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Expected Distances in Grid Networks:

Can the Manhattan Metric be used in Manhattan?

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Abstract. Urban and logistics models often approximate shortest-path travel on grid-like street networks using the Manhattan metric. When origins and destinations are continuously distributed along street segments, however, travelers cannot cut through blocks and the Manhattan distance can underestimate true network travel. We study rectilinear travel on a rectangular grid network with movement restricted to edges and demand distributed uniformly with respect to edge length. We characterize exactly when the network shortest-path distance differs from the Manhattan metric: a discrepancy occurs only when the two points lie on distinct parallel edges within the same horizontal or vertical strip, in which case the shortest route requires a detour via a strip boundary. Using a strip-based decomposition, we derive the distribution of the resulting detour penalty and a closed-form expression for its mean, yielding an explicit additive correction to the expected Manhattan distance. We show that this expected penalty is strictly positive for any finite grid but vanishes under grid refinement over a fixed physical rectangle, and we obtain a simple expression for symmetric square grids. Beyond mean-distance bias, we show via a traveling salesman tour example that small per-pair underestimation can accumulate into substantial objective errors in routing-type decision problems.

Keywords: Manhattan distance; grid networks; expected distance; detour penalty; urban accessibility; 15-minute city

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1. Introduction

The purpose of this paper is to compute expected distances in networks exhibiting a grid topology. According to Rose-Redwood (2018) and Kostof (2018), grid patterned towns have existed for centuries. Many cities throughout the world are constructed according to a grid pattern, never in their entirety, but often in some large areas. Chicago, Barcelona, Kyoto, New York (Manhattan), Melbourne, and Montréal are such examples (see Figure 1). The case of the *Eixample* (extension) in Barcelona is particularly interesting since that part of the city was designed with precise street and city block sizes and specifications with the aim of providing hygienic conditions and facilitating vehicular and pedestrian traffic (Aibar & Bijker, 1997; Pallares-Barbera, Badia, & Duch, 2011). Grid patterns are also frequently encountered in warehouses (Pansart, Catusse, & Cambazard, 2018; Schiffer, Boysen, Klein, Laporte, & Pavone, 2022).

Our interest in computing distance distributions in networks is partly motivated by the study of 15-minute cities (Arslan & Laporte, 2025; Moreno, Allam, Chabaud, Gall, & Pratlong, 2021) in which residents should access most services within 15 minutes on foot or by bicycle. In this context, it is relevant to be able to compute expected travel times in order, for example, to suitably locate facilities. In the same vein, urban transit planners are interested in the distribution of trip lengths between their origin and the closest point of entry in the transit system (Chandra & Quadrifoglio, 2013). The knowledge of trip length distributions and expectations is also useful for the computation of urban space consumption by cars and buses (Roca-Riu, Menendez, Dakic, Buehler, & Ortigosa, 2020).

The computation of expected distances in grid networks is closely related to that of expected distances between two random points in a closed geometrical figure. Marsaglia, Narasimhan, and Zaman (1990) provide an interesting account of this field of research, starting with the work of Crofton (1876) who computed the expected distance between two random points taken from two disjoint rectangles. Ghosh (1943a) studied the same problem, as well as the case where the two points belong to the same rectangle (Ghosh, 1943b). Multiple authors have since computed expected distances in circles (Eilon, Watson-Gandy, & Christofides, 1971) and in arbitrarily shaped polygons (Pure & Durrani, 2015). Hall (1984) developed analytic travel-distance expressions for movements on a rectangular grid, while Gaboune, Laporte, and Soumis (1993) computed expected distances in rectangles and rectangular parallelepipeds under three metrics: Manhattan, Euclidean, and Chebychev. Along similar lines, Christofides and Eilon (1969) studied expected distances arising in distribution and routing problems, establishing relationships between radial distances and optimal tour lengths. In another work that demonstrates that the network structure can materially change rectilinear travel, Drezner and Wesolowsky (1995) study distances and location models on an alternating one-way rectilinear grid, where directionality constraints complicate the induced travel distances. More recently, Hale, Huq, Hipkin, and Tucker (2013) proposed models for estimating expected inter-nodal distances (arc lengths) in network representations when detailed geometric information is unavailable.

Here we compute expected distances not in geometrical shapes, but in grid networks, which means that travel exclusively takes place on the edges of the network, i.e., shortcuts are not allowed. Consider for example the simple network (a rectangle) depicted in Figure 2. A trip from point p to point q must follow the edges of the rectangle. Hence, in this case, the Manhattan (rectilinear) distance cannot be directly applied to measure the trip length in a Manhattan-like city because a detour penalty

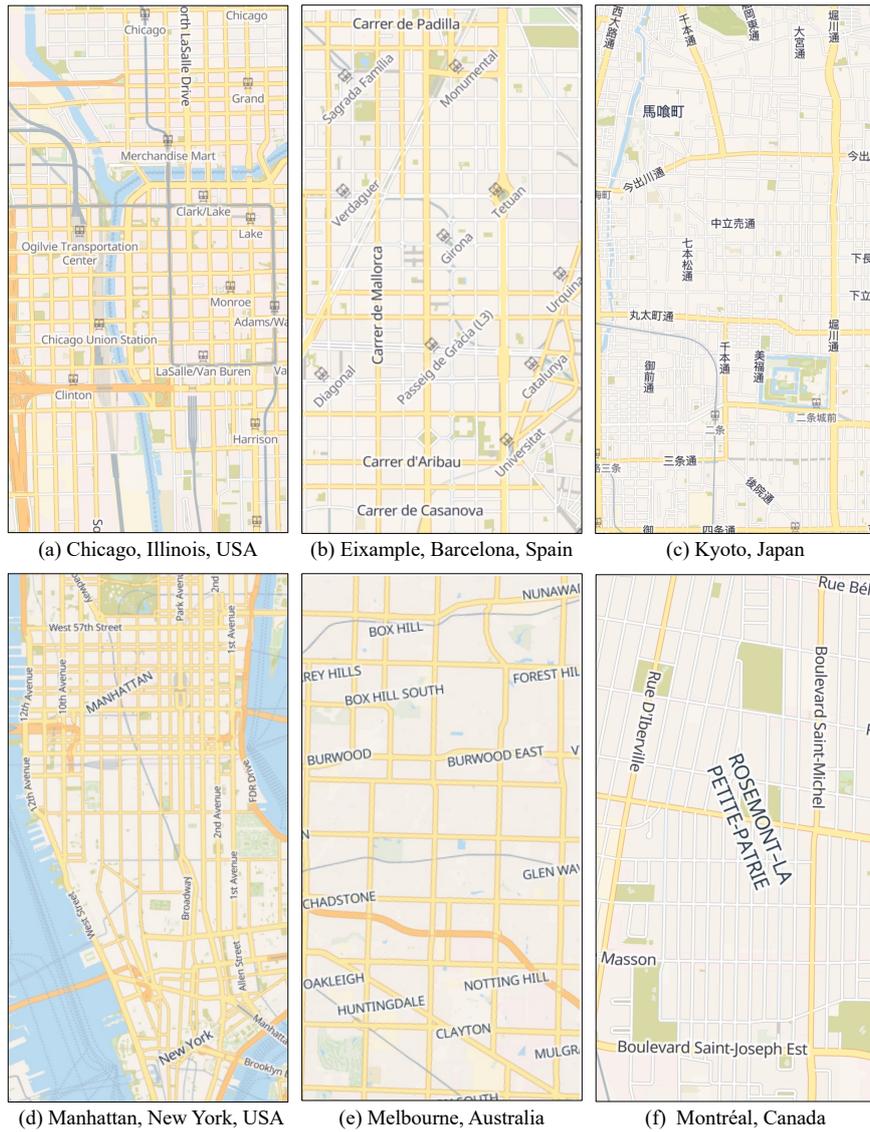


Figure 1.: Partial maps of six grid cities. ©OpenStreetMap contributors (Retrieved February 5, 2026).

must be added to the Manhattan distance. Note that the length of the detour can be made arbitrarily large by increasing the height of the rectangle.

The remainder of this paper is organized as follows. In Section 2 we present some preliminaries. Section 3 describes the computation of distances in grid networks. In Section 4 we compute the distribution of detour penalties. Section 5 is devoted to the computation of expected distances between random points on a grid network and to the measure of the departure from expected Manhattan distances. This is followed by some illustrative numerical results in Section 6 and conclusions in Section 7.

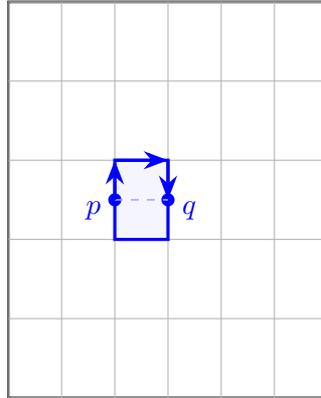


Figure 2.: Rectangular grid. The Manhattan distance between the p and q on both sides of the blue cell is not usable, but there is an around-the-block detour shown by arrows.

