8,931 |
| | UB | 8,934 | 9,090 | 8,947 | 8,934 | 8,934 | 8,934 |
| | Gap (%) | 7.78 | 14.11 | 3.02 | 0.38 | 0.36 | 0.33 |
| | Time (s) | – | 2,651 | 2,097 | 936 | 911 | 861 |
| | # of optimals | 6 | 20 | 33 | 56 | 58 | 59 |
| Medium (28 insts.) | LR | – | 35,855 | 50,174 | 50,174 | 51,308 | 51,303 |
| | LB | 61,850 | 50,018 | 59,569 | 64,659 | 64,860 | 64,910 |
| | UB | 67,071 | N/A | N/A | 68,022 | 67,743 | 67,641 |
| | Gap (%) | 9.88 | N/A | N/A | 8.46 | 6.75 | 6.37 |
| | Time (s) | – | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 |
| | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| Large (32 insts.) | LR | – | 93,089 | 119,158 | 119,158 | 120,654 | 120,646 |
| | LB | 133,224 | 109,045 | 123,967 | 136,450 | 136,825 | 139,211 |
| | UB | 147,140 | N/A | N/A | N/A | N/A | 153,291 |
| | Gap (%) | 10.36 | N/A | N/A | N/A | N/A | 13.28 |
| | Time (s) | – | 3,602 | 3,600 | 3,600 | 3,600 | 3,600 |
| | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| All (131 insts.) | LR | – | 33,188 | 43,471 | 43,471 | 44,192 | 44,187 |
| | LB | 50,373 | 41,599 | 47,752 | 51,991 | 52,126 | 52,720 |
| | UB | 55,120 | N/A | N/A | N/A | N/A | 56,745 |
| | Gap (%) | 8.85 | N/A | N/A | N/A | N/A | 4.78 |
| | Time (s) | – | 3,086 | 2,786 | 2,156 | 2,142 | 2,116 |
| | # of optimals | 6 | 20 | 33 | 56 | 58 | 59 |

Table 2: Results of the MIP solver for different CFs.

and "# of BKUBs found" respectively indicate the number of instances to which the corresponding CF found an LB that was better than the BKLB or an UB that was as good as the BKUB. Figure 5 presents the average number of constraints, variables (any kind), and binary variables for each formulation ("# of constraints", "# of variables", and "# of binary variables", respectively). The results are aggregated by size and presented for the overall solution, as in Table 2.

Regarding the LR, it is clear that the CF1 presents the lowest values. On average, ICF1 has an LR bound that is 30.98% higher than that of CF1. Moreover, despite having a stronger LR than ICF1, CF2 presents the same result as ICF1 for all tested instances. The average value of the LR of CF3 is 1.65% higher than that of CF2, even though their LRs are not comparable. ICF2 has an average LR bound that is 1.66% higher than that of CF2, but only 0.01% higher than that of CF3. This indicates that, although ICF2 has a stronger LR than CF3, their difference may not be significant in practice, since in the test instances they showed very similar results. Another impressive result is the fact that, for the medium and large instances, the average optimal values of LRs of ICF1, CF2, ICF2, and CF3 are higher than the average LB found by the solver after one hour of runtime with CF1 (this is also true for the overall average).

Comparing the number of variables and constraints, it is clear that ICF1 has a slightly larger number of general variables and smaller number of binary variables when compared to CF1. The number of constraints, however, is 81.21% smaller on average, significantly reducing the size of the linear programming problem solved in each branch-and-bound node. When comparing formulations CF2 and ICF2, the number of constraints increases again, being closer to that of CF1 and much larger than that of ICF1. The number of variables, however, is drastically reduced, going from being 88.95% smaller for the small-sized instances to being 98.22% smaller in the large-sized instances (for the binary variables the numbers are 93.57% and 99.07%, respectively). Finally, CF3 has the smallest