2. Preliminaries

We consider a rectangular grid network $G = (m, n, a, b)$ formed by straight line segments connecting parallel sets of equally spaced horizontal and vertical lines, where $m, n \in \mathbb{N}$ with $m, n \geq 2$ represent the number of grid lines in the horizontal and vertical directions, respectively, with spacings $a, b > 0$. The corresponding height and width of the rectangle are then $(m - 1)a$ and $(n - 1)b$. The lines are numbered starting from 0, so that the i^{th} horizontal line is at $y = ia$ and the j^{th} vertical line is at $x = jb$, for $i \in \{0, \dots, m - 1\}$ and $j \in \{0, \dots, n - 1\}$, respectively.

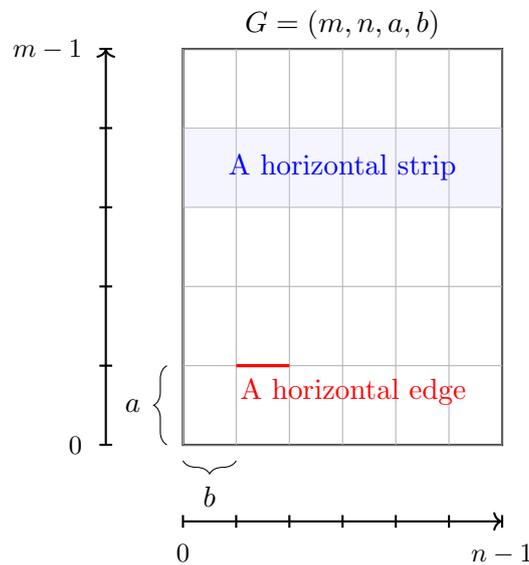


Figure 3.: A sample grid network with a highlighted strip (blue) and edge (red).

Definition 1 (Strip). A horizontal strip of the rectangular grid network $G = (m, n, a, b)$ is the region between two consecutive horizontal lines i and $i + 1$ of the grid for $i \in \{0, \dots, m - 2\}$. A vertical strip is defined analogously between consecutive

vertical lines.

Definition 2 (Edge). A *horizontal edge* $[i, j]$ of G is the segment along the i^{th} horizontal line, between the j^{th} and $(j + 1)^{\text{th}}$ vertical lines of the grid, for $i \in \{0, \dots, m - 1\}$ and $j \in \{0, \dots, n - 2\}$. A *vertical edge* is defined analogously between consecutive vertical lines.

Edges are the basic continuous segments along which demand is uniformly distributed.

Definition 3 (Cell). A *cell* $[i, j]$ on the rectangular grid $G = (m, n, a, b)$ is the smallest rectangle within G , delineated by the i^{th} and $(i + 1)^{\text{th}}$ horizontal lines and the j^{th} and $(j + 1)^{\text{th}}$ vertical lines, for $i \in \{0, \dots, m - 2\}$ and $j \in \{0, \dots, n - 2\}$.

A cell is thus the rectangle spanned by the smallest network grid $G^* = (2, 2, a, b)$ contained within the larger network $G = (m, n, a, b)$.

3. Distances on grid networks

Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be two distinct fixed points lying on the edges of G (not necessarily at intersections). We are interested in calculating the length of the shortest path length between the given points p and q , which we will denote by $d_N(p, q)$. If movement were allowed everywhere within the plane outlined by the grid $G = (m, n, a, b)$, the length of the shortest path between p and q would simply be the Euclidean distance. However, the grid network imposes the constraint of horizontal and vertical movements along segments. In light of this, we begin by recalling the definition of the Manhattan distance metric, which considers movements along the axes of a grid.

Definition 4. The *Manhattan distance* (or ℓ_1 norm) between two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ in a plane is defined by

$$d_M(p, q) := \|p - q\|_1 = |x_2 - x_1| + |y_2 - y_1|.$$

If all points were located at the intersections of the grid G , then the shortest path length between p and q , constrained to movement along the horizontal and vertical segments, would coincide with the Manhattan distance $d_M(p, q)$. Indeed, in this case, one could first move horizontally from x_1 to x_2 , and then vertically from y_1 to y_2 , or vice-versa. In the grid network, however, demand is uniformly distributed along the horizontal and vertical edges. Moreover, motion is restricted to the edges of the network, that is, all paths must follow existing “streets” (edges) and cannot pass through arbitrary points in the interior of any “block” (cell). As a consequence, the network shortest-path length $d_N(p, q)$ need not coincide with the Manhattan distance. In fact, the next proposition shows that these two quantities agree in all but one specific geometric configuration.

Proposition 1. *The shortest distance between points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ in the grid $G = (m, n, a, b)$ coincides with the Manhattan distance, i.e., $d_N(p, q) = d_M(p, q)$, except when p and q lie on separate parallel edges within the same strip (that is, except when p and q fall along separate horizontal edges of the same vertical strip, or, analogously, along separate vertical edges in the same horizontal strip).*

Proof. If the condition is satisfied, the shortest path can move horizontally from x_1 to x_2 and vertically from y_1 to y_2 in a stepwise manner, giving $d_N(p, q) = |x_2 - x_1| + |y_2 - y_1|$. \square

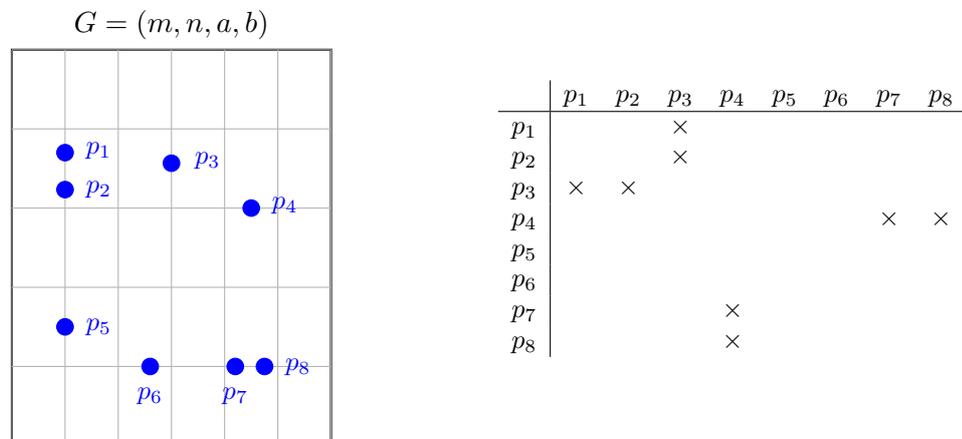


Figure 4.: Example points on the grid (left) and the corresponding non-Manhattan pair indicator matrix (right), where an \times marks pairs with $d_N \neq d_M$.

Figure 4 depicts eight randomly selected points and the corresponding non-Manhattan pair indicator matrix. For instance, the distance between p_1 and p_4 is equal to the Manhattan distance, while the distance between p_1 and p_3 is non-Manhattan. Note that when the two points fall along the same edge (for example, point pairs (p_1, p_2) or (p_7, p_8)), the distance between them is again Manhattan, as one coordinate distance will simply be zero. Therefore, when the point pairs do not lie on separate parallel edges within the same strip, the distance between these two points remains the ℓ_1 norm (Manhattan distance). However, when the points lie on separate edges within the same strip (for example, point pairs (p_1, p_3) or (p_4, p_8)), a detour along the network is required as movement through a cell is not possible, in which case $d_N(p, q) \geq |x_2 - x_1| + |y_2 - y_1|$. A more precise result is presented in the following proposition.

Proposition 2. Consider p and q lying on separate vertical edges within the same horizontal strip between the i^{th} and $(i+1)^{\text{th}}$ horizontal lines, for $i \in \{0, \dots, m-2\}$. Without loss of generality, assume $y_1 \leq y_2$ (which can always be achieved by renaming the points). Then,

$$d_N(p, q) = |x_2 - x_1| + |y_2 - y_1| + 2 \min\{y_1 - ia, (i+1)a - y_2\}.$$

Proof. Any path that involves crossing a vertical strip must do so along the grid lines. Since p and q lie within the same horizontal strip between the i^{th} and $(i+1)^{\text{th}}$ lines, the only available paths are at the $y = ia$ and $y = (i+1)a$ grid lines. This imposes a detour penalty requiring to either go *up* to $y = (i+1)a$, or *down* to the $y = ia$. Hence, the path has length $|x_2 - x_1| + |y_1 - s| + |y_2 - s|$ for some $s \in \{ia, (i+1)a\}$. As we are interested in the *shortest* path, we have that $d_N(p, q) = \min_{s \in \{ia, (i+1)a\}} (|x_2 - x_1| + |y_1 - s| + |y_2 - s|)$. Since, $y_1 \leq y_2$, for $s = ia$ we have

$$|y_1 - ia| + |y_2 - ia| = (y_2 - y_1) + 2(y_1 - ia),$$

while for $s = (i + 1)a$, we have

$$|y_1 - (i + 1)a| + |y_2 - (i + 1)a| = (y_2 - y_1) + 2\{(i + 1)a - y_2\}.$$

The shortest path then involves the minimum of these two expressions, which gives the detour penalty of $2 \min\{y_1 - ia, (i + 1)a - y_2\}$ in the expression for $d_N(p, q)$. \square

Corollary 1. *Consider p and q lying on separate horizontal edges within the same vertical strip between the j^{th} and $(j + 1)^{\text{th}}$ vertical lines, for $j \in \{0, \dots, n - 2\}$. Without loss of generality, assume $x_1 \leq x_2$. Then $d_N(p, q) = |x_2 - x_1| + |y_2 - y_1| + 2 \min\{x_1 - jb, (j + 1)b - x_2\}$.*

Proof. This follows immediately, using the same arguments as for Proposition 2, with the roles of x and y , and a and b , swapped. \square

Proposition 2, along with Corollary 1, shows that the shortest distance $d_N(p, q)$ is the sum of the usual Manhattan distance and a detour penalty term, where the latter is due to the constraint of moving along the edges defined by G . Some example point pairs are shown in Figure 4. Note that in either case, if at least one point lies at the intersection of the edges on the strip boundary, one of the two quantities evaluated in the minimum function is 0, so that the penalty term vanishes and $d_N(p, q) = d_M(p, q)$.

4. Distribution of the detour penalty

While Section 3 focused on calculating the distance between two given fixed points in G in a deterministic manner, we will now explore the grid network distance d_N from a probabilistic perspective, with a particular focus on the detour penalty. In particular, the distribution of the detour penalty involved in moving between two random points on the grid G will be studied in detail. Throughout, we assume that demand is uniformly distributed along the edges of the grid network G , with respect to the edge lengths, and that the random points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ are independently selected.

As established in Section 3, the network shortest distance d_N entails the addition of a detour penalty whenever p and q lie on distinct parallel edges within the same strip. Write

$$d_N(p, q) = d_M(p, q) + \delta(p, q),$$

where $\delta(p, q)$ denotes the detour penalty involved in moving between the random points p and q on G . Then $\delta(p, q) > 0$ in two different configurations:

- (i) both points lie on separate vertical edges in the same horizontal strip;
- (ii) both points lie on separate horizontal edges in the same vertical strip.

We consider the two cases separately, focusing on the grid cell level.

4.1. Case 1: common horizontal strip detour penalty

Suppose that the random points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ fall within the same horizontal strip, that is, between two consecutive horizontal grid lines such that $ia <$

$Y_k \leq (i + 1)a$, for each $k \in \{1, 2\}$, and some $i \in \{0, \dots, m - 2\}$. In this setting, both x -coordinates lie on vertical grid lines, with $X_1 = j_1 b$ and $X_2 = j_2 b$, for some $j_1 \neq j_2 \in \{0, \dots, n - 1\}$. It is then clear that the horizontal contribution to the network distance is determined in the same way as the Manhattan distance, specifically here, $|X_2 - X_1| = b|j_2 - j_1|$. Note that whenever $j_1 = j_2$ there is no detour penalty as the horizontal distance is null and one can move from Y_1 to Y_2 along the edge without a detour.

The vertical contribution to the network distance, on the other hand, involves a detour penalty as movement through the cells of a grid is not possible. Indeed, the shortest vertical path between Y_1 and Y_2 involves first reaching a horizontal grid line (at an intersection), and then returning to the height of the other point. That is, one either has to travel *up-then-down*, or *down-then-up*, along the grid network.

Within the strip delineated by the grid lines $y = ia$ and $y = (i + 1)a$, each vertical edge can be partitioned at the midline $y = ia + a/2$, yielding two half-segments each of length $a/2$. For each component $k = 1, 2$, we then consider the relative position of Y_k within its respective half-segment. In particular, let Z_k denote the vertical distance between the closest half-segment *below* Y_k and the point Y_k itself. The remaining portion of the half-segment is thus of length $a/2 - Z_k$ and represents the vertical distance between Y_k and its closest upper half-segment. This is illustrated in Figures 5 and 6, from which it is clear that $|Y_2 - Y_1| = |Z_2 - Z_1|$.

The conditional distribution of Z_1 and Z_2 can easily be established from the uniformity of demand across the edges of G . In particular, conditional on $ia \leq Y_k \leq (i + 1)a$, $k = 1, 2$, the random variables Z_1 and Z_2 are themselves independent and uniformly distributed on the interval $[0, a/2]$, that is,

$$Z_1, Z_2 \stackrel{iid}{\sim} \mathcal{U}[0, a/2].$$

The shortest vertical distance between Y_1 and Y_2 on the network can then be expressed in terms of the random variables Z_1 and Z_2 , by choosing the shorter of the two possible detours via the upper or lower horizontal grid line.

Consider first the case where p and q lie on *different* vertical half-segments within the same horizontal strip, as illustrated in Figure 5. In the configuration with Z_1 in the upper half-segment and Z_2 in the lower half-segment, the vertical separation on the network entails $|Z_2 - Z_1|$ in addition to the detour which considers the minimum between $a - 2Z_1$ and $2Z_2$. On the other hand, when the positions are flipped such that Z_1 is located the lower half-segment and Z_2 in the upper half-segment, by symmetry, the detour becomes the minimum of $2Z_1$ and $a - 2Z_2$.

Next consider the setting where p and q lie on the *same* vertical half-segment within a horizontal strip, as illustrated in Figure 6. As before, the vertical detour can be realized in two ways: either moving *up* to the $y = (i + 1)a$ grid line, or moving *down* to the $y = ia$ grid line. Clearly, if both points are positioned in the upper half-segment portion of the vertical edge, i.e., if $ia + a/2 < Y_k \leq (i + 1)a$ for $k = 1, 2$, then the shortest path is via the upper horizontal grid line of the strip and the detour penalty is the minimum of $a - 2Z_1$ and $a - 2Z_2$. On the other hand, when $ia < Y_k \leq ia + a/2$ for $k = 1, 2$, movement along the lower horizontal grid line is optimal and the penalty becomes the minimum of $2Z_1$ and $2Z_2$.

To summarize, the penalty term in the common horizontal strip configuration can take on four forms, depending on the relative position of Y_1 and Y_2 within the half-

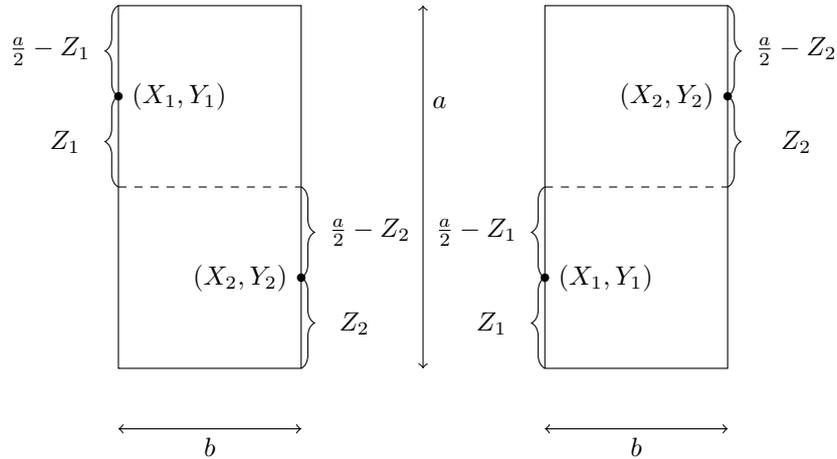


Figure 5.: Case 1: two points in the same horizontal strip, on distinct vertical half-segments.