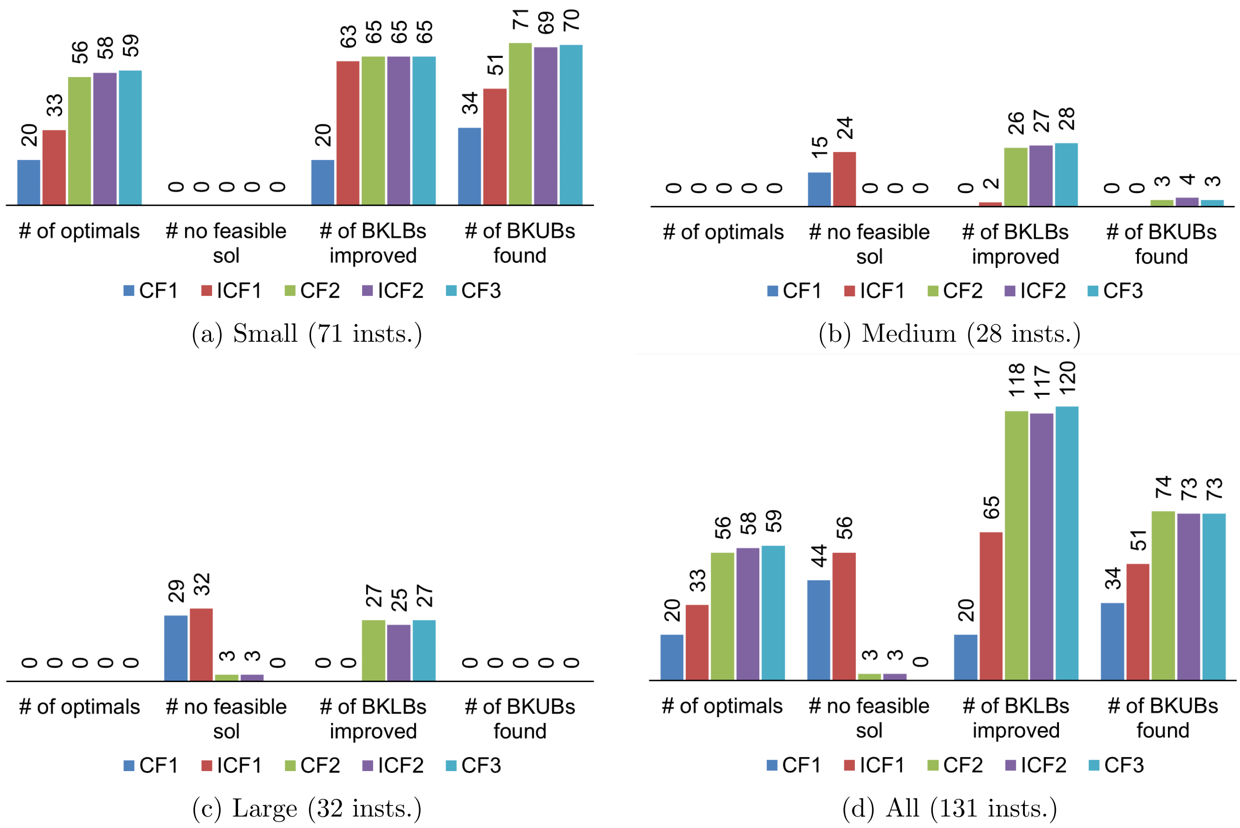
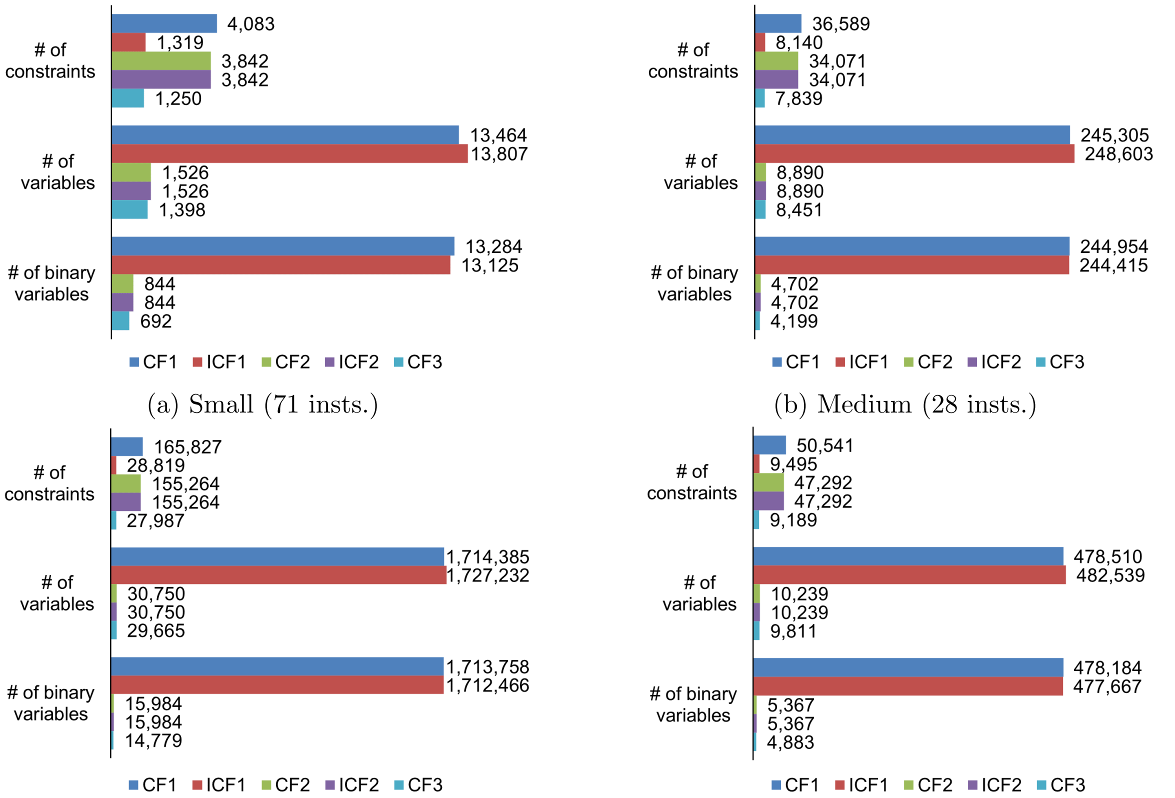



Figure 4: Results of the MIP solver for different CFs and sizes.

number of variables and constraints of all formulations. Compared to ICF2, CF3 has 80.57% fewer constraints, 4.18% fewer variables, and 9.02% fewer binary variables, being the smallest formulation, while preserving almost all strength of the LR. This translates into the results of the MIP solver, since this is the CF with the best overall performance.

For the small instances, CF1 allows the solver to prove optimality for only 20 out of 71 of them, presenting an average gap of 14.11%. These results are considerably improved by ICF1, as the solver proves optimality to a total of 33 instances, lowering the average gap to 3.02% and improving both the lower and the upper bounds. The three formulations with two-index arc variables (CF2, ICF2, and CF3) promote good results, with very similar UBs. The BKLB is improved in 65 instances, all of them with unknown optimal solutions in the literature. The solver finds the BKUB for all the 71 instances using CF2, for 69 instances using ICF2, and for 70 instances using CF3. The best LB is obtained when using CF3, which proves optimality for 59 of the instances (83.10% of them) and presents the best gap. The optimal solutions of 53 of these instances are not reported in the literature.

For medium and large instances, the solver can no longer find feasible solutions for all instances when using formulations with vehicle index variables. In fact, despite leading to better LBs on average, ICF1 results in feasible solutions for fewer instances than CF1. For medium-sized instances, the average gap for CF2 is 8.46%, whereas for ICF2 it is 1.71% lower, as a result of attaining better LBs and UBs. This fully justifies the modification in constraints (45) that yield ICF2 with sparser and stronger constraints. CF3 yields even better bounds and gap, with solutions that are only 0.85% worse than the BKUB on average, while improving the BKLB by 4.95%. CF3 also results in better BKLB for all




(a) Small (71 insts.) (b) Medium (28 insts.)

(c) Large (32 insts.) (d) All (131 insts.)

Figure 5: Average number of constraints, variables, and binary variables for different CFs and sizes.

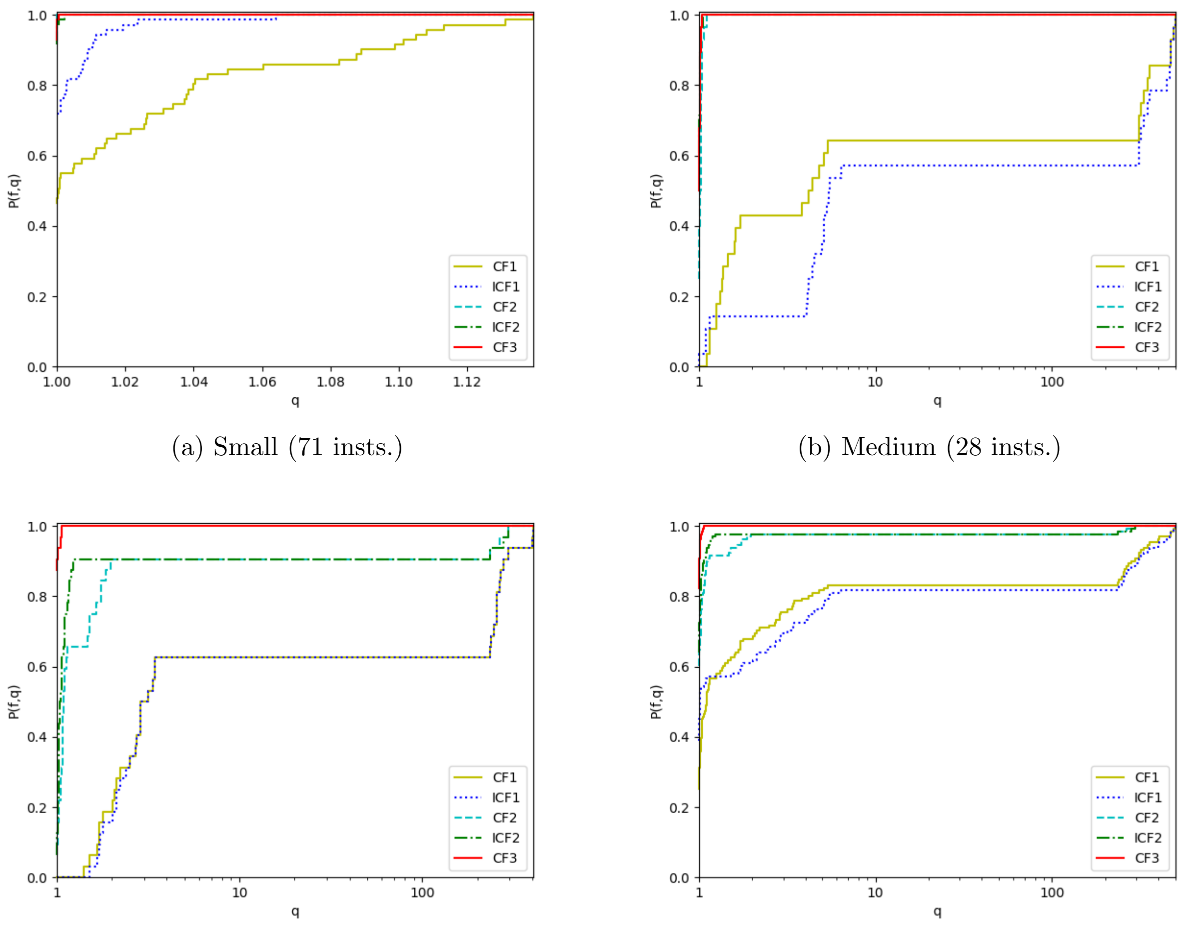
of these instances, while finding the BKUB for three of them.

For the large instances, only CF3 leads to feasible solutions for all instances. The performance is not as good as for the small- and medium-sized instances, since it presents 13.28% average gap. However, the obtained solutions are only 4.18% away from the average BKUB, which is an excellent result for a compact formulation in large-sized instances. Furthermore, the LB obtained with CF3 is 4.49% higher than the BKLB on average. In fact, the BKLB is improved for 27 out of 32 instances.

Overall, CF3 is the best performing CF and the only one that results in feasible solutions for all instances. The average gap is 4.78%, the UB is only 2.95% higher than the BKUB (which was obtained by tailored metaheurisitics) and the BKLB is improved in 4.66%, which is significant since it is a CF. It also leads to improved BKLB for 120 instances (91.60% of them), while resulting in the BKUB for 73 instances (55.73%).