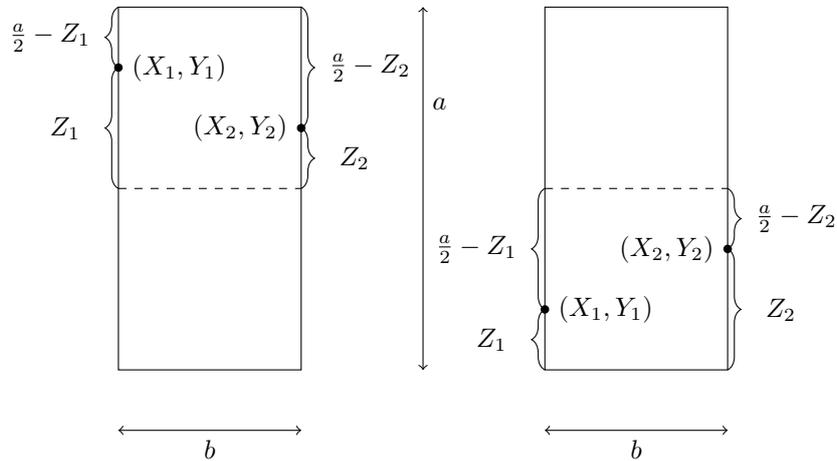


Figure 6.: Case 1: two points in the same horizontal strip, on the same vertical half-segment.

segments. Specifically,

$$\delta(p, q) = \begin{cases} \min\{a - 2Z_1, 2Z_2\} & \text{if } Y_1 \in [ia + a/2, (i + 1)a], Y_2 \in [ia, ia + a/2] \\ \min\{2Z_1, a - 2Z_2\} & \text{if } Y_1 \in [ia, ia + a/2], Y_2 \in [ia + a/2, (i + 1)a] \\ \min\{a - 2Z_1, a - 2Z_2\} & \text{if } Y_1 \in [ia + a/2, (i + 1)a], Y_2 \in [ia + a/2, (i + 1)a] \\ \min\{2Z_1, 2Z_2\} & \text{if } Y_1 \in [ia, ia + a/2], Y_2 \in [ia, ia + a/2]. \end{cases}$$

Note that while these paths lead to different expressions, they are all equivalent in distribution. Indeed, for $k \in \{1, 2\}$, the random variable Z_j follows a uniform distribution on the interval $[0, a/2]$ and thus $2Z_j \sim \mathcal{U}[0, a]$, which in turn implies that $a - 2Z_j \sim \mathcal{U}[0, a]$. As we have independence in the components Z_1 and Z_2 , each possible term for $\delta(p, q)$ involves the minimum of two independent $\mathcal{U}[0, a]$ random variables.

4.2. Case 2: common vertical strip detour penalty

The random behavior of the detour penalty in the setting where the random points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ fall within the same vertical strip is analogous to the case of a common horizontal strip, but will nonetheless now be detailed for completeness. Suppose that p and q lie between two consecutive vertical grid lines such that $jb < X_k \leq (j+1)b$ for $k = 1, 2$ and some $j \in \{0, \dots, n-2\}$. In this setting, both y -coordinates lie on horizontal grid lines, with $Y_1 = i_1a$ and $Y_2 = i_2a$ for some $i_1 \neq i_2 \in \{0, \dots, m-1\}$. Accordingly, the vertical contribution to the network distance is simply $|Y_2 - Y_1| = a|i_2 - i_1|$. Note that, as before, no detour penalty occurs if $i_1 = i_2$ as the vertical distance is then null and movement from X_1 to X_2 can proceed directly along the edge.

Analogously to case 1, a detour penalty occurs in the horizontal distance as the shortest path between X_1 and X_2 must reach a vertical line (an intersection) and then return to the horizontal position of the other point: one either has to travel *east-then-west* or *west-then-east*. As before, we partition each horizontal edge within the vertical strip at the midline $x = jb + b/2$, yielding two half-segments of equal length $b/2$. For $k = 1, 2$, let Z_k^\dagger denote the horizontal distance from the left endpoint of the half-segment containing X_k , as illustrated in Figures 7 and 8. As before, it is clear that $|X_2 - X_1| = |Z_2 - Z_1|$. Under the uniform along-edge model, conditional on the strip, these distances are independent and uniformly distributed on $(0, b/2)$, that is,

$$Z_1^\dagger, Z_2^\dagger \stackrel{iid}{\sim} \mathcal{U}[0, b/2].$$

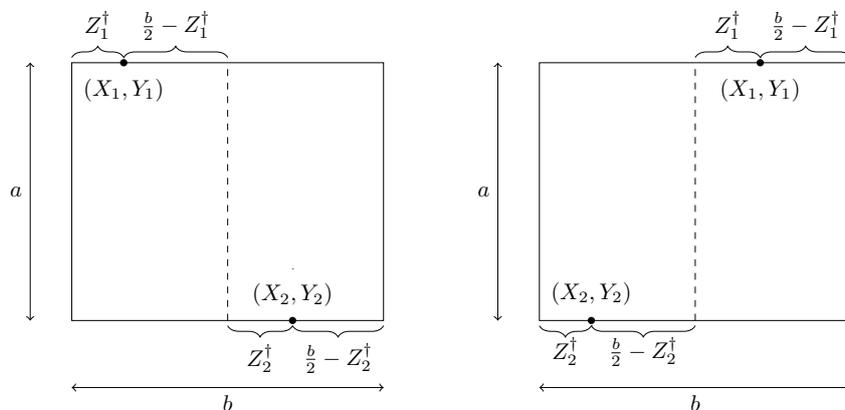


Figure 7.: Case 2: two points in the same vertical strip, on different horizontal half-segments.

The shortest horizontal distance on the network between X_1 and X_2 can then be expressed in terms of Z_1^\dagger and Z_2^\dagger . In the particular case where the points p and q lie on different horizontal half-segments in the same vertical strip, as in Figure 7, the detour penalty is given by

$$\delta(p, q) = \begin{cases} \min \left\{ 2Z_1^\dagger, b - 2Z_2^\dagger \right\} & \text{if } X_1 \in [jb, jb + b/2], X_2 \in [jb + b/2, (j+1)b] \\ \min \left\{ b - 2Z_1^\dagger, 2Z_2^\dagger \right\} & \text{if } X_1 \in [jb + b/2, (j+1)b], X_2 \in [jb, jb + b/2]. \end{cases}$$

On the other hand, when the points lie on the same horizontal half-segment within a vertical strip, as illustrated in Figure 8, the detour in the horizontal direction is

$$\delta(p, q) = \begin{cases} \min \left\{ 2Z_1^\dagger, 2Z_2^\dagger \right\} & \text{if } X_1 \in [jb, jb + b/2], X_2 \in [jb, jb + b/2] \\ \min \left\{ b - 2Z_1^\dagger, b - 2Z_2^\dagger \right\} & \text{if } X_1 \in [jb + b/2, (j+1)b], X_2 \in [jb + b/2, (j+1)b]. \end{cases}$$

As before, while the expressions for $\delta(p, q)$ differ across the four distinct scenarios, they are equal in distribution since, for $j \in \{1, 2\}$, Z_j^\dagger and $\frac{b}{2} - Z_j^\dagger$ are both uniformly distributed on the interval $[0, b/2]$, and hence $2Z_j^\dagger$ and $b - 2Z_j^\dagger$ are both $\mathcal{U}[0, b]$. Then, similarly to the case in the common horizontal strip, the penalty terms here all involve the minimum of two independent $\mathcal{U}[0, b]$ random variables.

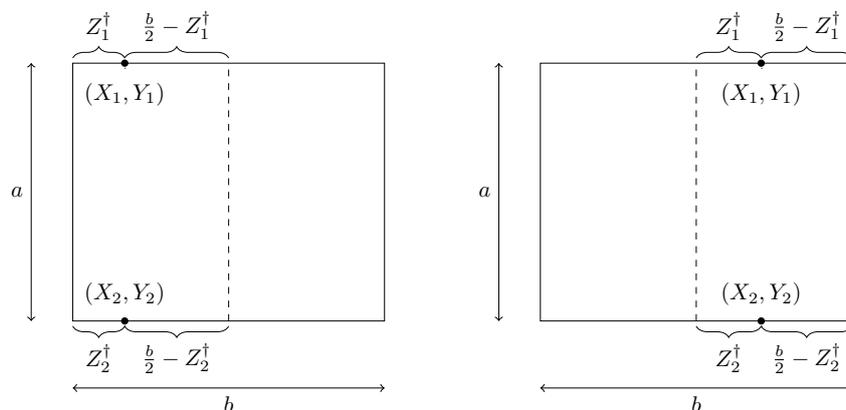


Figure 8.: Case 2: two points in the same vertical strip, on the same horizontal half-segment (left: both in the left half; right: both in the right half).

4.3. Combining the two cases: penalty distributions

We are interested in characterizing the random behavior of the detour induced by the grid network. In both case 1 (subsection 4.1) and case 2 (subsection 4.2), all possible configurations of p and q along the half-segments result in a detour penalty which involves the minimum of two independent uniform random variables. Specifically, let $U_1, U_2 \stackrel{iid}{\sim} \mathcal{U}[0, a]$ and $U_1^\dagger, U_2^\dagger \stackrel{iid}{\sim} \mathcal{U}[0, b]$, we then have that

$$\delta(p, q) = \begin{cases} \min\{U_1, U_2\} & \text{in the vertical detour} \\ \min\{U_1^\dagger, U_2^\dagger\} & \text{in the horizontal detour.} \end{cases}$$

The following two results establish the distribution of these two penalty terms.

Lemma 1. *Let $U_1, U_2 \stackrel{iid}{\sim} \mathcal{U}[0, a]$ and set $V := \min\{U_1, U_2\}$. The distribution function of the random variable V is then given by*

$$F_V(v) = \Pr(V \leq v) = 1 - \left(1 - \frac{v}{a}\right)^2, \quad 0 \leq v \leq a.$$

Proof. Since $U_1, U_2 \stackrel{iid}{\sim} \mathcal{U}[0, a]$, then for any $v \in [0, a]$

$$\begin{aligned} F_V(v) &= \Pr(V \leq v) = \Pr(\min\{U_1, U_2\} \leq v) \\ &= 1 - \Pr(\min\{U_1, U_2\} > v) = 1 - \Pr(U_1 > v) \Pr(U_2 > v) \\ &= 1 - \left(1 - \frac{v}{a}\right)^2. \end{aligned}$$

□

Lemma 2. Let $U_1^\dagger, U_2^\dagger \stackrel{iid}{\sim} \mathcal{U}[0, b]$ and set $V^\dagger := \min\{U_1^\dagger, U_2^\dagger\}$. The distribution function of the random variable V^\dagger is then given by

$$F_{V^\dagger}(v) = \Pr(V^\dagger \leq v) = 1 - \left(1 - \frac{v}{b}\right)^2, \quad 0 \leq v \leq b.$$

The proof for Lemma 2 is omitted, as it is analogous to that of Lemma 1.

We can further characterize the random behavior of the detour penalties in terms of its expectation, as detailed in the following two results.

Corollary 2. The random variable V has expectation $\mathbb{E}(V) = a/3$.

Proof. As detailed in Lemma 1, V takes values in $[0, a]$ and has distribution function

$$F_V(v) = \Pr(V \leq v) = 1 - \left(1 - \frac{v}{a}\right)^2, \quad 0 \leq v \leq a.$$

Since V is a nonnegative random variable, we can write its expectation by integrating the corresponding survival function, viz.

$$\mathbb{E}(V) = \int_0^\infty \Pr(V > v) dv = \int_0^a (1 - F_V(v)) dv = \int_0^a \left(1 - \frac{v}{a}\right)^2 dv.$$

Evaluating the integral, we obtain $\mathbb{E}(V) = a/3$, as claimed. □

Corollary 3. The random variable V^\dagger has expectation $\mathbb{E}(V^\dagger) = b/3$.

Proof. The argument is identical to that of Corollary 2, with a replaced by b . □

5. Expected distances on grid networks

In this section, we compare the expected shortest-path distance in the grid network $G = (m, n, a, b)$, i.e., $\mathbb{E}[d_N(p, q)]$, with the corresponding expected Manhattan distance $\mathbb{E}[d_M(p, q)]$. Throughout this section, the random points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ are assumed to be independent and uniformly distributed along the edges of G , with respect to edge lengths. More specifically, the grid network $G = (m, n, a, b)$ is comprised of $(m-1)n$ vertical edges, each of length a , and $m(n-1)$ horizontal edges, each of length b , yielding a total edge length of $L = (m-1)na + m(n-1)b$. The probability that a random point lies on a given edge e is then proportional to its length $\ell(e)$ with respect to the entire grid length, that is, $\Pr(e) = \ell(e)/L$. Moreover, given that a point p lies along the edge e , its relative position along the edge is uniformly distributed on the interval $[0, a]$ in the case of vertical edge, and $[0, b]$ in the case of a horizontal edge.

We first compute the expected Manhattan distance under this edge-uniform demand model.

Proposition 3 (Expected Manhattan distance). *Consider the grid network $G = (m, n, a, b)$, whose bounding rectangle is $[0, (n-1)b] \times [0, (m-1)a]$. Let p and q be two independent points uniformly distributed along the edges of G , with respect to edge lengths. Let $L := (m-1)na + m(n-1)b$ denote the total edge length of G . Then the expected Manhattan distance on G is given by*

$$\mathbb{E}[d_M(p, q)] = \frac{1}{3L^2} \left\{ a^3 n^2 (m-1)^3 + a^2 b n (m-1)(n-1)(2m^2 + mn - n - 1) \right. \\ \left. + ab^2 m (m-1)(n-1)(2n^2 + mn - m - 1) + b^3 m^2 (n-1)^3 \right\}.$$

Proof. The proof proceeds by conditioning on the type of edge on which each of p and q lie (namely, whether each edge is horizontal or vertical, and whether or not the edges fall within the strip), and using the fact that each edge is selected with probability proportional to its length. In each case, the Manhattan distance decomposes into one-dimensional absolute differences whose expectations can be computed explicitly. Summing over all edge-type configurations for p and q yields the stated expression. The details are straightforward, but algebraically tedious, and are thus omitted here for brevity. A comprehensive proof is provided in Appendix A. \square

In contrast, the network distance $d_N(p, q)$ in G must respect the available edges and cannot traverse inside blocks. As detailed in Proposition 1, the only settings in which $d_N(p, q)$ exceeds $d_M(p, q)$ are when p and q lie on distinct parallel edges within the same strip. In this case, the shortest grid distance between p and q entails a detour penalty. We now quantify the expected value of this detour penalty on the grid network, using the results from Section 4.3.

5.1. Frequency of detours and the expected grid distance

Recall that under the edge-uniform demand model on the grid network $G = (m, n, a, b)$, the probability that a random point (X, Y) lies on a given edge e is given by $\Pr(e) = \ell(e)/L$, where $\ell(e)$ corresponds to the length of the edge e and L denotes the total edge length, i.e., $L = (m-1)na + m(n-1)b$.

Let \mathcal{H} denote the event that independent points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ lie on distinct vertical edges of the same horizontal strip. The grid network $G = (m, n, a, b)$ has a total of $(m-1)$ horizontal strips, each containing n distinct edges. As such, there are $n(n-1)$ possible configurations in which two points can fall along distinct edges within any given horizontal strip. The edge-uniformity model implies that the probability of falling along any vertical edge is a/L , and thus

$$\Pr(\mathcal{H}) = (m-1)n(n-1) \left(\frac{a}{L} \right)^2,$$

Analogously, let \mathcal{V} denote the event that p and q lie on distinct horizontal edges of the same vertical strip. Using similar arguments,

$$\Pr(\mathcal{V}) = (n-1)m(m-1) \left(\frac{b}{L} \right)^2.$$

As established in Proposition 1, a detour penalty is induced in either of the events \mathcal{H} and \mathcal{V} ; otherwise $d_N(p, q) = d_M(p, q)$. Corollaries 2 and 3 determined that the average detour penalty is $a/3$ in the horizontal-strip case and $b/3$ in the vertical-strip case. Hence, the expected excess distance required for traveling on the grid network, in comparison to the Manhattan distance, is given by

$$\mathbb{E}[d_N(p, q) - d_M(p, q)] = \frac{a}{3} \Pr(\mathcal{H}) + \frac{b}{3} \Pr(\mathcal{V}),$$

Combining these results yields the following proposition.

Proposition 4 (Expected penalty distance). *Let p and q be independent random points on the grid network $G = (m, n, a, b)$, distributed uniformly with respect to edge length, and let $d_M(p, q)$ denote the Manhattan distance between p and q . Then*

$$\mathbb{E}[d_N(p, q)] = \mathbb{E}[d_M(p, q)] + \Delta(m, n, a, b),$$

where

$$\Delta(m, n, a, b) = \frac{(m-1)(n-1)(a^3n + b^3m)}{3\{an(m-1) + bm(n-1)\}^2}. \quad (1)$$

In particular, $\Delta(m, n, a, b) > 0$ for $m, n \geq 2$, and thus

$$\mathbb{E}[d_N(p, q)] > \mathbb{E}[d_M(p, q)].$$

Proof. First, write $d_N(p, q) = d_M(p, q) + \delta(p, q)$, where $\delta(p, q)$ is used to denote the detour penalty induced by the grid network. Recall that $\delta(p, q)$ is non-zero whenever p and q lie on distinct parallel edges within the same strip, that is, in the event $\mathcal{H} \cup \mathcal{V}$. As the events \mathcal{H} and \mathcal{V} are disjoint, it follows by the law of total expectation that

$$\mathbb{E}[\delta(p, q)] = \mathbb{E}[\delta(p, q)|\mathcal{H}] \Pr(\mathcal{H}) + \mathbb{E}[\delta(p, q)|\mathcal{V}] \Pr(\mathcal{V}).$$

The results of Corollaries 2 and 3, respectively, establish that $\mathbb{E}[\delta(p, q) | \mathcal{H}] = a/3$ and $\mathbb{E}[\delta(p, q) | \mathcal{V}] = b/3$. Simple algebra then yields

$$\mathbb{E}[\delta(p, q)] = \frac{(m-1)(n-1)(a^3n + b^3m)}{3(an(m-1) + bm(n-1))^2}.$$

It is clear that the above expression is strictly positive whenever $m, n \geq 2$. □

The above results imply that, on average, the shortest distance between two arbitrary points on a grid network G exceeds the Manhattan distance. However, this penalty vanishes as the grid cells become arbitrarily small, that is, as the grid becomes finer. This is formalized in the following Proposition.