To further compare the performance of the MIP solver for different formulations, Figures 6 and 7 present performance profiles (Dolan and Moré, 2002) for the UB and optimality gap, respectively. For the UB, for example, given a set of instances and a set of CFs, denote by $UB_{fp}$ the UB for instance $p$ when solved with formulation $f$. In these graphs, for a value $q > 0$, $P(f, q)$ indicates the fraction of instances for which CF $f$ finds solutions with an UB that lies within a factor $q$ of the best obtained UB. Hence, the value of $P(f, 0)$ indicates the fraction of instances for which CF $f$ finds the best UB among all CFs. For the gap, the definition is analogous. The graphs are presented with the horizontal axis in logarithmic scale.

For the UB, the performance profiles indicate that ICF1 outperforms CF1 for the small instances and is outperformed by CF1 for the medium ones, while for the large ones they are practically equivalent. On the overall average, CF1 slightly outperforms ICF1, which is coherent with the results in




(a) Small (71 insts.)

(b) Medium (28 insts.)

(c) Large (32 insts.)

(d) All (131 insts.)

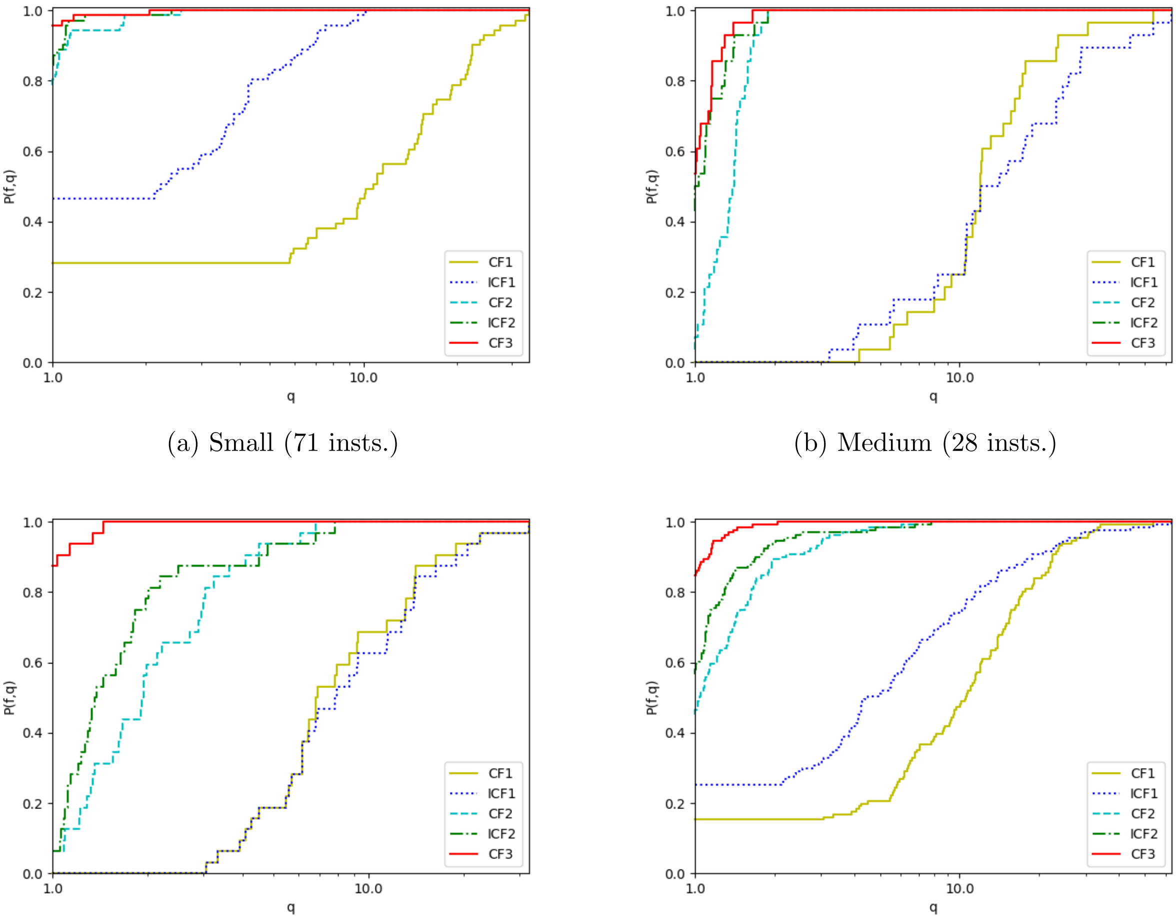
Figure 6: Performance profile of the MIP solver for different CFs and sizes considering the UB.

Figure 4, since there are more instances for which the solver does not find any feasible solution for ICF1 than for CF1. Nevertheless, for the gap, this behavior is not the same. Although for the small, medium, and large instances the comparison of CF1 and ICF1 is similar for both UB and gap, on the overall average, ICF1 outperforms CF1, since it provides much better LBs.

The performance profiles of CF2 and ICF2 for UB are very similar. They are practically the same for the small and medium instances, and, for the large instances, ICF2 outperforms CF2 for small values of $q$. For the gap, this difference is more significant. For the small instances, there is a small difference, which did not exist for the UB performance profiles. Moreover, for the medium and large instances, as well as for the overall average, this difference is more expressive due mostly to the improvements in the LR and the LB documented in Table 2.

Finally, the performance profiles confirm the results presented in Table 2 that CF3 is the best performing CF. For the UB, $P(f, q)$ is equal to one for almost every possible value of $q$, greatly outperforming the other CFs for the overall average. For the gap, the results of the MIP solver for CF3 are also much better than those for the other CFs.