Proposition 5 (Vanishing expected grid penalty under refinement). *Consider a rectangle of width $W > 0$ and height $H > 0$, and a sequence of grids $G_k = (m_k, n_k, a_k, b_k)$*

whose vertex sets lie on this rectangle, with

$$a_k = \frac{H}{m_k - 1}, \quad b_k = \frac{W}{n_k - 1},$$

and $m_k, n_k \rightarrow \infty$ as $k \rightarrow \infty$. Let $\Delta(m, n, a, b)$ be the expected detour penalty from Proposition 4. Then

$$\Delta(m_k, n_k, a_k, b_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

so that the expected grid distance converges to the expected Manhattan distance:

$$\mathbb{E}[d_N(p, q)] \rightarrow \mathbb{E}[d_M(p, q)].$$

Proof. Write $m = m_k, n = n_k, a = a_k, b = b_k$ for brevity, and note that $(m-1)a = H$ and $(n-1)b = W$ by construction. Substituting $a = H/(m-1)$ and $b = W/(n-1)$ into $\Delta(m, n, a, b)$ in Proposition 4 yields

$$\Delta(m, n, a, b) = \frac{H^3 n(n-1)}{3(m-1)^2 (Hn + Wm)^2} + \frac{W^3 m(m-1)}{3(n-1)^2 (Hn + Wm)^2}.$$

Using the bounds $n-1 \leq n$ and $m-1 \leq m$, and observing that $Hn + Wm \geq Hn$ and $Hn + Wm \geq Wm$, we obtain

$$\Delta(m, n, a, b) \leq \frac{H^3 n^2}{3(m-1)^2 H^2 n^2} + \frac{W^3 m^2}{3(n-1)^2 W^2 m^2} = \frac{H}{3(m-1)^2} + \frac{W}{3(n-1)^2}.$$

Since $m_k, n_k \rightarrow \infty$, both terms on the right-hand side tend to zero, hence $\Delta(m_k, n_k, a_k, b_k) \rightarrow 0$. Finally, by Proposition 4 we have $\mathbb{E}[d_N(p, q)] = \mathbb{E}[d_M(p, q)] + \Delta(m, n, a, b)$, so $\mathbb{E}[d_N(p, q)] \rightarrow \mathbb{E}[d_M(p, q)]$. This completes the proof. \square

5.2. A symmetric square grid

The expression (1) simplifies considerably for a square grid with $m = n$ and $a = b$.

Corollary 4 (Square grid). *Consider a square grid with $m = n \geq 2$ equally spaced horizontal and vertical lines and a common spacing $a = b > 0$. Then*

$$\mathbb{E}[d_N(p, q)] = \mathbb{E}[d_M(p, q)] + \frac{a}{6m}.$$

Proof. Plugging in the values for $\Delta(m, m, a, a)$ gives the desired result. \square

Corollary 4 highlights that, for a fixed spacing a , the expected detour penalty $\Delta(m, m, a, a)$ decreases like $1/m$ as the number of streets increases. Equivalently, for a fixed overall side length $(m-1)a$, the expected grid distance approaches the expected Manhattan distance as the grid is refined, in agreement with the intuitive behaviour discussed in Section 6.

6. Illustrative examples: detour penalties for real city grids

This section provides concrete numerical illustrations of Proposition 4, using block dimensions reported for several well-known grid cities. The goal is not to produce a calibrated urban model, but rather to translate the closed-form average penalty $\Delta(m, n, a, b)$ into interpretable magnitudes in meters under plausible grid parameters.

6.1. From city maps to graphs

Recall that $G = (m, n, a, b)$ is a rectangular grid with $m \geq 2$ horizontal lines and $n \geq 2$ vertical lines, with spacings $a > 0$ (between consecutive horizontal lines) and $b > 0$ (between consecutive vertical lines). The bounding rectangle therefore has height $(m - 1)a$ and width $(n - 1)b$.

To connect to urban settings, we proceed as follows. Fix a neighborhood window of height H and width W (here we use $H = W = 1000$ meters for all cities, for comparability). Given a typical north–south block length a and an east–west block length b , we select

$$m = 1 + \text{round}\left(\frac{H}{a}\right), \quad n = 1 + \text{round}\left(\frac{W}{b}\right).$$

where $\text{round}(x)$ denotes the integer closest to x and we round down when there is a tie (at 0.5). Therefore, $(m - 1)a$ and $(n - 1)b$ are close to (H, W) . This makes the examples reflect a 1 km \times 1 km district containing a realistic number of blocks. We refer to this selected area as a *window*.

Table 1 reports the grid settings (m, n, a, b) constructed from published block sizes (Chicago Department of Transportation, 2007; Eggimann, 2025; Wikipedia, 2026), together with the resulting expected penalty Δ . For context, we also report the ratio $\Delta/\mathbb{E}[d_M]$, which quantifies the average relative increase beyond the Manhattan distance induced by the detour constraint.

City	a (meters)	b (meters)	m	n	$\mathbb{E}[d_M]$ (meters)	Δ (meters)	$\Delta/\mathbb{E}[d_M]$ (%)
Manhattan (New York)	274.00	80.00	5	13	725.99	11.91	1.64
Chicago	200.00	100.00	6	11	709.11	5.42	0.76
Melbourne (CBD blocks)	200.00	100.00	6	11	709.11	5.42	0.76
Barcelona (Eixample)	113.33	113.33	10	10	715.87	1.89	0.26
Barcelona (superilla, 3 \times 3 blocks)	339.99	339.99	4	4	779.14	14.17	1.82
Barcelona (superilla, 400m crossings)	400.00	400.00	3	3	644.44	22.22	3.45

Table 1.: Expected detour penalty $\Delta(m, n, a, b)$ for several city-grid parameterizations over a ≈ 1 km² window.

6.1.1. Manhattan (New York): canonical blocks

The Manhattan grid is a canonical case where “Manhattan distance” is a mathematical reference point. A “standard block” is approximately 80 m by 274 m. Interpreting the long dimension as north–south spacing ($a = 274$ m) and the short dimension as east–

west spacing ($b = 80$ m), a one km square window contains about $m = 5$ horizontal lines and $n = 13$ vertical lines.

The resulting expected detour penalty is 11.91 m. While this is small relative to the overall expected Manhattan distance in the same window (about 1.64% on average), this is not negligible in absolute terms when one aggregates many trips, or when travel times are sensitive at the tens-of-meters scale (e.g., first/last-mile access in dense districts).

6.1.2. Chicago and Melbourne: the 100×200 m benchmark

A typical Chicago block is about 100 m by 200 m, and the same dimensions are frequently used to describe central Melbourne blocks. Using $(a, b) = (200, 100)$ m, a 1 km by 1 km window contains $(m, n) = (6, 11)$ horizontal and vertical lines, respectively, thus yielding an expected penalty of 5.42 m.

Compared to Manhattan, the penalty is smaller in absolute terms, largely because the aspect ratio is less extreme and the number of blocks within the window is slightly larger in the short dimension, reducing the probability that two independently sampled edge points land in a configuration requiring a detour.

6.1.3. Barcelona (Eixample): near-square blocks and the square-grid corollary

The Eixample expansion is famously regular; the Cerdà plan is commonly summarized by a square block structure of about 113.33 m. Modeling this as $a = b = 113.33$ m and again using an approximately one km² window gives $m = n = 10$. The expected detour penalty becomes approximately 1.89 m. Indeed, according to the results of Corollary 4, for square grids with $m = n$ and $a = b$, the penalty simplifies to

$$\Delta(m, m, a, a) = \frac{a}{6m}.$$

Plugging $a = 113.33$ and $m = 10$ yields $\Delta \approx 113.33/60 \approx 1.89$ m, matching the result. In other words, in a district with many small, near-square blocks, the average extra travel beyond Manhattan distance is only a couple meters.

6.1.4. Barcelona superilles: a coarsened grid

Barcelona's *superilla* (superblock) concept is often described as grouping 3×3 city blocks (nine blocks) into a larger cell, with through travel shifted to the perimeter and major crossings roughly every 400 m. While the physical street fabric remains, many mobility policies (especially for motorized through traffic) behave as if the internal network were partially suppressed. We consider a stylized parameterization: when using the Eixample block length 113.33 m, a 3×3 grouping yields an effective spacing of about 340 m, corresponding to $m = n = 4$, and the expected penalty increases to 14.17 m. When the spacing is at approximately 400 m as a coarse arterial grid, the same window yields $m = n = 3$ and $\Delta \approx 22.22$ m.

These numbers are still modest in relative terms for long trips, but they are an order of magnitude larger than the Eixample baseline, highlighting a general mechanism: *coarsening the grid increases detour penalties* because strips become wider and there are fewer strips in the window, thereby increasing the chance that two random points fall on edges with a shared strip.

6.2. Scaling remarks and practical interpretation

As established in Proposition 5, refining the grid drives Δ to 0. The examples above illustrate this: dense, fine-grained grids (Example-scale blocks) yield penalties of the order of a few meters, whereas coarse superblock-scale grids yield penalties on the order of tens of meters.

In applications, the right interpretation depends on what “random points” represent. If p and q represent uniformly sampled curbside origins/destinations (deliveries, pickups, micro-mobility docking, etc.), then Δ is a direct correction term to the expected Manhattan distance. If instead demand concentrates near intersections or certain corridors, the relevant penalty may be smaller than the edge-uniform benchmark, since the detour vanishes. Nonetheless, the closed form $\Delta(m, n, a, b)$ remains useful as a compact diagnostic: it quantifies the average “price” (in terms of distance, e.g., meters) of respecting the street graph rather than allowing continuous rectilinear travel through blocks.

These magnitudes can look very small on a per-trip basis, especially when normalized by the expected Manhattan distance (often below a few percent), but two mechanisms make them potentially *non-negligible* for combinatorial routing problems such as the Traveling Salesman Problem (TSP) over a set of customers. In particular, the detour is sparse, but conditionally large and systematic. The penalty arises only in specific geometric configurations (points on distinct parallel edges within the same strip), yet when it arises it is *order-of-block-length*. Moreover, the detour length Δ is driven by (a) the probability of falling in a strip-sharing configuration and (b) a conditional detour whose mean is $a/3$ in the horizontal-strip case and $b/3$ in the vertical-strip case. Thus, the surrogate metric d_M is not merely a noisy approximation of d_N ; it is a *biased, structure-dependent underestimator* that particularly underprices certain short “across-the-block” moves that must in fact detour to a strip boundary. Therefore, combinatorial optimization can amplify small metric distortions.

In a TSP, for example, the optimal tour is determined by an ordering of edges among a huge number of feasible permutations. Even if the *average* pairwise discrepancy is small, the *relative ranking* of competing edges can change, especially when many candidate edges have similar Manhattan lengths. This is most acute when the solution uses many *short* edges (as in last-mile delivery), because the detour mechanism is strip-local and a fixed additive error is proportionally larger for short moves. This phenomenon has been observed empirically in closely related settings. Thus, Boyacı, Dang, and Letchford (2021) study the impact of replacing true road-network distances by planar Euclidean distances in Steiner TSP instances built from real city road graphs. They report that the Euclidean approximation can be accurate on average, yet can also yield tours that are *noticeably different and worse* in certain cities (e.g., Mexico City). Importantly, they provide a concrete explanation that is directly analogous to our grid-vs-Manhattan setting: in an optimal Euclidean TSP solution, consecutive required nodes tend to be close, while the ratio between true and surrogate distances tends to be higher when the surrogate distance is small. They also quantify the potential scale of the resulting mismatch: for Steiner TSP instances, average ratios between the true and surrogate distances exceed 1.3 in multiple configurations.

Our city-scale examples should therefore be read as *mean-distance assessments*, not as a guarantee that Manhattan-based routing will be near-optimal on the true grid network. Although the average relative increase can be small, the absolute meters can become meaningful when aggregated over many movements or when decisions are sensitive. In other words, for TSP-style applications on grids, one should prefer using

d_N when constructing the cost matrix. To this end, we present a simple TSP instance where d_M can be misleading. Consider a grid network $G = (m, n, a, b)$. Assume that the number of horizontal strips ($m - 1$) and the number of vertical strips ($n - 1$) are both even (equivalently, m and n are odd). Place customers at the *midpoint of every vertical edge*, i.e.,

$$p_{i,j} := (jb, ia + \frac{a}{2}), \quad i = 0, \dots, m - 2, \quad j = 0, \dots, n - 1,$$

so there are $(m - 1)n$ customers, one at the middle of each vertical edge. Note that midpoints force a full around-the-block detour. Consider a strip spanning from the gridline $y = ia$ to $y = (i + 1)a$, with distinct x -axis locations at $j \neq j'$. The two customers $p_{i,j}$ and $p_{i,j'}$ lie on *distinct vertical edges within the same horizontal strip*. By Proposition 2, any such pair satisfies

$$d_N(p_{i,j}, p_{i,j'}) = d_M(p_{i,j}, p_{i,j'}) + 2 \min \left\{ \frac{a}{2}, \frac{a}{2} \right\} = d_M(p_{i,j}, p_{i,j'}) + a.$$

In particular, even the nearest horizontal neighbor in the same strip has

$$d_M(p_{i,j}, p_{i,j+1}) = b \quad \text{but} \quad d_N(p_{i,j}, p_{i,j+1}) = b + a.$$

Clearly, d_M underapproximates every ‘‘around-the-block’’ move between midpoints by an *additive* a .

Consider now the natural snake tour T that, in each strip i , visits the customers $(p_{i,0}, p_{i,1}, \dots, p_{i,n-1})$ consecutively when i is even, and $(p_{i,n-1}, p_{i,n-2}, \dots, p_{i,0})$ consecutively when i is odd, connecting consecutive strips by a vertical edge at the current end column. Since $(m - 1)$ is even, this snake closes with a final purely vertical move back to the starting customer. Along this tour:

- there are exactly $(m - 1)(n - 1)$ within-strip horizontal steps, and each such step incurs the full midpoint penalty a (since it connects two midpoints on distinct vertical edges within the same strip);
- all inter-strip connections are vertical moves along a grid line, hence for those edges $d_N = d_M$.

Therefore, the total grid network length of the tour T , denoted by $L_N(T)$, is given by

$$L_N(T) = L_M(T) + a(m - 1)(n - 1),$$

where $L_M(T)$ is used to represent the Manhattan length of the tour T . Moreover, the Manhattan length of this snake tour is easy to count:

$$L_M(T) = (m - 1)(n - 1)b + 2(m - 2)a,$$

where $(m - 1)(n - 1)b$ is the total horizontal travel inside strips and $2(m - 2)a$ is the total vertical travel (one a -step between each consecutive pair of strips plus one closing vertical move of length $(m - 2)a$). Combining the two displays yields the explicit ratio

$$\frac{L_N(T)}{L_M(T)} = 1 + \frac{a(m - 1)(n - 1)}{(m - 1)(n - 1)b + 2(m - 2)a}.$$

When n is large, the grid is wide and the tour travels overwhelmingly more distance

moving horizontally than vertically. So, the horizontal contribution dominates L_M , and therefore

$$\frac{L_N(T)}{L_M(T)} \approx 1 + \frac{a}{b}.$$

Hence, by choosing elongated blocks with $a \gg b$ (tall blocks compared to their width), the Manhattan-based tour length can underestimate the true network tour length by an *arbitrarily large factor*. This illustrates that, in the simplest possible TSP construction, d_M can be a very poor surrogate for d_N .

7. Conclusions

We have studied rectilinear travel on rectangular grid networks $G = (m, n, a, b)$ and quantified, in closed form, how shortest-path distances on the embedded graph deviates from the famous Manhattan metric. The main contribution is an explicit expression for the expected detour penalty $\Delta(m, n, a, b) = \mathbb{E}[d_N(p, q) - d_M(p, q)]$, which isolates the geometric and combinatorial drivers of excess travel: common strip membership and the need to reach a strip boundary before crossing. The resulting formulas clarify how anisotropy (unequal spacings a and b) and coarseness (small m, n) amplify detours, while grid refinement drives Δ to zero for a fixed physical rectangle.

Our illustrative examples translate these expressions into interpretable magnitudes for real city-like block dimensions. Fine-grained grids (e.g., near-square blocks) yield detour penalties of the order of a few meters, whereas coarsened “effective” grids (as in superblock-style layouts) can increase penalties by an order of magnitude, into the tens of meters, even over the same neighborhood-scale window. Beyond offering a sharp analytical benchmark, these results provide a compact assessment for comparing grid designs and mobility policies: $\Delta(m, n, a, b)$ is a single, physically meaningful correction term that captures the average cost of respecting the street graph rather than permitting continuous rectilinear motion.