Therefore, the presented results make clear that the use of formulations with vehicle index variables does not lead to good results in a general-purpose MIP solver. Moreover, in medium- and large-sized instances, it may not possible to obtain even feasible solutions. The two-index arc variables are more suited to solve the 2E-LRP, leading to much better results. Nevertheless, CF2, which is based in a




(a) Small (71 insts.)

(b) Medium (28 insts.)

(c) Large (32 insts.)

(d) All (131 insts.)

Figure 7: Performance profile of the MIP solver for different CFs and sizes considering the optimality gap.

formulation found in the literature, has too many constraints and binary variables, deteriorating the performance of the solver. CF3 is much smaller, while preserving most of the quality of the LR and being, therefore, the best option to represent the 2E-LRP with a compact formulation.

### 5.2 The impact of multiple platforms

Table 3 presents a closer look at how the number of platforms may affect the performance of the solver according to the addressed CFs. As discussed in Section 5, there are two instance sets with a single platform in each instance and three sets with multiple platforms. In Table 3, this information is presented in column $|\mathcal{P}|$. The results presented in the table clearly indicate that the number of platforms significantly affects the performance of the solver. In general, instances with a single platform are easier to solve than the ones with multiple platforms. This characteristic affects the solver's ability to find feasible solutions and prove optimality. For the small instances, the use of CF2, ICF2, and CF3 leads to optimal solutions for all single-platform instances, which is not true for the multiple-platforms ones. Likewise, for these instances, the solver performs better with both CF1 and ICF1 in the single-platform instances. Moreover, for the large instances, both CF2 and ICF2 result in feasible solutions for all single-platform instances, while for three multiple-platforms instances no solution is found.

| Size | $\|\mathcal{P}\|$ | Metric | BKS | CF1 | ICF1 | CF2 | ICF2 | CF3 |
|---|---|---|---|---|---|---|---|---|
| Small | Single (8 insts.) | LB | 69,840 | 64,636 | 71,576 | 73,071 | 73,071 | 73,071 |
| | | UB | 73,071 | 74,274 | 73,165 | 73,071 | 73,071 | 73,071 |
| | | Gap (%) | 4.06 | 12.44 | 2.00 | 0.00 | 0.00 | 0.00 |
| | | # of optimals | 0 | 0 | 2 | 8 | 8 | 8 |
| | | # no feasible sol. | 0 | 0 | 0 | 0 | 0 | 0 |
| | Multiple (63 insts.) | LB | 717 | 673 | 762 | 785 | 785 | 786 |
| | | UB | 790 | 813 | 793 | 790 | 790 | 790 |
| | | Gap (%) | 8.25 | 14.33 | 3.15 | 0.43 | 0.40 | 0.37 |
| | | # of optimals | 6 | 20 | 31 | 48 | 50 | 51 |
| | | # no feasible sol. | 0 | 0 | 0 | 0 | 0 | 0 |
| Medium | Single (16 insts.) | LB | 107,320 | 86,878 | 103,429 | 112,188 | 112,531 | 112,606 |
| | | UB | 116,324 | N/A | N/A | 117,915 | 117,455 | 117,277 |
| | | Gap (%) | 7.73 | N/A | N/A | 4.83 | 4.16 | 3.99 |
| | | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | | # no feasible sol. | 0 | 5 | 12 | 0 | 0 | 0 |
| | Multiple (12 insts.) | LB | 1,224 | 872 | 1,089 | 1,287 | 1,300 | 1,315 |
| | | UB | 1,400 | N/A | N/A | 1,499 | 1,460 | 1,460 |
| | | Gap (%) | 12.73 | N/A | N/A | 13.31 | 10.21 | 9.54 |
| | | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | | # no feasible sol. | 0 | 10 | 12 | 0 | 0 | 0 |
| Large | Single (20 insts.) | LB | 212,188 | 173,755 | 197,485 | 217,353 | 217,950 | 221,762 |
| | | UB | 234,320 | N/A | N/A | 273,690 | 258,576 | 244,036 |
| | | Gap (%) | 9.28 | N/A | N/A | 17.49 | 14.16 | 8.80 |
| | | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | | # no feasible sol. | 0 | 17 | 20 | 0 | 0 | 0 |
| | Multiple (12 insts.) | LB | 1,618 | 1,194 | 1,437 | 1,612 | 1,617 | 1,626 |
| | | UB | 1,841 | N/A | N/A | N/A | N/A | 2,050 |
| | | Gap (%) | 12.15 | N/A | N/A | N/A | N/A | 20.74 |
| | | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | | # no feasible sol. | 0 | 12 | 12 | 3 | 3 | 0 |
| All insts. | Single (44 insts.) | LB | 148,173 | 122,324 | 140,390 | 152,878 | 153,274 | 155,034 |
| | | UB | 162,094 | N/A | N/A | 180,568 | 173,531 | 166,857 |
| | | Gap (%) | 7.77 | N/A | N/A | 9.71 | 7.95 | 5.45 |
| | | # of optimals | 0 | 0 | 2 | 8 | 8 | 8 |
| | | # no feasible sol. | 0 | 22 | 32 | 0 | 0 | 0 |
| | Multiple (87 insts.) | LB | 912 | 772 | 900 | 968 | 971 | 975 |
| | | UB | 1,019 | N/A | N/A | N/A | N/A | 1,056 |
| | | Gap (%) | 9.41 | N/A | N/A | N/A | N/A | 4.45 |
| | | # of optimals | 6 | 20 | 31 | 48 | 50 | 51 |
| | | # no feasible sol. | 0 | 22 | 24 | 3 | 3 | 0 |

Table 3: The impact of the number of potential platforms on the CFs performances.

Therefore, in addition to promoting the best overall performance, CF3 is the least sensitive to the existence of multiple platforms. For medium-sized instances, the average gaps for CF2 and ICF2 are 8.48% and 6.05% worse in the multiple-platforms instances than in the single-platform ones, while for CF3 this number is only 5.55%. Additionally, CF3 yields feasible solutions for the large multiple-platforms instances that CF2 and ICF2 do not.

## 5.3 Experiments with VIs

We ran experiments with the VIs to assess their impact on the solver performance. These experiments were performed with formulations CF1, ICF1, ICF2, and CF3. The impact of VIs in CF2 was not assessed because this formulation is very similar to ICF2. Since the impacts of the VIs on the performances of the formulations is highly dependent on the instances sizes, the results are all presented aggregated by size, not the overall average.

The experiments consisted of grouping the VIs based on similar characteristics. First, the performances were evaluated including all VIs. Then, each VI group was removed to evaluate how it affected the performance, resulting in six different configurations. Table 4 presents which VIs are included in each configuration. Note that ICF1 does not consider configuration VI2, since variables $w$ are not defined in this formulation and hence configurations VI2 and All would be the same. Also, the formulations with two-index arc variables (ICF2 and CF3) do not consider symmetry breaking constraints and, thus, do not have configuration VI5. The results of the VI experiments are summarized in Tables 5 to 8.

| Configuration | Meaning | CF1 VIs | ICF1 VIs | ICF2 VIs | CF3 VIs |
|---|---|---|---|---|---|
| VI1 | All VIs except for the lower bounds on the number of facilities | (27)–(28) (30)–(37) | (27)–(28) (30)–(35) | (36)–(37) (50)–(57) | (50)–(55) (64)–(65) |
| VI2 | All VIs except for the ones that relate assignment and opening of facilities | (27)–(35) | – | (29), (50)–(55) | (29), (50)–(55) |
| VI3 | All VIs except for the lower bounds on the number of vehicles | (27)–(29) (31)–(37) | (27)–(29) (31)–(35) | (29), (36)–(37) (51)–(57) | (29), (51)–(55) (64)–(65) |
| VI4 | All VIs except for those that relate the opening of facilities with their visit | (27)–(30) (36)–(37) | (27)–(30) | (29), (36)–(37) (50), (56)–(57) | (29), (50) (64)–(65) |
| VI5 | All VIs except for those that break symmetry | (29)–(37) | (29)–(35) | – | – |
| All | All VIs | (27)–(37) | (27)–(35) | (29), (36)–(37) (50)–(57) | (29), (50)–(55) (64)–(65) |

Table 4: Different VI configurations for each formulation.

Table 5 presents the results for CF1 and the six VI configurations. Only the results for the small instances are shown, since for medium and large instances no configuration of CF1 is able to find feasible solutions for all instances. Moreover, the numbers of general and binary variables are not included in this table since they do not change when including or removing VIs.

| | CF1 | CF1–VI1 | CF1–VI2 | CF1–VI3 | CF1–VI4 | CF1–VI5 | CF1–All |
|---|---|---|---|---|---|---|---|
| LB | 7,880 | 8,152 | 8,183 | 7,895 | 8,185 | 8,259 | 8,187 |
| UB | 9,090 | 8,969 | N/A | N/A | N/A | 8,980 | N/A |
| Gap (%) | 14.11 | 10.73 | N/A | N/A | N/A | 7.44 | N/A |
| Time (s) | 2,651 | 2,626 | 2,594 | 2,664 | 2,598 | 2,379 | 2,587 |
| # of optimals | 20 | 21 | 22 | 20 | 22 | 27 | 22 |
| # no feasible sol. | 0 | 0 | 3 | 1 | 2 | 0 | 1 |
| # of BKLBs improved | 20 | 24 | 29 | 17 | 22 | 40 | 27 |
| # of BKUBs found | 34 | 40 | 38 | 32 | 35 | 44 | 37 |
| # of constrains | 4,083 | 4,873 | 4,734 | 4,873 | 4,560 | 4,543 | 4,875 |

Table 5: The impact of including VIs in CF1 for small instances (71 instances).

It is clear that the inclusion of all VIs is not beneficial for CF1 because the solver cannot find feasible solutions for all instances. Indeed, the only cases in which the inclusion of VIs is beneficial are the ones that do not include the lower bound on the number of facilities (29) or the symmetry breaking constraints (27)–(28). The best performing VI configuration is CF1–VI5, i.e., the one that includes all VIs except for the symmetry breaking ones. Compared to CF1 without VIs, this CF has 11.27% more constraints, leading to 4.81% LB improvement, 1.21% UB reduction, and 6.67% decrease in the average gap. Moreover, the number of instances proved optimal increased from 20 to 27, a 35% improvement. Configuration CF1–VI5 doubles the number of BKLBs improved and increases the number of BKUBs found in 29.41% compared to CF1. These improvements, however, do not get to the quality of ICF1, which outperformed them even without VIs.