Several extensions are natural. One may replace edge-uniform sampling by demand models concentrated near intersections or along corridors, incorporate weighted edges to reflect heterogeneous speeds or turn penalties, and study more realistic perturbations of the grid (missing streets, cul-de-sacs, one-way constraints). Our framework suggests that many such variants will retain a tractable “strip-based” structure, making it possible to preserve interpretability while moving closer to empirical urban networks.

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Appendix A. Proof of Proposition 3

We wish to show that for two random points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ following the edge-uniform demand model on $G = (m, n, a, b)$, the expected Manhattan distance is

$$\begin{aligned} \mathbb{E}[d_M(p, q)] &= \frac{1}{3L^2} \left(a^3 n^2 (m-1)^3 + a^2 b n (m-1)(n-1)(2m^2 + mn - n - 1) \right. \\ &\quad \left. + ab^2 m (m-1)(n-1)(2n^2 + mn - m - 1) + b^3 m^2 (n-1)^3 \right), \end{aligned}$$

where $L := (m-1)na + m(n-1)b$ denotes the total edge length of G .

The grid network $G = (m, n, a, b)$ is such that the total length of all horizontal edges is $L_H = m(n-1)b$, and the total length of all vertical edges is $L_V = n(m-1)a$. Let p_H and p_V denote the probability that an arbitrary random point falls on a horizontal and vertical edge, respectively. It then follows that

$$p_H := \frac{L_H}{L} = \frac{m(n-1)b}{L}, \quad p_V := \frac{L_V}{L} = \frac{n(m-1)a}{L},$$

with $p_H + p_V = 1$. Let H_j denote the event that the random point (X_j, Y_j) falls on a horizontal edge, $j \in \{1, 2\}$. The complement, $V_j := H_j^c$, $j \in \{1, 2\}$, then corresponds to the event that (X_j, Y_j) falls along a vertical edge. Since the random points $p = (X_1, Y_1)$ and $q = (X_2, Y_2)$ are independent and identically distributed, it follows that H_1, H_2 are also independent, with corresponding probability p_H .

Recall that the expected Manhattan distance is given by

$$\mathbb{E}[d_M(p, q)] = \mathbb{E}[|X_1 - X_2|] + \mathbb{E}[|Y_1 - Y_2|].$$

We begin with the first component of the expectation, and for ease of notation set $X = |X_1 - X_2|$. We have

$$\mathbb{E}(X) = \mathbb{E}(X|H_1, H_2)p_H^2 + \{\mathbb{E}(X|H_1, V_2) + \mathbb{E}(X|V_1, H_2)\}p_H p_V + \mathbb{E}(X|V_1, V_2)p_V^2.$$

We now consider each case separately.

Case 1: H_1, H_2

Conditional on H_1, H_2 , both X_1 and X_2 fall uniformly along any of the horizontal edges. Since each of the $n-1$ horizontal edges has equal length b , it follows that

$X_1, X_2 | H_1, H_2 \stackrel{iid}{\sim} \mathcal{U}[0, (n-1)b]$, as the projection of the x -axis covers the whole interval without gaps. As demonstrated in e.g., Gaboune, Laporte, and Soumis (1993), it can then be shown that

$$\mathbb{E}(X | H_1, H_2) = \frac{(n-1)b}{3}.$$

Case 2: H_1, V_2 and V_1, H_2

For $j \in \{1, 2\}$, $X_j | V_j$ is discrete uniform, taking on values in $\{0, b, 2b, \dots, (n-1)b\}$ with equal probability, that is, $\Pr(X_j = rb | V_j) = 1/n$ for $r \in \{0, \dots, n-1\}$. This follows since each vertical line at $x = rb$ carries $(m-1)$ edges of length a and $(m-1)a/L_V = 1/n$. Without loss of generality, suppose that we are in the setting H_1, V_2 , so that $X_1 | H_1 \sim \mathcal{U}[0, (n-1)b]$ independent of $X_2 | V_2$, with $\Pr(X_2 = rb | V_2) = 1/n$ for $r \in \{0, \dots, n-1\}$. Then

$$\begin{aligned} \mathbb{E}(X | H_1, V_2) &= \sum_{r=0}^{n-1} \mathbb{E}(X | H_1, V_2, X_2 = rb) \Pr(X_2 = rb | V_2) \\ &= \frac{1}{n} \sum_{r=0}^{n-1} \mathbb{E}(|X_1 - rb| | H_1). \end{aligned}$$

Now for an arbitrary $U \sim \mathcal{U}[0, L_U]$ and a fixed constant $c \in (0, L_U)$, it is straightforward to see that

$$\mathbb{E}(|U - c|) = \int_0^{L_U} \frac{|u - c|}{L_U} du = \frac{c^2 + (L_U - c)^2}{2L_U}.$$

It follows that for any $r \in \{0, \dots, n-1\}$,

$$\mathbb{E}(|X_1 - rb| | H_1) = \frac{(rb)^2 + \{(n-1)b - rb\}^2}{2(n-1)b} = \frac{(n-1)b}{2} + \frac{r^2b}{(n-1)} - rb.$$

Putting this together, we obtain

$$\mathbb{E}(X | H_1, V_2) = \frac{1}{n} \sum_{r=0}^{n-1} \mathbb{E}(|X_1 - rb| | H_1) = \frac{1}{n} \sum_{r=0}^{n-1} \left(\frac{(n-1)b}{2} + \frac{r^2b}{(n-1)} - rb \right).$$

Using the identities $\sum_{r=0}^{n-1} r = n(n-1)/2$ and $\sum_{r=0}^{n-1} r^2 = n(n-1)(2n-1)/6$, simple algebra yields

$$\mathbb{E}(X | H_1, V_2) = \frac{b(2n-1)}{6}.$$

By symmetry, we also have that $\mathbb{E}(X | V_1, H_2) = b(2n-1)/6$.

Case 3: V_1, V_2

As previously established, conditional on V_1, V_2 , both X_1 and X_2 are discrete uniform with support $\{0, b, 2b, \dots, (n-1)b\}$. Independence between the components implies that $\Pr(X_1 = j_1b, X_2 = j_2b | V_1, V_2) = 1/n^2$ for any $j_1, j_2 \in \{0, \dots, n-1\}$. Thus,

$$\mathbb{E}(X | V_1, V_2) = \frac{1}{n^2} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} |j_1b - j_2b| = \frac{b}{n^2} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} |j_1 - j_2|.$$

It can be shown that $\sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} |j_1 - j_2| = n(n^2 - 1)/3$ and thus

$$\mathbb{E}(X | V_1, V_2) = \frac{b(n^2 - 1)}{3n}.$$

Combining results

Combining the three cases, we obtain

$$\mathbb{E}(X) = \left(\frac{b(n-1)}{3}\right) p_H^2 + \left(\frac{b(2n-1)}{3}\right) p_{HPV} + \left(\frac{b(n^2-1)}{3n}\right) p_V^2.$$

Upon substituting $p_H = m(n-1)b/L$ and $p_V = n(m-1)a/L$, this further simplifies to

$$\mathbb{E}(|X_1 - X_2|) = \frac{b}{3L^2} \left(b^2 m^2 (n-1)^3 + abm(m-1)n(n-1)(2n-1) + a^2(m-1)^2 n(n^2-1) \right).$$

One can proceed in an analogous manner to obtain $\mathbb{E}(|Y_1 - Y_2|)$. Indeed, by the symmetry of the problem, one finds that

$$\mathbb{E}(|Y_1 - Y_2|) = \frac{a}{3L^2} \left(a^2 n^2 (m-1)^3 + abn(n-1)m(m-1)(2m-1) + b^2(n-1)^2 m(m^2-1) \right).$$

The expected Manhattan distance $\mathbb{E}[d_M(p, q)] = \mathbb{E}|X_1 - X_2| + \mathbb{E}|Y_1 - Y_2|$ is then obtained by summing the two expectations, viz.

$$\begin{aligned} \mathbb{E}[d_M(p, q)] = & \frac{1}{3L^2} \left(b^3 m^2 (n-1)^3 + ab^2 m(m-1)n(n-1)(2n-1) \right. \\ & + a^2 b(m-1)^2 n(n^2-1) + a^3 n^2 (m-1)^3 \\ & \left. + a^2 bn(n-1)m(m-1)(2m-1) + ab^2(n-1)^2 m(m^2-1) \right). \end{aligned}$$

Further simplifications yield the result stated in Proposition 3.