Table 6 presents the results for the ICF1. The results show that only ICF1–VI5 (without the symmetry breaking constraints) has better results than ICF1 and showing a limited improvement. The LB increases 0.33%, the UB decreases 0.10%, the gap reduces 0.12%, the average runtime is 2.96% smaller, and two new instances have their solutions proved optimal, while four new instances have their BKUBs found.

| | ICF1 | ICF1–VI1 | ICF1–VI3 | ICF1–VI4 | ICF1–VI5 | ICF1–All |
|---|---|---|---|---|---|---|
| LB | 8,741 | 8,675 | 8,645 | 8,729 | 8,770 | 8,684 |
| UB | 8,947 | N/A | N/A | N/A | 8,938 | N/A |
| Gap (%) | 3.02 | N/A | N/A | N/A | 2.90 | N/A |
| Time (s) | 2,097 | 2,397 | 2,349 | 2,317 | 2,035 | 2,352 |
| # of optimals | 33 | 30 | 31 | 30 | 35 | 29 |
| # no feasible sol. | 0 | 4 | 3 | 2 | 0 | 3 |
| # of BKLBs improved | 63 | 62 | 60 | 61 | 63 | 63 |
| # of BKUBs found | 51 | 45 | 45 | 45 | 55 | 42 |
| # of constrains | 1,319 | 1,968 | 1,968 | 1,655 | 1,639 | 1,970 |

Table 6: The impact of including VIs in ICF1 for small instances (71 instances).

Table 7 presents the results for the inclusion of VIs with ICF2 for small and medium instances. The large instances are not presented since neither the base formulation nor any of the VI scenarios found results for all instances.

| Size | Metric | ICF2 | ICF2–VI1 | ICF2–VI2 | ICF2–VI3 | ICF2–VI4 | ICF2–All |
|---|---|---|---|---|---|---|---|
| Small (71 insts.) | LB | 8,930 | 8,930 | 8,930 | 8,930 | 8,930 | 8,931 |
| | UB | 8,934 | 8,934 | 8,934 | 8,934 | 8,934 | 8,934 |
| | Gap (%) | 0.36 | 0.37 | 0.35 | 0.35 | 0.33 | 0.33 |
| | Time (s) | 911 | 843 | 876 | 823 | 845 | 851 |
| | # of optimals | 58 | 58 | 58 | 58 | 59 | 58 |
| | # no feasible sol. | 0 | 0 | 0 | 0 | 0 | 0 |
| | # of BKLBs improved | 65 | 65 | 65 | 65 | 65 | 65 |
| | # of BKUBs found | 69 | 68 | 70 | 69 | 70 | 69 |
| | # of constraints | 3,842 | 4,321 | 4,161 | 4,321 | 4,008 | 4,323 |
| Medium (28 insts.) | LB | 64,860 | 65,068 | 65,202 | 65,019 | 65,004 | 65,104 |
| | UB | 67,743 | 67,693 | 67,603 | 67,343 | 67,720 | 67,476 |
| | Gap (%) | 6.75 | 7.08 | 6.84 | 6.66 | 6.54 | 6.21 |
| | Time (s) | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 |
| | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | # no feasible sol. | 0 | 0 | 0 | 0 | 0 | 0 |
| | # of BKLBs improved | 27 | 25 | 28 | 28 | 27 | 28 |
| | # of BKUBs found | 4 | 4 | 4 | 6 | 5 | 3 |
| | # of constraints | 34,071 | 35,602 | 35,091 | 35,602 | 34,588 | 35,604 |

Table 7: The impact of including VIs in ICF2 for small and medium instances.

For ICF2, the inclusion of VIs is overall beneficial. For the small instances, the average LB and UB do not vary significantly. The best gaps come from ICF2–VI4 and ICF2–All, and ICF2–VI4 results in proved optimal solutions to most instances (59 against 58 from the other approaches). The running times do not vary much, even though the VIs help improving them on average. For the medium-sized instances, the best UB is achieved in configuration ICF2–VI3, the best LB in ICF2–VI2, and the best gap in ICF2–All. The number of constraints in ICF2–All increases in 4.49% with respect to ICF2, but this clearly pays off.

It is worth noticing that, with the inclusion of VIs, the performance related to ICF2–All is better than that of CF3 (the best performing formulation so far) for the medium-sized instances, improving the LB in 0.30%, the UB in 0.24%, and the gap in 0.16%. For the small instances, ICF2–All matches CF3 in LB, UB, and gap, but loses in the number of instances with optimality proved. ICF2–VI4, however, lead to the same number of instances with proved optimal solution as CF3.

Finally, Table 8 presents the results regarding CF3 and the different VI configurations. For the small instances, the average LB and UB do not change significantly. However, considering the optimality gap and the number of instances with proved optimal solution, the best configuration is CF3–VI1, which shows an improvement of 0.06% in the gap and provides proved optimal solutions for two extra instances compared to CF3.

| Size | Metric | CF3 | CF3–VI1 | CF3–VI2 | CF3–VI3 | CF3–VI4 | CF3–All |
|---|---|---|---|---|---|---|---|
| Small (71 insts.) | LB | 8,931 | 8,931 | 8,930 | 8,930 | 8,931 | 8,931 |
| | UB | 8,934 | 8,934 | 8,934 | 8,934 | 8,934 | 8,934 |
| | Gap (%) | 0.33 | 0.27 | 0.34 | 0.33 | 0.32 | 0.31 |
| | Time (s) | 861 | 834 | 820 | 834 | 834 | 830 |
| | # of optimals | 59 | 61 | 58 | 58 | 59 | 58 |
| | # no feasible sol. | 0 | 0 | 0 | 0 | 0 | 0 |
| | # of BKLBs improved | 65 | 65 | 65 | 65 | 65 | 65 |
| | # of BKUBs found | 70 | 70 | 70 | 69 | 71 | 70 |
| | # of constraints | 1,250 | 1,824 | 1,569 | 1,824 | 1,510 | 1,826 |
| Medium (28 insts.) | LB | 64,910 | 65,050 | 65,136 | 65,087 | 65,210 | 65,219 |
| | UB | 67,641 | 67,532 | 67,513 | 67,647 | 67,658 | 67,412 |
| | Gap (%) | 6.37 | 6.32 | 5.57 | 6.17 | 6.14 | 6.10 |
| | Time (s) | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 |
| | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | # no feasible sol. | 0 | 0 | 0 | 0 | 0 | 0 |
| | # of BKLBs improved | 28 | 28 | 28 | 28 | 28 | 28 |
| | # of BKUBs found | 3 | 8 | 3 | 6 | 6 | 5 |
| | # of constraints | 7,839 | 9,734 | 8,858 | 9,734 | 8,720 | 9,736 |
| Large (32 insts.) | LB | 139,211 | 139,062 | 138,903 | 138,746 | 139,197 | 139,074 |
| | UB | 153,291 | N/A | N/A | 154,833 | 155,202 | N/A |
| | Gap (%) | 13.28 | N/A | N/A | 14.11 | 14.81 | N/A |
| | Time (s) | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 | 3,600 |
| | # of optimals | 0 | 0 | 0 | 0 | 0 | 0 |
| | # no feasible sol. | 0 | 1 | 2 | 0 | 0 | 2 |
| | # of BKLBs improved | 27 | 29 | 31 | 30 | 28 | 31 |
| | # of BKUBs found | 0 | 0 | 0 | 0 | 0 | 0 |
| | # of constraints | 27,987 | 32,581 | 30,413 | 32,581 | 30,161 | 32,583 |

Table 8: The impact of including VIs in CF3.

For the medium-sized instances, the inclusion of VIs is mainly beneficial. The LB is improved in every configuration compared to the base formulation CF3 and the UB is improved in most of them. The best LB and UB are obtained from the inclusion of all VIs (CF3–All), improving the LB in 0.47% and the UB in 0.34% compared to CF3. CF3–All also outperforms ICF2–All for these instances, improving the LB in 0.18%, the UB in 0.09%, and the gap in 0.11%. Thus, the 24.20% increase in the number of constraints compared to CF3 is worth it for these instances.

Nonetheless, for the large instances, the inclusion of VIs worsens the performance of CF3. Indeed, with CF3–VI1, CF3–VI2, and CF3–All it is not possible to find feasible solutions for some instances. Furthermore, in the configurations that do find feasible solutions for all instances (CF3–VI3 and CF3–VI4), the average LB, UB, and gap are worse than the corresponding values for CF3. Possibly this happens because these models already have large numbers of variables and constraints due to the instance size, and the inclusion of these VIs make it even harder for the MIP solver to process the branch-and-bound nodes, on top of possible effects on the solver heuristics. It is worth noticing, however, that the inclusion of VIs in CF3 for large instances allows for the improvement of another four BKLBs.

In conclusion, the effect of including valid inequalities heavily depends on the instance size and the base formulation. For the formulations with vehicle index variables (CF1 and ICF1), the inclusion of VIs helped the performance of the solver depending on which VIs were included, since in some configurations they prevented the solver from finding feasible solutions to some instances. For the two-index arc variables formulations (ICF2 and CF3), the inclusion of VIs was beneficial for the small- and medium-sized instances. In fact, for the medium-sized instances, the best performing approach for both formulations was to include all of the VIs that were compatible with the corresponding formulation. For the large instances, however, the inclusion of VIs had negative effects in the solution quality in all evaluated scenarios.

Finally, considering all experiments performed, we have found lower bounds that are better than the BKLBs reported in the literature for 125 out of the 131 benchmark instances evaluated, which encompasses all instances with unknown optimal solutions in the literature so far. Furthermore, we have proved optimality for 55 instances for the first time. The detailed results are available in the supplementary material.

## 6 Conclusion

In this paper, we have compared mixed-integer programming (MIP) compact formulations for the two-echelon location-routing problem (2E-LRP). We have discussed a formulation with vehicle index variables from the literature and provided improvements to it. Additionally, we have introduced two novel formulations based on two-index arc variables. From a theoretical perspective, we have demonstrated that the formulations with two-index variables have stronger linear programming relaxations. We have also showed, from extensive computational experiments, that these formulations perform much better in practice when solved with a general-purposed MIP solver.

This suggests that, although the literature on the 2E-LRP is mostly based on compact formulations with a vehicle index, the future use of two-index variables formulations would be beneficial both for defining variants and evaluating the performance of tailored algorithms. Furthermore, for ad hoc methods based on mathematical formulations such as branch-and-cut schemes, decomposition-based algorithms, and matheuristics, the formulations with two-index arc variables are likely to be a better starting point than the formulations based on variables with a vehicle index.

We have also discussed the impacts of including valid inequalities in these formulations, both novel and literature-based. Our experiments suggest that their utility depends on the instance size and type of formulation. On the one hand, for small and medium instances (up to 75 customers) they help the MIP solver. On the other hand, for large instances, they actually worsen the solver performance.

Considering all experiments performed, we have improved the best known lower bounds for 125 out of the 131 benchmark instances evaluated (the other six had the optimal solution as lower bound).

We have also obtained the optimal solutions of 55 instances for the first time.

Interesting research developments are available for future work. For instance, one may focus on extending the addressed formulations to the numerous 2E-LRP variants present in the literature. Moreover, the development of branch-and-cut schemes and other ad hoc solution methods on top of these formulations could further improve the best known lower and upper bounds for these instances.

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